ASYMPTOTIC SOLUTIONS TO A TENSOR-SCALAR THEORY OF GRAVITY

J.-M. Gérard and I. Mahara

SCW 9440

Institut de Physique Théorique
Université catholique de Louvain
B-1348 Louvain-la-Neuve, Belgium

ABSTRACT

Most cosmological solutions to a minimal tensor-scalar theory of gravity are only defined on a finite time interval. Moreover, to each exceptional solution whose asymptotic limit is pure tensor gravity correspond an infinite number of solutions with a sizeable scalar component. Consequently, this tensor-scalar theory does not generically contain an attractor mechanism towards General Relativity.

Unification of gravity with the three other fundamental interactions seems to require a modification of the Einstein classical tensor theory of General Relativity. A common feature shared by the current attempts is the appearance of scalar degrees of freedom which, if coupled to ordinary matter, should mediate sizeable long range forces. On the other hand, we know that the pure tensor theory of gravity is successfully tested at the 10^{-3} level in our solar system. One natural way to avoid any fine tuning of couplings would be to treat General Relativity as the cosmological attractor of a genuine tensor-scalar theory of gravity [1]. In this letter, we study in detail the possibility to implement such a mechanism within the minimal extension [2] of the Einstein action containing one massless helicity-2 tensor field ($g_{\mu\nu}$) and one massless helicity-0 scalar field (φ):

$$S_{gr} = -\frac{1}{2\kappa} \int d^4x \sqrt{-g} \{R - 2g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi\} \quad (1)$$

with $\kappa \equiv 8\pi G$. As long as gravity is not coupled to matter, General Relativity (GR) is obviously recovered if φ is taken constant

$$\varphi_{GR} = \varphi_0. \quad (2)$$

The action (1) can be expressed in the so-called Jordan frame via the following Weyl rescaling

$$\begin{aligned} \tilde{g}_{\mu\nu} &= A^2(\varphi) g_{\mu\nu} \\ &= \frac{1}{G\Phi} g_{\mu\nu}. \end{aligned} \quad (3)$$

In this new frame, the metric also depends on the scalar degree of freedom and we obtain the well-known action

$$\tilde{S}_{gr} = -\frac{1}{16\pi} \int d^4x \sqrt{-\tilde{g}} \left\{ \Phi \tilde{R} - \frac{\omega(\Phi)}{\Phi} \tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \right\}$$

which has been used to trigger (extended) inflation [1] or space-time compactification [3] around the Planck scale. If we define

$$\alpha(\varphi) \equiv \frac{\partial}{\partial \varphi} \ln A(\varphi) = [2\omega(\Phi) + 3]^{-\frac{1}{2}} \quad (4)$$

the Brans-Dicke (BD) model [4] for gravitation corresponds to the special case where ω and, consequently, α are constant such that

$$A_{BD}(\varphi) = A_0 e^{\alpha\varphi}. \quad (5)$$

The functional $\alpha(\varphi)$ has a rather simple interpretation once matter is coupled to gravity.

In the “physical” Jordan frame, matter fields Ψ only couple to the metric

$$\tilde{S}_m = \int d^4x \sqrt{-\tilde{g}} \mathcal{L}[\Psi, \tilde{g}_{\mu\nu}] \quad (6)$$

such that the (weak) equivalence principle applies at the classical level [5], leading to the energy-momentum conservation law

$$\tilde{T}^{\mu\nu}_{|\nu} = 0. \quad (7)$$

In the Einstein frame, matter fields couple both to the metric and the scalar, the corresponding matter action is more involved

$$S_m = S_m[\Psi, A^2 g_{\mu\nu}] \quad (8)$$

but the equations of motion are straightforward to derive from the variation with respect to $g^{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2 \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \partial_\sigma \varphi \partial^\sigma \varphi + \kappa T_{\mu\nu} \quad (9a)$$

and to φ

$$\square \varphi = -\frac{\kappa}{2} \alpha(\varphi) T. \quad (9b)$$

In this frame, the energy-momentum tensor

$$T_{\mu\nu} = 2\sqrt{-g} \frac{\partial \mathcal{L}[\Psi, A^{-2}g^{\mu\nu}]}{\partial g^{\mu\nu}} = A^2(\varphi) \tilde{T}_{\mu\nu} \quad (10)$$

is not conserved anymore since

$$T^{\mu\nu}_{|\nu} = T\alpha(\varphi)\partial^\mu \varphi \quad (11)$$

with T standing for its trace $g^{\mu\nu}T_{\mu\nu}$. From Eq.(9b), we conclude that $\alpha(\varphi)$ can be treated as the basic coupling strength between the scalar field φ and ordinary matter. The tensor theory of General Relativity is then the Brans-Dicke model in the limit where ω goes to infinity, namely

$$\alpha_{GR}^2 = 0. \quad (12)$$

On the other hand we expect α to be typically of the order of unity in unified theories including gravity. For illustration, the (truncated) effective action for string cosmology [6] corresponds to the Brans-Dicke model with $\omega = -1$, namely

$$\alpha_{String}^2 = 1. \quad (13)$$

Current tests [7] of General Relativity based on light-deflection and time-delay experiments lead however to the rather stringent limit $\omega > 500$ such that

$$\alpha_{Today}^2 < 10^{-3}. \quad (14)$$

A way to reconcile Eqs (13) and (14) would be to find a dynamical attractor mechanism [1] driving the early unified tensor-scalar theory towards General Relativity.

Such a scenario has been recently advocated [8] to happen *generically* for a single scalar field propagating in an homogeneous, spatially flat Einstein metric

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= dt^2 - a^2(t) d\vec{x}^2 \end{aligned} \quad (15)$$

with a perfect-fluid matter distribution characterized by the *positive* density ρ and the pressure p :

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}. \quad (16)$$

The purpose of this letter is to analyse the asymptotic behaviour of the time-dependent solutions of this minimal tensor-scalar cosmological model. We work in the Einstein frame

but any physical observable quantity should of course be independent of the field redefinition defined in Eq.(3).

Using Eqs (15) and (16), the equations of motion given in (9) and (11) reduce to three independent relations

$$\dot{H} + 3H^2 = \left(\frac{1-\lambda}{2}\right) \kappa \rho \quad (17a)$$

$$\dot{H} + \frac{3}{2}(1+\lambda)H^2 = -\left(\frac{1-\lambda}{2}\right) \dot{\varphi}^2 \quad (17b)$$

$$\dot{\rho} + 3(1+\lambda)\rho H = (1-3\lambda)\rho\alpha\dot{\varphi} \quad (17c)$$

where an overdot denotes the (cosmic) time derivative and $H(t)$ is the logarithmic derivative of the expansion scale factor $a(t)$:

$$H(t) \equiv \frac{\dot{a}}{a}. \quad (18)$$

The constant values $\lambda = -1, 0$ and $\frac{1}{3}$ for the pressure-to-density ratio

$$\lambda = \frac{p}{\rho} \quad (19)$$

correspond to the physically interesting cases of inflationary, matter-dominated and radiative universes, respectively. In what follows, we shall restrict ourselves to the range

$$-1 \leq \lambda \leq \frac{1}{3}. \quad (20)$$

The three non-linear coupled equations (17) contain four unknown functions of time: ρ, φ, H and (see Eq.(4)) A . This system can however be solved by integration in quadrature, in terms of a single arbitrary function. Indeed, if we make the following change of variable

$$H(t) \equiv \frac{1}{x(t)} \quad , \quad x(t) \neq 0 \quad (21)$$

the first two equations in (17) become respectively

$$\begin{aligned} \dot{x} - 3 + \frac{1}{2}(1-\lambda)\kappa\rho x^2 &= 0 \\ \dot{x} - \frac{3}{2}(1+\lambda) - \frac{1}{2}(1-\lambda)\dot{\varphi}^2 x^2 &= 0. \end{aligned} \quad (22)$$

We obtain in particular the relation

$$(\kappa\rho + \dot{\varphi}^2)x^2 = 3 \quad (23)$$

which, together with the positivity of the density ρ , allows us to define

$$\begin{aligned} \dot{\varphi}x &= \sqrt{3}\sin\theta(t) \\ \kappa\rho x^2 &= 3\cos^2\theta(t) \end{aligned} \quad (24)$$

with $\theta(t)$ the *arbitrary* function. As we shall see, this parametrization turns out to be quite useful to study the time-dependent cosmological solutions of a tensor-scalar theory. In particular, General Relativity (see Eq.(2)) simply corresponds to the choice

$$\theta_{GR}(t) = n\pi. \quad (n \text{ integer}) \quad (25)$$

The system (22) reduces now to a study of the differential equation

$$\dot{x}(t) = \frac{3}{2}(1 + \lambda) + \frac{3}{2}(1 - \lambda)\sin^2\theta(t) \quad (26)$$

whose solutions are monotonic functions of time

$$\dot{x}(t) > 0 \quad (27)$$

if we disregard an inflationary universe ($\lambda = -1$) within General Relativity at all time ($\theta = n\pi$). Consequently, $x(t)$ vanishes at most once at a time t_0 in the existence interval $]t_1, t_2[$ of $\theta(t)$. Due to Eq.(21), if a solution $x(t)$ of Eq.(26) vanishes for $t_0 \neq t_{1,2}$, it gives then rise to two different solutions for the Hubble expansion function $H(t)$, one defined in $]t_1, t_0[$ and the other one in $]t_0, t_2[$. The function $\theta(t)$ being arbitrary, we conclude that most of the cosmological solutions are only defined on a *finite* time interval and we may, without loss of generality, restrict the analysis to intervals of type $]t_0, t_2[$.

To analyse asymptotic solutions to the tensor-scalar theory, we therefore have to *assume* that $\theta(t)$ is defined on the whole real line of $\mathbb{R}$. In such a case, $x(t)$ vanishes exactly once at time t_0 and a simple translation allows us to write the general solution of Eq.(26) as

$$x(t) = \frac{3}{2}(1 + \lambda)t + \frac{3}{2}(1 - \lambda) \int_0^t \sin^2 \theta(\tau) d\tau \quad (28)$$

for all cosmic time $t \geq 0$. For example, the Brans-Dicke cosmological solution corresponds to the special case

$$\theta_{BD}(t) = \theta_0. \quad (29)$$

In other words, if $\theta(t)$ is well-defined for all positive time, so is $x(t)$ with the constraints

$$\frac{3}{2}(1 + \lambda)t \leq x(t) \equiv \frac{1}{H(t)} \leq 3t \quad (30)$$

the lower and upper bounds being saturated for $\theta(t) = n\pi$ (General Relativity) and $\theta(t) = (2n + 1)\frac{\pi}{2}$, respectively.

The time-dependence of the scalar field φ , of the density ρ and of the expansion scale factor a are determined by Eqs (24) and (28). In particular we notice from Eqs. (24) and (30) that the time-derivative of φ always goes to zero when t goes to infinity:

$$\dot{\varphi}(t \rightarrow \infty) = 0. \quad (31)$$

However, as we shall show, this result does *not* imply that all cosmological solutions are attracted towards General Relativity.

We have seen that the tensor theory of General Relativity with gravity coupled to matter requires a vanishing scalar coupling strength α . In that case, $A(\varphi)$ is the non-zero constant A_0 defined in Eq.(5):

$$A_{GR} = A_0 \neq 0 \quad (32)$$

which can always be rescaled to one such that the Einstein metric $g_{\mu\nu}$ is indeed identical to the physical $\tilde{g}_{\mu\nu}$ one with G , the measured Newton constant. To select the cosmological

solutions whose asymptotic limit is pure-tensor gravity, it is therefore necessary to integrate Eq.(17c) using

$$\frac{\dot{A}}{A} = \alpha \dot{\varphi} \quad (33)$$

and to require a non-vanishing constant value for $A(t \rightarrow \infty)$.

For an empty universe ($\rho = p = 0$) we have

$$\theta(t) = (2n + 1) \frac{\pi}{2} \quad (34a)$$

while for a radiation-dominated universe ($\rho = 3p$) we obtain

$$\theta(t) = ctn^{-1} \frac{a(t)}{k} \quad (34b)$$

with k a non-zero arbitrary constant. For these universes, $A(t)$ is not determined but exact analytical solutions defined in the Jordan frame have been discussed in the literature [9]. They correspond to relatively simple (exponential or trigonometric) functions for $A(\varphi)$.

So, let us assume that $p \neq \frac{\rho}{3} \neq 0$, with λ constant. In that case, Eq.(17c) becomes

$$\sqrt{3} \left\{ \frac{2}{3} \frac{\dot{\theta} x}{\cos \theta} + (1 - \lambda) \sin \theta \right\} = -(1 - 3\lambda)\alpha. \quad (35)$$

If we define $\varphi' = \dot{\varphi} x$, with $x(t)$ monotonic in time (see Eq.(27)), the prime denotes the derivative with respect to the new evolution parameter $\ln[a(t)]$ and the dynamics contained in Eq.(35) is analogous [8] to the damped motion along the φ -axis of a massive particle in the external potential

$$V(\varphi) = (1 - 3\lambda) \ln A(\varphi). \quad (36)$$

In this nice mechanical analogy, the existence of a damping force seems to require a constant asymptotic value for A corresponding to a minimum of the potential V .

In our parametrization (24), the integration of Eq.(35) provides us with the following time-dependence of A :

$$\frac{[A(t)]^{1-3\lambda}}{\cos^2 \theta(t)} = \frac{[A(t_0)]^{1-3\lambda}}{\cos^2 \theta(t_0)} \exp \left\{ -3(1 - \lambda) \int_{t_0}^t \frac{\sin^2 \theta(\tau)}{x(\tau)} d\tau \right\} \quad (37)$$

for all $t_0 > 0$ with $\cos \theta(t_0) \neq 0$. We then conclude that $A(t)$ is indeed always tending towards a constant

$$A(t \rightarrow \infty) = A_0 \quad (38)$$

which is zero (non zero) if the integral in Eq.(37) diverges (converges) respectively. A necessary condition for the integral in Eq.(37) to converge is that $\lim_{t \rightarrow \infty} \theta(t) = n\pi, n \in \mathbb{Z}$. In that sense, the divergent case is generic and, contrary to a recent claim [8], General Relativity is not a cosmological attractor for the minimal tensor-scalar theory considered. In other words the associated mechanical potential V in Eq.(36) is generically not bounded from below. This feature could already be suspected from the study of the asymptotic solutions in the Brans-Dicke model defined by Eq.(29):

Example 1. : $\theta(t) = \theta_0 \neq (2n+1) \frac{\pi}{2}, n \text{ integer.}$

If $\theta_0 = n\pi$, we have General Relativity at all time and $A_{GR}(t) = A(t_0)$ in agreement with Eq.(32). But if θ_0 is different from $(2n+1) \frac{\pi}{2}$ to avoid the undetermination inherent to a vanishing density (see Eq.(34a)), the integral in Eq.(37) diverges logarithmically when t goes to infinity and we obtain the Brans-Dicke solution

$$\begin{aligned} x(t) &= 3 \left[1 - \left(\frac{1-\lambda}{2} \right) \cos^2 \theta_0 \right] t \\ A_{BD}(t \rightarrow \infty) &= 0 \\ \alpha_{BD}(t \rightarrow \infty) &= -\sqrt{3} \left(\frac{1-\lambda}{1-3\lambda} \right) \sin \theta_0. \end{aligned} \quad (39)$$

The special value $\theta_0 = \cos^{-1} \sqrt{\frac{2}{3(1-\lambda)}}$ implies $x(t) = 2t$ and $|\alpha| = \sqrt{\frac{1-\lambda}{1-3\lambda}}$ and corresponds to the critical case of a static universe in the Jordan physical basis. Notice also that the asymptotic absolute value of the scalar coupling strength is always smaller than $\sqrt{3}$ for a matter-dominated universe ($\lambda = 0$).

Another simple example of vanishing asymptotic value for A is provided by assuming a linear time-dependence for θ :

Example 2. : $\theta(t) = t$.

This example nicely illustrates the way the constraints given in Eq.(30) are fulfilled at all time. We obtain indeed

$$\begin{aligned} x(t) &= \frac{9t}{4} - \frac{3}{8} \sin 2t \\ A(t \rightarrow \infty) &= 0 \\ |\alpha(t \rightarrow \infty)| &= +\infty \end{aligned} \tag{40}$$

such that the tensor-scalar theory is attracted towards pure-scalar gravity although $\varphi(t \rightarrow \infty)$ is a constant.

We can of course find models which are attracted towards pure-tensor gravity. However to each of them always corresponds another one which is certainly not. Let us illustrate this statement by the following simple example:

Example 3. : $\theta(t) = ctn^{-1}t$ for $\lambda = 0$.

We obtain the solution

$$\begin{aligned} x(t) &= \frac{3t}{2} + \frac{3}{2} \tan^{-1}t \\ A(t \rightarrow \infty) &\neq 0 \\ \alpha(t \rightarrow \infty) &= 0 \end{aligned} \tag{41a}$$

with relaxation towards General Relativity. However an anti-complementary solution

$$\begin{aligned}
x_{\frac{\pi}{2}}(t) &= 3t - \frac{3}{2} \tan^{-1} t \\
A_{\frac{\pi}{2}}(t \rightarrow \infty) &= 0 \\
|\alpha_{\frac{\pi}{2}}(t \rightarrow \infty)| &= 3\sqrt{3}
\end{aligned} \tag{41b}$$

not attracted towards General Relativity is automatically associated to it through the simple substitution

$$\theta(t) \rightarrow \theta(t) + \frac{\pi}{2}. \tag{42}$$

This association is valid for all exceptional solution with General Relativity as an attractor since the integral

$$\int_{t_0}^{\infty} [\sin^2 \theta + \sin^2(\theta + \frac{\pi}{2})](\tau) \frac{d\tau}{x(\tau)} = \int_{t_0}^{\infty} \frac{d\tau}{x(\tau)} \tag{43}$$

always diverges. We know that General Relativity is restored if the asymptotic value of $\theta(t)$ is $n\pi$. The new solution based on the substitution (42) is therefore characterized by the ambiguous value $\theta(t \rightarrow \infty) = (2n + 1)\frac{\pi}{2}$ and the coupling strength α can take any possible non vanishing value, depending on the specific model considered. The following example shows us that this value can even be infinite:

Example 4. : $\theta(t) = \cos^{-1}(\tanh t)$ for $\lambda = 0$.

The solution

$$\begin{aligned}
x(t) &= \frac{3t}{2} + \frac{3}{2} \tanh t \\
A(t \rightarrow \infty) &\neq 0 \\
\alpha(t \rightarrow \infty) &= 0
\end{aligned} \tag{44a}$$

is attracted towards pure tensor gravity while the anti-complementary one

$$\begin{aligned}
x_{\frac{\pi}{2}}(t) &= 3t - \frac{3}{2} \tanh t \\
A_{\frac{\pi}{2}}(t \rightarrow \infty) &= 0 \\
|\alpha_{\frac{\pi}{2}}(t \rightarrow \infty)| &= \infty
\end{aligned} \tag{44b}$$

is attracted towards pure scalar gravity.

All the cosmological solutions to the minimal tensor-scalar theory of gravity have been parametrized in terms of the arbitrary function $\theta(t)$. Consequently we can generalize the substitution (42) to an arbitrary constant shift θ_0

$$\theta(t) \rightarrow \theta(t) + \left(\theta_0 \neq (2n+1) \frac{\pi}{2} \right), n \text{ integer} \tag{45}$$

to avoid the ambiguity for α . We are therefore allowed to conclude that to each General Relativity attracted solution characterized by the following asymptotic behaviour

$$\begin{aligned}
x(t \rightarrow \infty) &\sim \frac{3}{2}(1+\lambda)t \\
A_{GR}(t \rightarrow \infty) &\neq 0 \\
\alpha_{GR}(t \rightarrow \infty) &= 0
\end{aligned} \tag{46a}$$

always corresponds an infinite number of Brans-Dicke attracted solutions characterized by the following asymptotic behaviour

$$\begin{aligned}
x_{\theta_0}(t \rightarrow \infty) &\sim 3 \left[1 - \left(\frac{1-\lambda}{2} \right) \cos^2 \theta_0 \right] t \\
A_{\theta_0}(t \rightarrow \infty) &= 0 \\
\alpha_{\theta_0}(t \rightarrow \infty) &= -\sqrt{3} \left(\frac{1-\lambda}{1-3\lambda} \right) \sin \theta_0.
\end{aligned} \tag{46b}$$

It is then clear that the fine-tuning required nowadays on α (see Eq.(14)) is not more natural

within this minimal tensor-scalar model than in the usual Brans- Dicke cosmology given in Example 1.

Finally we would like to emphasize that the asymptotic conditions $\theta(t \rightarrow \infty) = 0$ and $A(t \rightarrow \infty) \neq 0$ are not sufficient to ensure a restoration of General Relativity at infinite time. Our last example illustrates that point very well:

Example 5. : $\theta(t) = \frac{\sin t}{t}$ for $\lambda = 0$.

In that case, $\theta(t)$ oscillates towards a vanishing value with

$$\begin{aligned} x(t \rightarrow \infty) &\sim \frac{3}{2}t \\ A(t \rightarrow \infty) &\neq 0 \end{aligned} \tag{47a}$$

such that the mechanical potential defined in Eq.(36) admits a minimum. However the scalar coupling strength does not go to zero

$$\alpha(t \rightarrow \infty) \sim -\sqrt{3} \cos t . \tag{47b}$$

This last example reinforces if necessary our main conclusion, namely that a minimal tensor-scalar theory does not generically contain a natural attractor mechanism towards General Relativity. On the other hand, we believe that the method presented here to solve the equations of motion by integration in quadrature can be extended to analyse asymptotic solutions in the case of a non-vanishing spatial curvature.

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