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**GENERALIZATION
OF THE TODA CHAIN SYSTEM
TO ELLIPTIC CURVE CASE**

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Abstract

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We propose a system of two equations which when some of its parameters vanish separates into two equations describing independent one-dimensional Toda chains. The system has its foundation in the discrete transformations of the Landau-Lifshitz equation which is closely connected with elliptic curves. Nontrivial solutions of the system are found in an explicit form.

Аннотация

Белов Н.А. и др. Обобщение системы уравнений цепочки Toda на случай эллиптической кривой: Препринт ИФВЭ 94-55. – Протвино, 1994. – 8 с., библиогр.: 4.

Предложена система двух уравнений, которая в предельном случае (при определенном выборе искомых параметров) расщепляется в две независимые одномерные цепочки Toda. Система описывает дискретные преобразования уравнения Ландау-Лифшица, как известно, тесно связанного с эллиптической кривой. Найдены нетривиальные решения, изучаемые системы которых позволяют построить n -солитонные решения уравнения Ландау-Лифшица в явном виде.

INTRODUCTION

In our previous paper [1] we presented an explicit solution of the chain of discrete transformations for the Landau-Lifshitz equation describing classical anisotropic Heisenberg ferromagnets. Our solution was derived from direct calculations which were rather tedious and involved many steps. One of the aims of this paper is to simplify these calculations. Another more general and important aim is to present two possible generalisations of the usual one-dimensional Toda system to the case of "elliptic curve". For specific boundary conditions we can present their solutions in an explicit form.

Most of the difficulties of the calculations reported in [1] came from the fact that many intermediate expressions were in the form of a ratio of two rather complicated expressions containing factors which were cancelled between the numerator and the denominator. In this paper we overcome these difficulties by considering expressions which do not contain such over-all factors.

1. DISCRETE TRANSFORMATION OF THE LANDAU-LIFSHITZ EQUATION

The chain of equations which describe discrete transformation[1,3] of the Landau-Lifshitz (L-L) equation may be written in the form

$$\frac{[\ln((\dot{v}_n)^2 + P(v_n))]}{2\dot{v}_n} = \frac{1}{v_{n-1} + v_n} + \frac{1}{v_n + v_{n+1}}, \quad (1)$$

where n is a natural number, $P(v) = \alpha v^4 + \gamma v^2 + \alpha$ and v_n is the unknown lattice function dependent on x , the differentiation with respect to which is denoted by $\dot{}$.

For the derivation of this equation, its connection with the L-L equation and some of its applications in physics see [1].

In [1] we have used another form of the equations of discrete transformation. The equations given in [1] can be obtained from above equations (1) by the following changes of variables

$$v_n = \exp -s \frac{1 + \sigma_n}{1 - \sigma_n}, \quad \left(\frac{\partial s}{\partial x}\right)^2 + F(s) = 0, \quad (2)$$

$$P(s) \equiv \alpha(\exp 2s + \exp -2s) + \gamma, \quad \phi \equiv \frac{\partial \phi}{\partial x} = \sqrt{-P} \frac{\partial \phi}{\partial s} \equiv \sqrt{-P} \phi'.$$

After these changes the equations of discrete transformation take the form

$$(\ln \theta_n)' = -2\sigma_n' \left[\frac{\sigma_{n-1}}{1 - \sigma_{n-1}\sigma_n} + \frac{\sigma_{n+1}}{1 - \sigma_{n+1}\sigma_n} \right] + \frac{\sigma_{n-1} - \sigma_n}{1 - \sigma_{n-1}\sigma_n} + \frac{\sigma_{n+1} - \sigma_n}{1 - \sigma_{n+1}\sigma_n}, \quad (3)$$

where $\theta_n = P\sigma_n'(\sigma_n' + \sigma_n^2 - 1) + Q\sigma_n(\sigma_n + 1) - Q'\sigma_n^2$, and where $Q \equiv \alpha(\exp 2s - \exp -2s)$.

These equations are exactly equivalent to the recurrence relations of our previous paper [1], which connected three values of σ with indices $(n-1, n, n+1)$, respectively.

2. ELLIPTIC TODA LATTICES

Let us determine the functions

$$\exp \rho_n = \frac{[(\dot{v}_n)^2 + P(v_n)]}{(v_{n-1} + v_n)(v_n + v_{n+1})}, \quad (4)$$

which are constructed from an arbitrary solution of the chain (1).

Using (1) it is not difficult to show that the following equations are satisfied

$$\ddot{\rho}_n = \exp \rho_{n-1} + 2 \exp \rho_n - \exp \rho_{n+1} + \alpha(v_{n-1}^2 - 2v_n^2 + v_{n+1}^2), \quad (5)$$

where α is the parameter which appears in $P(x)$ (see (2)). The equation of the chain (1) is invariant with respect to the transformation $v_n \rightarrow \frac{1}{v_n}$ (it can be verified directly). Performing this transformation on the expression in (4) we find that the functions π_s defined by

$$\exp \pi_n = \frac{v_{n-1}v_{n+1}}{v_n^2} \frac{[(\dot{v}_n)^2 + P(v_n)]}{(v_{n-1} + v_n)(v_n + v_{n+1})} = \frac{v_{n-1}v_{n+1}}{v_n^2} \exp \rho_n \quad (6)$$

satisfy the equations:

$$\ddot{\pi}_n = \exp \pi_{n-1} + 2 \exp \pi_n - \exp \pi_{n+1} + \alpha(v_{n-1}^{-2} - 2v_n^{-2} + v_{n+1}^{-2}). \quad (7)$$

To check that the equations of system (7) are indeed satisfied it is convenient to use the realtions

$$(\ln v_n)' = -\exp \pi_n + \exp \rho_n - \alpha(v_n^2 - v_n^{-2}), \quad (8)$$

whose validity can be verified by direct calculations.

As a corollary we obtain the following chain for two unknown functions, say, ρ_s and v_s at each lattice point:

$$\begin{aligned} \ddot{\rho}_n &= \exp \rho_{n-1} + 2 \exp \rho_n - \exp \rho_{n+1} + \alpha(v_{n-1}^2 - 2v_n^2 + v_{n+1}^2), \\ \exp \pi_n &= \frac{v_{n-1}v_{n+1}}{v_n^2} \exp \rho_n, \end{aligned} \quad (9)$$

$$\ddot{\pi}_n = \exp \pi_{n-1} + 2 \exp \pi_n - \exp \pi_{n+1} + \alpha(v_{n-1}^{-2} - 2v_n^{-2} + v_{n+1}^{-2}).$$

The case $\alpha = 0$ (the cylindrically symmetric rotator) is sometimes also called the trigonometric subcase of the elliptic system. We see from (9) that this case is equivalent to the two noninteracting Toda chains.

Using the equations (9) we can reproduce the results of section 4 of [1] in a few lines.

Next we will need the following relation

$$(\ln(v_n \exp s \pm 1))' = \exp \rho_n - \frac{(v_{n+1} \exp s \mp 1)(v_{n-1} \exp s \mp 1)}{(v_n \exp s \pm 1)^2} \exp \rho_n \quad (10)$$

$$-\alpha v_n^2 + \alpha \exp(-2s) + (s \frac{v_n \exp s \mp 1}{v_n \exp s \pm 1}),$$

which can be proved using definitions (2) and performing the necessary differentiations.

Let us now define two new lattice functions θ_n and ϕ_n :

$$\begin{aligned} \exp \theta_n &= \frac{(v_{n+1} \exp s + (-1)^n)(v_{n-1} \exp s + (-1)^n)}{(v_n \exp s - (-1)^n)^2} \exp \rho_n, \\ \exp \phi_n &= \frac{(v_{n+1} \exp s - (-1)^n)(v_{n-1} \exp s - (-1)^n)}{(v_n \exp s + (-1)^n)^2} \exp \rho_n. \end{aligned} \quad (11)$$

Then from (4) and (11) we find that

$$\exp(\theta_n - \phi_n) = (\sigma_{n-1}\sigma_n^2\sigma_{n+1})^{(-1)^{n+1}}. \quad (12)$$

Using (10) and (11) and recalling (2) it is possible to convince oneself that the lattice functions θ_n and ϕ_n satisfy the following system of equations:

$$\begin{aligned} \dot{\theta}_n &= -\exp \theta_{n-1} + 2 \exp \theta_n - \exp \theta_{n+1} + (\dot{s}(\sigma_{n-1} - 2\sigma_n + \sigma_{n+1})), \\ \exp(\theta_n - \phi_n) &= (\sigma_{n-1}\sigma_n^2\sigma_{n+1})^{(-1)^{n+1}}, \\ \dot{\phi}_n &= -\exp \phi_{n-1} + 2 \exp \phi_n - \exp \phi_{n+1} + (\dot{s}(\sigma_{n-1}^{-1} - 2\sigma_n^{-1} + \sigma_{n+1}^{-1})). \end{aligned} \quad (13)$$

We will call this system EToda-2 lattice - the elliptic Toda lattice of the second kind.

Together with (13) it is natural to consider the following system of equations for the unknown functions x_n and y_n

$$\left(\frac{\dot{a}_n - \sqrt{P}b_n}{a_n}\right) = \frac{a_{n-1}a_{n+1}}{a_n^2} \quad \left(\frac{\dot{b}_n - \sqrt{P}a_n}{b_n}\right) = \frac{b_{n-1}b_{n+1}}{b_n^2}. \quad (14)$$

Then, performing the change of variables

$$\exp \theta_n = \frac{a_{n-1}a_{n+1}}{a_n^2}, \quad \exp \phi_n = \frac{b_{n-1}b_{n+1}}{b_n^2}$$

it is easy to check that the equations (13) are a direct consequence of equations (14).

Note that if lattice (13) is limited from the both sides its solution can always be represented in this last form; this, however, is not the case when the lattice is not limited.

3. AN EXPLICIT SOLUTION OF THE TODA-2 CHAIN WITH ONE FIXED END

The aim of this section is to show that under appropriate boundary conditions the equations of EToda-2 lattice possess nontrivial solutions which can be presented in an explicit form. The boundary conditions we have in mind are those that arise in the problem of the construction of n soliton solutions of the L-L equation [1] and so, the results of this section provide us with explicit solutions of the L-L equation.

The form of our solution is very similar to the corresponding solution of the ordinary one-dimensional Toda lattice with one fixed end [2] and so we hope

that EToda lattice-2 will play the same role in the case of elliptical curves (the elliptic parametrization of the spectral parameter) as the ordinary Toda chain (and its different modifications, Bäcklund transformations etc.) plays in the case of rational curve[3].

To proceed further we note that we want to find solutions of a system of equations (14) with the following boundary conditions imposed at the left end of the lattice (see [1]):

$$a_0 = 0, \quad b_0 = 1, \quad a_1 = Y, \quad b_1 = Y', \quad (15)$$

$$a_2 = \begin{pmatrix} Y & L \\ Y' & L' \end{pmatrix}, \quad b_2 = \begin{pmatrix} Y' & \sqrt{P}(Y'' - Y) \\ \dot{Y}' & (\sqrt{P}(Y'' - Y)) \end{pmatrix}.$$

Not all of these boundary conditions are mutually independent but, as is easy to check, they are not in contradiction with system (14).

Next, we need the following lemma. Take an (infinite) matrix C constructed as follows. Let the first row of the matrix be given by an arbitrary set of functions of one variable and its s -th row be derived from it by differentiating it s times. Denote the principal n -th order minor of C by C_n . Then we have the following result:

$$\frac{C_{n-1}C_{n+1}}{C_n^2} = \left(\tilde{C}_n \right), \quad (16)$$

where $\tilde{}$ over C , ie $\tilde{C}$, denotes that the last column of the matrix corresponding to C_n , namely, $(c_n, c_n^{(1)}, \dots, c_n^{(n-1)})$ has been replaced by the first n entries of the last column of the matrix corresponding to C_{n+1} ie by $(c_{n+1}, c_{n+1}^{(1)}, \dots, c_{n+1}^{(n-1)})$.

The proof of this lemma is straightforward.

The following generalisation is also easy to prove. Let the main minors of a matrix D denoted by D_n be represented in the form $D_n = t^n B_n$ where t is any function of the independent variable and let the matrix B satisfy the conditions of our lemma. Then equality (16) becomes

$$\frac{D_{n-1}D_{n+1}}{D_n^2} = \left(\frac{\tilde{B}_n}{B_n} \right). \quad (17)$$

A solution of (14) and (15) may be expressed in terms of the principal minors

of the following (infinite-dimensional) matrix AB

$$\begin{pmatrix} 0 & Y' & Y'' - Y & L' & L'' - L & L'_2 & L''_2 - L_2 & \dots \\ Y & L & P(Y''' - Y') & L_2 & P(L''' - L') & L_3 & P(L''_2 - L'_2) & \dots \\ Y' & L' & L'' - L & L'_2 & L''_2 - L_2 & L'_3 & L''_3 - L_3 & \dots \\ L & L_2 & P(L''' - L') & L_3 & P(L''_2 - L'_2) & L_4 & P(L''_3 - L'_3) & \dots \\ L' & L'_2 & L''_2 - L_2 & L'_3 & L''_3 - L_3 & L'_4 & L''_4 - L_4 & \dots \\ L_2 & L_3 & P(L''_2 - L'_2) & L_4 & P(L''_3 - L'_3) & L_5 & P(L''_4 - L'_4) & \dots \\ L'_2 & L'_3 & L''_3 - L_3 & L'_4 & L''_4 - L_4 & L'_5 & L''_5 - L_5 & \dots \end{pmatrix} \quad (18)$$

where $L \equiv L_1 = LY = PY'' - QY', L_n = LL \dots L^{n-\text{times}} Y$.

It is easy to understand, from the explicit form (18), the law of the construction of the matrix elements of the matrix AB and the periodicity properties of its rows and columns (note that the differences between (18) and the corresponding expression of [1] are very minor and drop out when we consider the final expressions).

Then, finally, the solution to our problem is obtained by taking

$$b_n = (AB)_n^+, \quad a_n = (AB)_n^-, \quad (19)$$

where by $(AB)_n^\pm$ we denote the n -th order minors of the matrix AB with elements Y', Y in their left and upper corners, respectively.

To prove this we note that the determinants which determine a_n, b_n may be expressed in terms of matrices which satisfy the conditions of our lemma (16). Namely, the first row of the matrix which determines b_n is given by

$$b^1 = (Y', \sqrt{P}(Y'' - Y), L', \sqrt{P}(L'' - L), L'_2, \sqrt{P}(L''_2 - L_2), \dots), \quad (20)$$

while a_n can be represented in the form: $a_n = P^{-\frac{n-1}{2}} \tilde{a}_n$, where the first row of the matrix $\tilde{a}$, which satisfies the conditions of our lemma, is given by

$$\tilde{a}^1 = (Y, L, P^{\frac{3}{2}}(Y''' - Y'), L_2, P^{\frac{3}{2}}(L''' - L'), L_3, P^{\frac{3}{2}}(L''_2 - L'_2), \dots). \quad (21)$$

If we now substitute (20) and (21) into (16) we find that (19) is indeed a solution of our system (14).

4. COMMENTS ON THE CASE WITH BOTH ENDS FIXED

From the explicit form of the solution of the last section it is easy to see that if we choose the initial function Y in the form

$$Y = \sum_1^n c_i \psi_i,$$

where functions ψ_i are eigenfunctions of the operator L , ie $L\psi_i = \lambda_i\psi_i$, then a_{2n} or b_{2n} will vanish. This fact is due to the linear dependence among the n columns of our fundamental matrix and it implies that the chain is also interrupted at the other end.

CONCLUDING REMARKS

One of the main results of this paper is the derivation of the equations of elliptic Toda chains in which the interaction potential is expressed in terms of the Jacobi elliptic functions (in the case of EToda-2). We have also shown that the chain possesses nontrivial solutions which, as usual in the theory of discrete symmetries of completely integrable systems [3], may be expressed in terms of some determinants. If we impose definite boundary conditions at the either end of the chain then the method allows us to construct n -soliton solutions of the L-L equation in an explicit form.

Our work has posed many interesting questions for which we have no answers at the moment. Is our elliptic Toda chain with fixed ends an exactly integrable dynamical system and what boundary conditions are necessary for this integrability? What are the fundamental properties of the substitution [4] which lead to the elliptic EToda-2 equations? Is it possible to generalise elliptic Toda chains to two dimensions and/or to the case of an arbitrary semisimple algebra? In the case of an ordinary Toda chain all these questions have positive answers. We hope to be able to answer some of these questions soon.

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