Lyapunov Exponent and Plasmon Damping Rate
in Nonabelian Gauge Theories

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Abstract

We argue that the maximal positive Lyapunov exponent of classical SU(N) gauge theory coincides with (twice) the damping rate of a plasmon in the thermal gauge theory.

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I. INTRODUCTION

Numerical studies of Hamiltonian SU($N$) lattice gauge theory in (3+1) dimensions have shown that the gauge fields exhibit chaotic behavior in the classical limit [1]. The numerical value of the largest positive Lyapunov exponent $\lambda_0$ has been obtained for SU(2) and SU(3) with the result [1,2]

$$\lambda_0 = c_N g^2 E_p,$$

(1)

where $E_p$ is the average energy per plaquette, $c_2 \approx 0.17$ for SU(2), and $c_3 \approx 0.10$ for SU(3). For the SU(2) gauge theory the complete spectrum of Lyapunov exponents was obtained on small lattices [3]. These calculations, which follow the evolution of a classical gauge field configuration in Minkowski space, also showed that the energy density distribution on the lattice rapidly approaches a thermal distribution [4]. This finding confirms the expectation of a finite growth rate of the coarse-grained entropy density of the gauge field, which follows from the observation that the sum over all positive Lyapunov exponents at fixed energy density grows like the volume [3]. Hence, at any given level of coarse-graining, the classical gauge field “self-thermalizes” on a time scale of the order of the inverse Lyapunov exponent.

In order to determine the value of the maximal Lyapunov exponent $\lambda_0$, the evolution of the gauge field configurations must be followed over periods $t_0 \gg \lambda_0^{-1}$. The Lyapunov exponent is therefore effectively obtained for gauge fields that are members of a thermal ensemble, and we can identify the average energy per plaquette $E_p$ in (1) with that of a thermalized lattice. At weak coupling the gauge field is a collection of weakly coupled harmonic oscillators, hence the average energy per independent degree of freedom of the classical gauge field is equal to the temperature $T$, yielding $E_p = \frac{2}{3}(N^2 - 1)T$ for SU($N$). The factor $\frac{2}{3}$ accounts for the restrictions imposed by Gauss’ law. We can therefore rewrite the result (1) as

$$\lambda_0 = \frac{2}{3} c_N (N^2 - 1) g^2 T \approx \begin{cases} 0.34 g^2 T & (N = 2), \\ 0.53 g^2 T & (N = 3). \end{cases}$$

(2)
As already noted in [4] these values for $\gamma_0$ coincide, apart from a factor 2, with those of the damping rate of a thermal plasmon at rest, obtained by Braaten and Pisarski [5] in the framework of thermal perturbation theory:

$$\gamma_0 \approx 6.635 \frac{N}{24\pi} g^2 T = \begin{cases} 0.176 g^2 T & (N = 2), \\ 0.264 g^2 T & (N = 3). \end{cases}$$ (3)

The goal of the present article is to establish this connection and to explain the origin of the factor $\lambda_0 / \gamma_0 = 2$.

II. COLLECTIVE PLASMA MODES

We begin by briefly reviewing the derivation of the plasmon damping rate. Nonabelian gauge field fluctuations in a thermal background have been studied extensively in the framework of perturbation theory [6-10]. The gauge field develops massive collective modes (plasmons) with frequency $\omega(k) > k$ due to interaction with “hard” thermal gauge bosons, i.e. excitations with energy of order $T$. The energy of a plasmon at rest is $m_\phi \equiv \omega(0) = \frac{1}{3} \sqrt{NG} T$ in $\text{SU}(N)$ gauge theory. For our purpose it is important that the dispersion relation $\omega(k)$ can be obtained in the framework of semiclassical transport theory, where classical field fluctuations $a_\mu$ are coupled to the quantized thermal excitations of the gauge field [11]. The gauge invariant description of the collective modes requires the introduction of effective $n$-point vertices [8], which can be systematically derived from the effective action [12,13]:

$$\mathcal{L}_{HTL}(a_\mu) = -\frac{3}{2} m_\phi^2 \int d^4 \tau \text{tr} \left( f^{\mu\nu} n_\alpha n_\beta \frac{f^2}{(n \cdot D)^2} f^{\gamma \delta}_{\mu} \right)$$ (4)

where $n_\alpha = (1, \hat{n})$ is a null four-vector, and the integral is over all directions of the spatial unit vector $\hat{n}$. $D^\nu$ stands for the gauge-covariant derivative. We have denoted the collective gauge potential $a_\mu$ and field strength $f_{\mu\nu}$ by lower-case letters to indicate that these describe fluctuations around a thermal background. Note that $\mathcal{L}_{HTL}$ is a classical construction, with the sole exception that the plasmon rest mass $m_\phi$ depends on the energy distribution $n(\omega) = (e^{\omega/T} - 1)^{-1}$ of quantized thermal excitations of the gauge field;
m^2_\phi = \frac{2}{3} N g^2 \frac{\hbar^2}{T} \int \frac{d^3k}{(2\pi)^3} n(\omega)(1 + n(\omega)) = \frac{N g^2}{9} \frac{T^2}{\hbar}.

(5)

At leading order in g, (5) is evaluated for hard thermal quanta with \omega = |\vec{k}|.

Braaten and Pisarski [5] showed that the collective plasmon modes are unstable due to the effective interaction (4). The plasmon damping rate \gamma(k) is defined as imaginary part of the plasmon pole in the Feynman propagator corresponding to decaying plane wave solutions. The rate of instability for a plasmon at rest can be expressed as the imaginary part of the polarization function of the gauge field at the plasmon pole [14]:

\gamma_0 \equiv \gamma(0) = \frac{1}{2m_\phi} Im \Pi_t(m_\phi + i0, 0),

(6)

where the transverse polarization function \Pi_t(\omega, \vec{k}) only depends on soft modes described by (4). The plasmon rest mass exactly cancels from the expression (6) and the result (3) is a pure number multiplied by \frac{g^2T}{\hbar}, which is a classical inverse length scale. In fact, the calculation explicitly makes use of the classical limit of the Bose distribution, \( n(\omega) \rightarrow T/\hbar \omega \), in the evaluation of the loop integral (see eq. (23) of ref. [5]).

Since the effective action (4) can be derived from classical considerations [15], assuming a given spectrum of thermal excitations, it also applies to the collective excitations of the classical gauge field on a lattice. The sole modification is that the spectrum of thermal fluctuations is now given by the limit of the Bose distribution. Denoting the lattice spacing by \( a \) we find

m^2_\phi \rightarrow \frac{2}{3} N g^2 T \sum_{k} \frac{1}{\omega^2} = \frac{1}{3\pi} \frac{N g^2 T}{a}

(7)

in the weak-coupling, large volume limit. The plasmon mass (7) is a purely classical quantity of dimension \( (length)^{-2} \) not containing \( \hbar \), but it diverges in the continuum limit \( a \rightarrow 0 \). This is not surprising, since the lattice spacing serves as a cut-off that is required to regularize the ultraviolet divergences of the classical thermal gauge theory. The exponential growth rate of small classical field fluctuations is not affected by this divergence because it does not depend on the value of \( m_\phi \), as mentioned above. The result (3) for the plasmon damping
rate $\gamma_0$ remains valid if the correct plasmon mass $m_\beta$ in the effective action (4) is replaced by the value (7) for the classical gauge field defined on a lattice.

More intuitively, the independence of $\gamma_0$ from the value of $m_\beta$ can be understood as follows. The cross section for scattering of a thermal gluon on a slow plasmon is:

$$\sigma \approx \frac{N^2}{N^2 - 1} \frac{g^4 \pi^2}{4 \mu^4_D}$$

(8)

where $\mu_D = \sqrt{3m_\beta}$ is the inverse Debye color screening length. The scattering rate is obtained by multiplying with the gluon density in the initial state and with the Bose factor in the final state, yielding:

$$\nu = 2(N^2 - 1) \int \frac{d^3k}{(2\pi)^3} n(\omega)(1 + n(\omega)) \sigma$$

$$= \frac{N^2 - 1}{N} \frac{T \mu^4_D \sigma}{g^2 \hbar^2} \approx \frac{N}{4\pi} g^2 T,$$

(9)

where we have made use of (4). From this result, which has the same structure as the expression (3) for $\gamma_0$, it is obvious that the plasmon mass $m_\beta$ as well as $\hbar$ cancel from the scattering rate.

### III. LYAPUNOV EXPONENTS

The Lyapunov exponents measure the growth rate of infinitesimal perturbations around an exact solution of the classical lattice Yang-Mills equations. Since the maximal Lyapunov exponent $\lambda_0$ was shown to be independent of the lattice spacing, we assume that we can work in the continuum limit whenever adequate. If $A_\mu(x, t)$ is an exact solution of the Yang-Mills equations, the linearized equation for a small perturbation $a_\mu(x, t)$ around $A_\mu$ is

$$D^2 a_\mu - D_\mu D_\nu a^\nu - 2i[F_{\mu\nu}, a^\nu] = 0.$$  

(10)

Here $D_\mu(A) = \partial_\mu - i[A_\mu, \ ]$ is the gauge covariant derivative where the bracket denotes the Lie algebra commutator, and $F_{\mu\nu}$ is the field strength tensor associated with the background field $A_\mu$.  

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The numerical approach to the determination of $\lambda_0$ proceeds by solving (10) for an arbitrary initial condition $a_\mu(x,0)$ and measuring the growth rate of the norm of $a_\mu(x,t)$. To be precise, the maximal Lyapunov exponent was determined in [1,2] from the logarithmic growth rate of the “distance” between neighboring field configurations, defined on the lattice as

$$D[U'_t, U_t] = \frac{1}{2N_p} \sum_p \left| tr \ U_p - tr \ U'_p \right|, \quad (11)$$

where $U_t$ are the group valued link variables, $U_p$ denotes the elementary plaquette operator, and $N_p$ is the total number of spatial plaquettes. In the continuum limit, the distance measure (11) takes the form

$$D[A'_\mu, A_\mu] \propto \int d^3x \left| tr \ B'(x)^2 - tr \ B(x)^2 \right|, \quad (12)$$

where $B(B')$ are the magnetic fields associated with the gauge potential $A_\mu(A'_\mu)$. In going from (11) to (12) we have suppressed the constant factor $(g^2a/2N_p)$, since we are interested only in the growth rate of $(\ln D)$. For an infinitesimal perturbation $a_\mu$ that is a solution of the linearized equation (10), we obtain:

$$D[a_\mu|A_\mu] \equiv D[A_\mu + a_\mu, A_\mu] \propto \int d^3x \left| tr \left( \frac{\partial (tr B^2)}{\partial A_\mu} a_\mu \right) + \frac{1}{2} tr \left( \frac{\partial^2 (tr B^2)}{\partial A_\mu \partial A_\nu} a_\mu a_\nu \right) \right|. \quad (13)$$

The maximal Lyapunov exponent is then formally defined as

$$\lambda_0[A_\mu] = \max_{a_\mu(0)} \lim_{t_0 \to -\infty} \frac{1}{t_0} \ln \frac{D[a_\mu(t_0)|A_\mu]}{D[a_\mu(0)|A_\mu]}. \quad (14)$$

In practice, every randomly chosen initial configuration $A_\mu(0)$ with a fixed average energy density has been found to yield the same value for the maximal Lyapunov exponent $\lambda_0$. The numerical calculations show that the maximal Lyapunov exponent depends only weakly on the lattice size and extrapolates smoothly to the limit of spatially homogeneous gauge potentials on a $1^3$ lattice. We take this as an indication that $\lambda_0$ is associated with long wavelength perturbations $a_\mu(x,t)$ in an appropriately chosen gauge.
IV. ERGODIC LIMIT

We now propose to make use of the fact, noted in the Introduction, that the background
gauge field $A_\mu(x,t)$ rapidly approaches thermal configurations, by replacing the long-time
average of the growth rate of $(\ln \mathcal{D})$ by the canonical average over background gauge fields
$A_\mu$, where the temperature $T$ is chosen such that the thermal energy density equals the
average energy density of the time-dependent background field $A_\mu(x,t)$. The replacement
of the temporal average by the canonical average relies on two conditions: The autocorrelation
function of the background field $A_\mu(x,t)$ must decay on a time scale that is short
compared with the time $t_0$ required for the calculation of the Lyapunov exponent, and the
time evolution of the background field must be ergodic on the time scale $t_0$.

The ergodicity of the background gauge field is assured by its dynamical chaoticity on
time scales long compared to the inverse of the positive Lyapunov exponents, hence the
second condition is fulfilled [16]. On the other hand, if the first condition were violated,
the Lyapunov exponent would depend on the starting configuration $A_\mu(x,t)$. In numerical
studies [1-4] we have found that this is not the case, therefore even without a direct study
of the autocorrelation function we conjecture that the first condition is also satisfied. These
conditions are in accordance with the $g^2T \ll gT \ll T$ hierarchy assumed in hot perturbative
gauge theory.

Next, we formally express the solution of (10) for $t > 0$ by means of the retarded
propagator $\Delta^R_{\mu\nu}$, in the presence of the time-dependent background field $A_\mu(x,t)$ as

$$a_\mu(x,t) = -\int d^3x' \frac{\partial}{\partial t} \Delta^R_{\mu\nu}(x,t,x',0) A_\nu(x',0).$$  \hspace{1cm} (15)

Now let us divide the total time interval $0 \leq t \leq t_0$ into a large number of smaller intervals
of length $\tau = t_0/n$ with $\tau$ larger than the autocorrelation time of the gauge field and $n \gg 1$.
Suppressing Lorentz indices and spatial coordinates and denoting the time derivative by a
dot, (15) can then be written in the form

$$a(t_0) = (-)^n \dot{\Delta}^R(\tau,0|A_n) \cdots \dot{\Delta}^R(\tau,0|A_1)a(0),$$  \hspace{1cm} (16)
where the field configurations $A_1, \ldots, A_n$ are uncorrelated. Moreover, as discussed in the Introduction, the $A_i$ can be considered as independent members of the classical canonical ensemble of gauge fields at finite temperature.

Because the maximal Lyapunov exponent is independent of the initial background field configuration, we are allowed to take the canonical average over the initial background field configuration $A_\mu(x,0)$ in the definition (14) of $\lambda_0$. Because of their uncorrelated nature, we can then also perform the independent canonical average of all the fields $A_i$ in the propagator in (16), obtaining

$$
\left\langle (-)^n \hat{\Delta}^R(\tau, 0|A_n) \cdots \hat{\Delta}^R(\tau, 0|A_1) \right\rangle_T = \left( -\left\langle \hat{\Delta}^R(\tau, 0|A) \right\rangle_T \right)^n \\
= -\hat{\Delta}^R(\tau, 0[T])^n = -\hat{\Delta}^R(t_0, 0[T]).
$$

(17)

where $\Delta^R_{\mu\nu}(\tau, 0[T])$ is the retarded gauge field propagator averaged over a thermal ensemble of background field configurations. We thus find that we can, for the purpose of calculating the maximal Lyapunov exponent, replace (15) by the analogous equation for the propagation of a small gauge field perturbation in a thermal background:

$$
a^{(T)}_{\mu}(x, t) = -\int d^3 x' \frac{\partial}{\partial t} \Delta^R_{\mu\nu}(x, t; x', 0[T]) a^{(T)}_{\nu}(x', 0) \quad \text{for } t > 0.
$$

(18)

The maximal Lyapunov exponent is then obtained from the relation

$$
\lambda_0 \approx \max_{a(0)} \frac{d}{dt} \ln \langle \mathcal{D}[a_{\mu}(t)] \rangle_T,
$$

(19)

where the distance measure (13) in a thermal background is

$$
\langle \mathcal{D}[a_{\mu}] \rangle_T \propto \int d^3 x \left[ \text{tr} \left( \frac{\partial^2 (tr B^2)}{\partial A_{\mu} \partial A_\nu} a^{(T)}_{\mu} a^{(T)}_{\nu} \right) + \frac{1}{2} \text{tr} \left( \frac{\partial^2 (tr B^2)}{\partial A_{\mu} \partial A_\nu} a^{(T)}_{\mu} a^{(T)}_{\nu} \right) \right].
$$

(20)

The first term in (20) vanishes, because the thermal average of any quantity transforming under the adjoint representation is zero. In the second term, the thermal average projects on to the singlet part of $\partial^2 (tr B^2)/\partial A_{\mu} \partial A_\nu$, yielding

$$
\langle \mathcal{D}[a_{\mu}] \rangle_T \propto \int d^3 x \left[ \text{tr} \left( \frac{\partial^2 (tr B^2)}{\partial A_{\mu} \partial A_\nu} \right) a^{(T)}_{\mu} a^{(T)}_{\nu} \right].
$$

(21)

$\langle \mathcal{D} \rangle_T$ grows exponentially with time because, as we will discuss next, the amplitude of classical fluctuations $a^{(T)}_{\mu}$ grows exponentially in the presence of thermal background fields.
V. THERMAL RETARDED PROPAGATOR

The general form of the retarded propagator is [18]

\[ i \Delta^R(k, t) = \Theta(t) \int_{-\infty}^{\infty} \frac{dw}{2\pi} \rho(k, \omega)e^{-i\omega t}, \]  

(22)

where \( \rho(k, \omega) \) is the spectral function. It contains a pole term picking up the zeroes \( \omega(k) \) of the inverse propagator corresponding to collective plasma modes and a cut term describing the effect of scattering on thermally excited modes:

\[ \rho(k, \omega) = Z(k)\delta(\omega^2 - \omega(k)^2) + \beta(k, \omega)\Theta(k^2 - \omega^2). \]  

(23)

The cut coefficient \( \beta(k, \omega) \) is related to the real and imaginary parts of the self energy \( \Pi(k, \omega) \):

\[ \beta(k, \omega) = \frac{\frac{1}{\pi} Im \Pi}{(k^2 - \omega^2 + Re \Pi)^2 + (Im \Pi)^2}. \]  

(24)

The respective self energies for the transverse and longitudinal excitations to leading order in hot perturbative QCD are [19]

\[ \Pi_t(k, \omega) = m^2x^2 \left(1 + \frac{1 - x^2}{2x} \ln \frac{1 + x}{1 - x}\right) + \frac{i\pi}{2}m^2x(1 - x^2) \]  

(25)

and

\[ \Pi_l(k, \omega) = k^2 + m^2 \left(2 - x \ln \frac{1 + x}{1 - x}\right) - i\pi m^2x \]  

(26)

with \( x = \omega/k \) and \( m^2 = 3m_g^2/2 \). Using these forms one obtains the following cut parts of the respective retarded propagators for small \( k/m \)

\[ \Delta^R_{cut,t}(k, t) \rightarrow -2\Theta(t) \frac{k}{m^2} \int_0^1 dx \frac{4x(1 - x^2) \sin kt x}{4x^4 \left(1 + \frac{1-x^2}{2x} \ln \frac{1+x}{1-x}\right)^2 + x^2(1 - x^2)^2} \]  

(27)

and

\[ \Delta^R_{cut,l}(k, t) \rightarrow -2\Theta(t) \frac{k}{2m^2} \int_0^1 dx \frac{x \sin kt x}{\left(2 - x \ln \frac{1+x}{1-x}\right)^2 + x^2}. \]  

(28)
Since the integrand is bounded, the cut contribution to the retarded propagator cannot grow exponentially with time and hence does not contribute to the maximal Lyapunov exponent (19). In fact, the cut contribution vanishes in the long wavelength limit $k \to 0$.

This leaves us with the pole part, which remains finite in this limit. Because $Im \Pi(k, \omega)$ is an odd function of $\omega$, the pole contribution contains both exponentially damped and exponentially growing collective plasma modes with frequency $\pm [\omega(k) - i\gamma(k)]$. The exponential growth of the solutions $a^{(T)}$ in (21) is dominated by the largest damping rate $\gamma(k)$ of the collective gauge field modes. Only $\gamma(0) = \gamma_0$ has been calculated so far in thermal perturbation theory. Assuming that $\gamma(k) \leq \gamma_0$ for all $k$, we tentatively conclude that $\langle \mathcal{D}[a] \rangle_T \propto \exp(2\gamma_0 t)$. This permits us to identify the largest Lyapunov exponent with twice the damping rate of a plasmon at rest: $\lambda_0 = 2\gamma_0$.

VI. SUMMARY

This concludes our argument establishing a connection between $\lambda_0$ and $\gamma_0$. We note that some elements of the argument are heuristic, in particular, the replacement of the long-time average of the growth rate of fluctuations around a specific field configuration by the thermal average. This reasoning assumes that the growth rate, or equivalently the plasmon damping rate, depends only on coarse-grained properties of the gauge field. We believe that this is so, because the one-loop calculation of the damping rate $\gamma_0$ only involves soft loop momenta [8] and hence does not depend on details of the short-distance fluctuations of the gauge field.

Because of the general nature of our argument, we conjecture that the complete spectrum of Lyapunov exponents obtained in [3] reflects the spectrum of damping rates $\gamma(k)$ of excitations in a thermal bath. If this were true, it would confirm our assumption that $\gamma(k) \leq \gamma_0$. Since, at present, it is not known whether $\gamma(k)$ is a quantity with a classical limit for $k \neq 0$, the identification with the Lyapunov spectrum remains a conjecture. We finally note that if the correspondence between ergodic and canonical averages holds up for other physical quantities, transport coefficients of nonabelian gauge fields at the classical
scale ($g^2 T$), such as magnetic screening [20] or color diffusion [21], could possibly also be calculated by real-time evolution of classical gauge fields on a lattice.

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[14] The sign conventions in ref. 5 are such that a positive imaginary part of ℏt implies a negative imaginary part of the pole energy. In writing (13) we have already continued ℏt to Minkowski space.


[16] Note that the infrared limit of the Yang-Mills equation has been shown not to be


