Unitarity of Quantum Theory and Closed Time-like Curves

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Abstract

Interacting quantum fields on spacetimes containing regions of closed timelike curves (CTCs) are subject to a non-unitary evolution $X$. Recently, a prescription has been proposed, which restores unitarity of the evolution by modifying the inner product on the Hilbert space of final states. We point out that this proposal restricts the class of observables to the class of self-adjoint operators commuting with the non-unitary part $(XX^*)^{1/2}$ of the evolution. Thus, their expectation values evolve only according to the unitary part $(XX^*)^{-1/2}X$.

We also propose an alternative method by which unitarity of the evolution may be regained, by extending $X$ to a unitary evolution on a larger (possibly indefinite) inner product space. The proposal removes the ambiguity noted by Jacobson in assigning expectation values to observables localised in regions spacelike separated from the CTC region. We comment on the physical significance of the possible indefiniteness of the inner product introduced in our proposal.

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1 Introduction

Various recent studies [1, 2, 3] of perturbative interacting quantum field theory in the presence of a compact region of closed timelike curves (CTCs) have concluded that the evolution from initial states in the far past of the CTCs to final states in their far future fails to be unitary, in contrast with the situation for free fields [1, 4]. This presents many problems for the usual Hilbert space framework of quantum theory: as we describe in Section 2, the Schrödinger and Heisenberg pictures are inequivalent and ambiguities arise in assigning probabilities to events occurring before [2], or spacelike separated from [5], the region of non-unitary evolution.

The main reaction to these difficulties has been to abandon the Hilbert space formulation in favour of a sum over histories approach such as the generalised quantum mechanics of Gell-Mann and Hartle (see, e.g., [6]). In particular, Hartle [7] has addressed the issue of non-unitary evolutions in generalised quantum mechanics. Nonetheless, it is of interest to see if the Hilbert space approach can be ‘repaired’ by restoring unitarity. Recently, Anderson [8] has shown how this may be accomplished by changing the inner product on the final Hilbert space. In this paper, we first consider the implications of this proposal for observables on the Hilbert spaces of initial and final states, and the evolution of their expectation values. We find that significant restrictions are placed on the class of physical observables: in order for a quantity to be measurable both before and after the evolution, its corresponding operator must commute with the non-unitary part of the evolution (in the sense of the polar decomposition [9]). Accordingly, the expectation values of such observables evolve only according to the unitary part.

Thus it is unclear whether familiar concepts such as position, momentum, spin, energy, etc., are valid observables. Indeed, it seems unlikely that momentum could be valid, for that would entail translational invariance of the non-unitary part of the evolution – which is not expected for an isolated compact region of CTCs. It therefore appears that, in Anderson’s proposal, the class of sensible physical questions and observations can be determined by the structure of spacetime in the future, or alternatively, that the class of observable quantities may change over time. We note in passing that the Heisenberg picture in the unmodified Hilbert space also suffers from the same problem (see Section 2).

It is therefore prudent to seek other means by which unitarity can be restored. In Section 4, we propose a method of unitarity restoration using the mathematical technique of unitary dilations. This is motivated by the simple geometric observation that any linear transformation of the real line is the projection of an orthogonal transformation (called an orthogonal dilation of the original mapping) in a larger (possibly indefinite) inner product space. To see this, note that any linear contraction on the line may be regarded as the projection of a rotation in the plane: the contraction in length along the $x$-axis, say, being balanced by a growth in the $y$-component. Similarly, a linear dilation on the line may be regarded as the projection of a Lorentz boost in two dimensional Minkowski space. This observation may be extended to operators on Hilbert spaces:
it was shown by Sz.-Nagy [10] that any contraction (i.e., an operator $X$ such that $\|X\psi\| \leq \|\psi\|$ for all $\psi$) has a unitary dilation acting on a larger Hilbert space. The theory was subsequently extended to non-contractive operators by Davis [11] at the cost of introducing indefinite inner product spaces. Unitary dilations have previously found physical applications in the quantum theory of open systems [12], and have also been employed by one of us in an inverse scattering construction of point-like interactions in quantum mechanics [13, 14].

Put concisely, starting with a non-unitary evolution $X$, we pass to a unitary dilation of $X$, mapping between enlarged inner product spaces whose inner product may (possibly generically) be indefinite. The signature of the inner product is determined by the operator norm $\|X\|$ of $X$: if $\|X\| \leq 1$, the enlarged inner product spaces are Hilbert spaces, whilst for $\|X\| > 1$, they are indefinite inner product spaces (Krein spaces). Within the context of our proposal, it is therefore important to determine $\|X\|$ for any given CTC evolution operator.

Essentially, the unitary dilation proposal performs the minimal book-keeping required to restore unitarity by asserting the presence of a hidden component of the wavefunction, which is naturally associated with the CTC region. These ‘extra dimensions’ are not accessible to experiments conducted outside the CTC region, but provide somewhere for particles to hide from view, whilst maintaining global unitarity. We will see that our proposal thereby circumvents the problems associated with non-unitary evolutions mentioned above.

Of course, it is a moot point whether or not one should require a unitary evolution of quantum fields in the presence of CTCs; one might prefer a more radical approach such as that advocated by Hartle [7]. Our philosophy here is to determine the extent to which the conventional formalism of quantum theory can be repaired.

## 2 Non-Unitary Quantum Mechanics

As we mentioned above, a non-unitary evolution raises many problems for the standard formalism and interpretation of quantum theory, some of which we now discuss.

Firstly, the usual equivalence of the Schrödinger and Heisenberg pictures is lost. Given an evolution $X$ of states and an observable $A$, we would naturally define the evolved observable $A'$ so that for all initial states $\psi$, the expectation value of $A'$ in state $\psi$ equals the expectation of $A$ in the evolved state $X\psi$. Explicitly, we require

$$\frac{\langle \psi \mid A' \psi \rangle}{\langle \psi \mid \psi \rangle} = \frac{\langle X\psi \mid AX\psi \rangle}{\langle X\psi \mid X\psi \rangle}$$

(2.1)

for all $\psi$ in the Hilbert space $\mathcal{H}$. If $X$ is unitary up to a scale\(^1\) (i.e., $X^*X = XX^* = \lambda I$, $\lambda \in IR^+$), then equation (2.1) is uniquely solved by the Heisenberg evolution $A' = $\(^1\)Clearly, the scale of $X$ carries no physical information.
$X^{-1}AX$. On the other hand, if $X$ is not unitary up to scale, then there is no operator $A'$ satisfying (2.1) unless $A$ is a scalar multiple of the identity.

For completeness, we give a proof of this fact. Defining $f(\psi)$ to equal the RHS of (2.1), and taking $\psi$ and $\varphi$ to be any orthonormal vectors, we note that linearity of $A'$ entails

$$f(\psi) + f(\varphi) = f(\psi + \varphi) + f(\psi - \varphi),$$

whilst linearity of $A$ implies

$$2(f(\psi)\|X\psi\|^2 + f(\varphi)\|X\varphi\|^2) = f(\psi + \varphi)\|X(\psi + \varphi)\|^2 + f(\psi - \varphi)\|X(\psi - \varphi)\|^2. \quad (2.3)$$

Multiplying $\varphi$ by a phase to ensure that $\langle X\psi \mid X\varphi \rangle$ is imaginary (and hence that $\|X(\psi \pm \varphi)\|^2 = \|X\psi\|^2 + \|X\varphi\|^2$), we combine these relations to obtain

$$(f(\psi) - f(\varphi))(\|X\psi\|^2 - \|X\varphi\|^2) = 0 \quad (2.4)$$

which is clearly insensitive to the phase of $\varphi$ and therefore holds for all orthonormal vectors $\psi$ and $\varphi$. If $X$ is not unitary up to scale, we choose $\varphi$ and $\psi$ so that $\|X\psi\| \neq \|X\varphi\|$. Thus $f(\psi) = f(\varphi) = F$ for some $F$. It follows that $f(\chi) = F$ for all $\chi \perp \text{span} \{\psi, \varphi\}$ (because $\|X\chi\|$ cannot equal both $\|X\psi\|$ and $\|X\varphi\|$) and hence for all $\chi \in \mathcal{H}$. Thus $A$ is a scalar multiple of the identity.

Thus, the conventional equivalence of the Schrödinger and Heisenberg pictures is radically broken. If there are evolved states, there are no evolved operators, and vice versa. In addition, the Heisenberg picture places restrictions on the class of allowed observables. In order to preserve the canonical commutation relations, we take the evolution to be $A \rightarrow X^{-1}AX$; however, we also want to preserve self-adjointness of observables under evolution. Combining these two requirements, we conclude that $A$ must commute with $XX^*$ – the non-unitary part of the evolution in the sense of the polar decomposition. Thus, the claim attributed to Dirac [15] that ‘Heisenberg mechanics is the good mechanics’ carries the price of a restricted class of observables when the evolution is non-unitary.

A second problem with non-unitary evolutions, noted by Jacobson [5] (see also Hartle’s elaboration [7]) is that one cannot assign unambiguous values to expectation values of operators localised in regions spacelike separated from the CTC region. Let $\mathcal{R}$ be a compact region spacelike separated from the CTCs, and which is contained in two spacelike slices $\sigma_+$ and $\sigma_-$, such that $\sigma_-$ passes to the past of the CTCs and $\sigma_+$ to their future. If $A$ is an observable which is localised within $\mathcal{R}$, one can measure its expectation value with respect to the wavefunction on either spacelike surface. In order for these values to agree, equation (2.1) must hold with $A' = A$. If $X$ is unitary up to scale, this is satisfied by any observable which commutes with $X$ – in particular by all observables localised in $\mathcal{R}$. However, if $X$ is not unitary up to scale, our arguments above show that there is no observable (other than multiples of the identity) for which unambiguous expectation values may be calculated. Jacobson concludes that a breakdown of unitarity is accompanied by a breakdown of causality.
Thirdly, Friedman, Papastamatiou and Simon [2] have pointed out related problems with the assignment of probabilities for events occurring before the region of CTCs. They consider a microscopic system which interacts momentarily with a measuring device before the CTC region and which is decoupled from it thereafter. The microscopic system passes through the CTC region, whilst the measuring device does not. However, the probability that a certain outcome is observed on the measuring device depends on whether it is observed before or after the microscopic system passes through the CTCs. This is at variance with the Copenhagen interpretation of quantum theory.

3 The Anderson Proposal

We begin with a brief résumé of Anderson’s proposal [8]. Let \( \mathcal{H} \) be a Hilbert space equipped with inner product \( \langle \cdot | \cdot \rangle \). Suppose the evolution operator, \( X \), is bounded with bounded inverse, but non-unitary, i.e., \( \langle X \psi | X \varphi \rangle \neq \langle \psi | \varphi \rangle \) for some \( \psi, \varphi \in \mathcal{H} \). We now define a new inner product, denoted as \( \langle \cdot | \cdot \rangle' \), by

\[
\langle \psi | \varphi \rangle' = \langle X^{-1} \psi | X^{-1} \varphi \rangle.
\]

(3.1)

Because \( (X^{-1})^* X^{-1} \) is positive\(^2\), it is clear that this defines a bona fide positive definite inner product. Moreover, the associated norm is complete by boundedness of \( X \) and \( X^{-1} \). Denoting \( \mathcal{H} \) with the new inner product by \( \mathcal{H}' \), the trivial calculation

\[
\langle X \psi | X \varphi \rangle' = \langle \psi | \varphi \rangle
\]

(3.2)

shows that \( X \) is unitary from \( \mathcal{H} \) to \( \mathcal{H}' \). Thus unitarity is restored by ‘renormalising’ the inner product.

We emphasise that, mathematically, there is nothing wrong with this procedure. However, one may question whether it is a physically reasonable prescription. To analyse this, we consider the definition and behaviour of observables. \textit{A priori}, it is not clear how observables are to be defined, for an operator which is self-adjoint on the initial Hilbert space of states will not in general be self-adjoint on the final Hilbert space (with the modified inner product). However, it is natural to restrict attention to those quantities which can be measured both before and after the CTC region, and we therefore define an observable to be an operator which is self-adjoint with respect to both inner products.

Pursuing this definition, the two calculations

\[
\langle A \psi | \varphi \rangle' = \langle X^{-1} A \psi | X^{-1} \varphi \rangle = \langle (X^{-1})^* X^{-1} A \psi | \varphi \rangle
\]

(3.3)

and

\[
\langle \psi | A \varphi \rangle' = \langle X^{-1} \psi | X^{-1} A \varphi \rangle = \langle \psi | (X^{-1})^* X^{-1} A \varphi \rangle,
\]

(3.4)

\(^2\)Here and elsewhere, the \( * \) denotes the adjoint with respect to \( \langle \cdot | \cdot \rangle \). We will not explicitly take adjoints with respect to \( \langle \cdot | \cdot \rangle' \).
show that $A$ is self-adjoint with respect to both inner products if and only if $A$ and $A(XX^*)^{-1}$ are self-adjoint with respect to the usual inner product on $\mathcal{H}$. It follows that $A$ must commute with $(XX^*)^{-1}$ (and its powers) in order to be an unambiguously defined observable. Noting the polar decomposition $[9] X = (XX^*)^{1/2} U$, where (because $\ker X^*$ is trivial) $U$ is unitary on $\mathcal{H}$, we have thus shown that all unambiguous observables must commute with the non-unitary part $(XX^*)^{1/2}$ of the evolution.

Now let us consider the evolution of the expectation value of an observable $A$ which is self-adjoint on both initial and final Hilbert spaces. In the initial normalised state $\psi$, we have expectation value $\langle A \rangle_\psi = \langle \psi | A \psi \rangle$, whilst in the evolved state $X\psi$ (which has unit norm in $\mathcal{H}'$), the expectation value is

$$\langle A \rangle'_{X\psi} = \langle X\psi | AX\psi \rangle' = \langle \psi | X^{-1}AX\psi \rangle$$

(3.5)

Now we use the fact that $A$ commutes with $(XX^*)^{1/2}$ to write

$$\langle \psi | X^{-1}AX\psi \rangle = \langle \psi | X^{-1}(XX^*)^{1/2}A(XX^*)^{-1/2}X\psi \rangle = \langle U^*AU \rangle_\psi$$

(3.6)

where $U = (XX^*)^{-1/2}X$ is unitary part of the evolution. Thus $\langle A \rangle'_{X\psi} = \langle U^*AU \rangle_\psi$, and we conclude that the expectation values of observables evolve according to the unitary part of the evolution only.

Another way of expressing the above is that equation (3.5) shows an equivalence between Anderson’s proposal and the Heisenberg picture in the original Hilbert space $\mathcal{H}$ and hence to the conclusion that $[A, XX^*] = 0$ by the arguments of Section 2.

We conclude that this proposal places heavy restrictions on the class of allowed observables. Let us emphasise that it is not simply the case that it abandons the non-unitary part of the evolution in favour of the unitary part (although this could be a separate proposal); the proposal of [8] restricts the class of observables, as well as (implicitly) modifying the evolution.

4 The Unitary Dilation Proposal

We begin by describing the theory of unitary dilations [10, 11]. Let $\mathcal{H}_1, \ldots, \mathcal{H}_4$ be Hilbert spaces and let $X$ be a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then an operator $\hat{X}$ from $\mathcal{H}_1 \oplus \mathcal{H}_3$ to $\mathcal{H}_2 \oplus \mathcal{H}_4$ is called a dilation of $X$ if $X = P_{\mathcal{H}_2} \hat{X} |_{\mathcal{H}_1}$, where $P_{\mathcal{H}_2}$ is the orthogonal projector onto $\mathcal{H}_2$. In block matrix form, $\hat{X}$ takes form

$$\hat{X} = \begin{pmatrix} X & P \\ Q & R \end{pmatrix}.$$ 

(4.1)

Our nomenclature follows that of Halmos [16].

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3In [8] a different expectation value computation is performed: namely the computation of the expectation value of the operator $XAX^{-1}$ in the evolved state $X\psi$, compared with the expectation value of $A$ in state $\psi$ (see note [14] in [8]).
Now let $X$ (bounded with bounded inverse and non-unitary) be our evolution operator, which evolves quantum states from an initial spacelike hypersurface to a final spacelike hypersurface. We assume that the Hilbert spaces of states on these surfaces are identical, so $X$ maps $\mathcal{H}$ to itself. If the initial hypersurface contains regions which are causally separate from the CTC region, we assume that $X$ has been normalised to be unitary on states localised in such regions. We point out that such exterior regions may not exist — even if the CTC region is itself compact. Consider, for example, a spacetime that is asymptotically (the universal cover of) anti-de Sitter space. In such a spacetime, hypersurfaces sufficiently far to the future and far to the past of the CTC region will be entirely contained within the CTC region’s light cone and there will be no exterior region on which to set up our normalisation. We may normalise the evolution operator on hypersurfaces for which an exterior region may be identified and extend arbitrarily to those surfaces where no such region exists. Indeed, it is entirely possible that every point in spacetime is contained in the light cone of the CTC region; in this case we give up any attempt to find a ‘physical’ normalisation for the evolution operator.

Given such an evolution operator, we may quantify its departure from unitarity with the operators $M_1 = I - XX^*$ and $M_2 = I - X^*X$. As a consequence of the spectral theorem, we have the intertwining relations

$$X^* f(M_1) = f(M_2) X^*; \quad X f(M_2) = f(M_1) X$$

for any continuous Borel function $f$. In particular, because $X^{-1}$ is bounded, we have $M_2 = X^{-1} M_1 X$ and so the closures $\mathcal{M}_i$ of the ranges of the $M_i$ are isomorphic.

For $i = 1, 2$, we now define $\mathcal{K}_i = \mathcal{H} \oplus \mathcal{M}_i$, equipped with the (possibly indefinite) inner product $\langle \cdot | \cdot \rangle_{\mathcal{K}_i}$ given by

$$\left[ \begin{pmatrix} \varphi \\ \Phi \end{pmatrix}, \begin{pmatrix} \psi \\ \Psi \end{pmatrix} \right]_{\mathcal{K}_i} = \langle \varphi | \psi \rangle + \langle \Phi | \text{sgn} M_i \Psi \rangle$$

where the inner products on the right are taken in $\mathcal{H}$ and $\text{sgn} M_i = |M_i|^{-1} M_i$ where $|M_i| = (M_i^* M_i)^{1/2}$. It is easy to show that $\text{sgn} M_i$ is positive if $\|X\| \leq 1$, in which case $\langle \cdot, \cdot \rangle_{\mathcal{K}_i}$ is positive definite; however, for $\|X\| > 1$, the inner products above are indefinite, and $\mathcal{K}_1$ and $\mathcal{K}_2$ are Krein spaces (for details on the theory of operators in indefinite inner product spaces, see the monographs [17, 18]). It is important to remember that the $\mathcal{K}_i$ also have a positive definite inner product from their original definition as a direct sum of Hilbert spaces$^4$. Thus a bounded linear operator $A$ from $\mathcal{K}_1$ to $\mathcal{K}_2$ has two adjoints: the Hilbert space adjoint $A^*$, and the Krein space adjoint, which we denote $A^\dagger$. It is a simple exercise to show that $A^\dagger$ is given by

$$A^\dagger = J_1 A^* J_2$$

where the operators $J_i$ defined on $\mathcal{K}_i$ are unitary involutions given by $J_i = \mathcal{H} \oplus \text{sgn} (M_i)$.

$^4$In fact, this inner product determines the topology of $\mathcal{K}_i$. 
Next, we define a dilation $\hat{X} : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ of $X$ by

$$\hat{X} = \begin{pmatrix} X & -sgn(M_1)|M_1|^{1/2} \\ |M_1|^{1/2} & X^*|M_1^{-1} \end{pmatrix}. \quad (4.5)$$

The adjoint $\hat{X}^\dagger$ is given by

$$\hat{X}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & sgn(M_1) \end{pmatrix} \begin{pmatrix} X^* & |M_1|^{1/2} \\ |M_1|^{1/2} & X^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & sgn(M_2) \end{pmatrix}. \quad (4.6)$$

It is then a matter of computation using the intertwining relations to show that $\hat{X}^\dagger \hat{X} = II_{\mathcal{K}_1}$ and $\hat{X} \hat{X}^\dagger = II_{\mathcal{K}_2}$. $\hat{X}$ is therefore a unitary dilation of $X$.

The construction we have given is not unique. For suppose that $\mathcal{N}_1$ and $\mathcal{N}_2$ are Krein spaces, and that $U_i : \mathcal{M}_i \rightarrow \mathcal{N}_i$ are unitary (with respect to the indefinite inner products). Then

$$\tilde{X} = \begin{pmatrix} 1 & 0 \\ 0 & U_2 \end{pmatrix} \hat{X} \begin{pmatrix} 1 & 0 \\ 0 & U_2^\dagger \end{pmatrix} \quad (4.7)$$

is also a unitary dilation of $X$, mapping between $\mathcal{H} \oplus \mathcal{N}_1$ and $\mathcal{H} \oplus \mathcal{N}_2$. Because this just amounts to a redefinition of the auxiliary spaces, it carries no additional physical significance. One may show that all other unitary dilations of $X$ require the addition of larger auxiliary spaces than the $\mathcal{M}_i$ (for example, one could dilate $\hat{X}$ further), and so $\hat{X}$ is the minimal unitary dilation of $X$ up to unitary equivalence of the above form.

The unitary dilation proposal is as follows: the non-unitary evolution $X$ is regarded as the projection onto $\mathcal{H}$ of the restriction of $\hat{X}$ to $\mathcal{H}$, where $\hat{X}$ is a unitary operator between enlarged inner product spaces. These enlarged spaces and $\hat{X}$ describe the ‘full’ physics of the situation; however, experiments performed on the initial and final surfaces reveal an apparently non-unitary evolution. The auxiliary space $\mathcal{M}$ represents degrees of freedom localised within the CTC region and therefore not directly accessible to experiments outside.

Let us point out that many features of this proposal can only be determined in the context of a particular evolution $X$ and therefore a particular CTC spacetime. There are, however, various model independent features of our proposal, which we discuss below.

**Predictability** It is clear that the overall evolution $\hat{X}$ is predictable; however, because the initial state involves degrees of freedom not present on the initial hypersurface (i.e., the component of the wavefunction in $\mathcal{M}$), it is clear that – as far as physical measurements are concerned – there is some loss of predictability in the final state. Of course, this could be restored if one decided axiomatically that the initial state should have no component in $\mathcal{M}$.

**Observables** We restrict ourselves to quantities measurable on the initial and final surfaces. Given a self-adjoint operator $A$ on $\mathcal{H}$, we define corresponding observables on $\hat{X}$ by

$$\hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

\[\text{In directly, we can infer their presence by analysing } X.\]
the $K_i$ with the block matrix form

$$\hat{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$  \hfill (4.8)

There is no requirement that observables should commute with the non-unitary part of $X$. On the premise that the initial state has no component in $M$ and takes the vector form $(\psi, 0)^T$, the initial expectation of $A$ is equal to the expectation of $A$ in state $\psi$. After evolution, the expectation value is

$$\left[ \hat{X} \left( \begin{array}{c} \psi \\ 0 \end{array} \right), \hat{A} \hat{X} \left( \begin{array}{c} \psi \\ 0 \end{array} \right) \right]_{K_2} = \frac{\langle X\psi | AX\psi \rangle}{\langle \psi | \psi \rangle} \hfill (4.9)$$

From this, it is clear that Jacobson’s ambiguity [5] is avoided for all observables $A$ such that $A = X^* AX$. This is satisfied if the range of $A$ is contained in $U = \ker M_1 \cap \ker M_2$ and $A$ commutes with the restriction $X|_U$ of $X$ to $U$. In particular, if $\mathcal{H}_R$ is the subspace of states supported in $\mathcal{R}$, and assuming (as in [5]) that $X$ acts as the identity on $\mathcal{H}_R$, it is easy to check that $\mathcal{H}_R \subseteq U$. There is therefore no ambiguity in assigning expectation values to local observables associated with $\mathcal{R}$, i.e., self-adjoint operators which vanish on the orthogonal complement of $\mathcal{H}_R$.

In addition, the breakdown of the Copenhagen interpretation noted in [2] is avoided as a direct consequence of the unitarity of $\hat{X}$.

**Time Reversal** From our comments on predictability, it seems that there is a distinction between initial and final states, either due to the loss of prediction in the forward evolution or by asserting that the initial state should have no component in $M$ (because, in general, the final state will have a non-zero component in $M$). We also note that our proposal is not symmetric between $X$ and $X^{-1}$; the adjoint operator $\hat{X}^\dagger$ is not a dilation of $X^{-1}$ unless $X$ is unitary. It appears that our proposal does not admit ‘time reversible time machines’.

## 5 Composition

We have described how a single non-unitary evolution may be dilated to a unitary evolution between enlarged inner product spaces. However, one might ask how a sequence of non-unitary evolutions can be chained together. To answer this, we return to our motivating example of contractions and dilations on the line.

Consider two contractions, given by multiplication by $\cos \theta$ and $\cos \varphi$. Each can be dilated to a rotation in the plane of angle $\theta$ and $\varphi$ respectively. If both dilations are taken to act in the $xy$-plane, their product is a rotation of $\theta + \varphi$, which is not generally a dilation of the product $\cos \theta \cos \varphi$ of our original contractions. On the other hand, if
the first dilation acts in the $xy$-plane and the second in the $xz$-plane, their product is a rotation of $I\mathbb{R}^3$ which does dilate $\cos \theta \cos \varphi$.

With this example in mind, let us consider two evolutions $X$ and $Y$ on $\mathcal{H}$ and their composition $XY$. We define the $M_i$ and $\mathcal{M}_i$ as before and introduce $N_1 = I - YY^*$, $N_2 = I - Y^*Y$ and $\mathcal{N}_i = \overline{\text{Ran } N_i}$ to be the closure of the range of $N_i$ for $i = 1, 2$. We also introduce $P_1 = I - XYY^*X^*$, $P_2 = I - Y^*X^*XY$ and $\mathcal{P}_i = \overline{\text{Ran } P_i}$. Note that

$$P_1 = M_1 + XN_1X^* \quad \text{and} \quad P_2 = N_2 + Y^*M_2Y.$$  

(5.1)

We define unitary dilations of $X$ and $Y$ as follows: $\dot{Y} : \mathcal{H} \oplus \mathcal{M}_1 \oplus \mathcal{N}_1 \rightarrow \mathcal{H} \oplus \mathcal{M}_1 \oplus \mathcal{N}_2$ is given by

$$\dot{Y} = \begin{pmatrix} Y & 0 & -\text{sgn } N_1 |N_1|^{1/2} \\ 0 & |N_2|^{1/2} & 0 \\ |N_2|^{1/2} & 0 & Y^*|\mathcal{N}_1| \end{pmatrix}$$  

(5.2)

and $\dot{X} : \mathcal{H} \oplus \mathcal{M}_1 \oplus \mathcal{N}_2 \rightarrow \mathcal{H} \oplus \mathcal{M}_2 \oplus \mathcal{N}_2$ is given by

$$\dot{X} = \begin{pmatrix} X & -\text{sgn } M_1 |M_1|^{1/2} & 0 \\ |M_2|^{1/2} & X^*|\mathcal{M}_1| & 0 \\ 0 & 0 & |\mathcal{N}_2| \end{pmatrix}. $$  

(5.3)

The product $\dot{X}\dot{Y}$ is given by

$$\dot{X}\dot{Y} = \begin{pmatrix} XY & -\text{sgn } M_1 |M_1|^{1/2} & -X \text{sgn } N_1 |N_1|^{1/2} \\ |M_2|^{1/2}Y & X^*|\mathcal{M}_1| & -|M_2|^{1/2} \text{sgn } N_1 |N_1|^{1/2} \\ |N_2|^{1/2} & 0 & Y^*|\mathcal{N}_1| \end{pmatrix}$$  

(5.4)

and is a unitary dilation of $XY$, mapping from $\mathcal{H} \oplus \mathcal{M}_1 \oplus \mathcal{N}_1$ to $\mathcal{H} \oplus \mathcal{M}_2 \oplus \mathcal{N}_2$. One may relate this to the dilation $\dot{X}\dot{Y}$ arising from the prescription (4.5) as follows. Let

$$Q_1 = \begin{pmatrix} |M_1|^{1/2} \\ |N_1|^{1/2} \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} |M_2|^{1/2}Y \\ |N_2|^{1/2} \end{pmatrix},$$  

(5.5)

and define $U_i (i = 1, 2)$ on $\text{Ran } P_i^{1/2} \subset \mathcal{P}_i$ by $U_i = Q_i |P_i|^{-1/2}$. The $U_i$ are easily seen to be isometries (with respect to the appropriate inner products) from their domains into $\mathcal{M}_i \oplus \mathcal{N}_i$ such that $Q_i |\text{Ran } P_i^{1/2} = U_i |P_i|^{-1/2}$. Provided that $Q_i = Q_i |\text{Ran } P_i^{1/2}$ is orthocomplemented in $\mathcal{M}_i \oplus \mathcal{N}_i$ (in the indefinite inner product), one may then show that

$$P_{\mathcal{H} \oplus \mathcal{Q}_2} \dot{X}\dot{Y} |\mathcal{H} \oplus \mathcal{Q}_1 = \begin{pmatrix} I & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} XY & -\text{sgn } P_1 |P_1|^{1/2} \\ |P_1|^{1/2} & (XY)^* |P_1| \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & U_1^* \end{pmatrix},$$  

(5.6)

where $P_{\mathcal{H} \oplus \mathcal{Q}_2}$ is the orthoprojector onto $\mathcal{H} \oplus \mathcal{Q}_2$. Thus $\dot{X}\dot{Y}$ is a dilation of an operator isometrically equivalent to $\dot{X}\dot{Y}$. The isometries act non-trivially only on the auxiliary spaces and have no physical significance. The extra dimensions introduced by the dilation are also to be expected because the combined evolution $Z = XY$ may be factorised
in many different ways; hence the two individual evolutions carry more information than their combination.

The assumption that the $Q_i$ are orthocomplemented is easily verified if the operators $U_i$ are bounded, for in this case, they may be extended to unitary operators on the whole of $\mathcal{P}_i$. Then $Q_i$ is the unitary image of a Krein space and is orthocomplemented by Theorem VI.3.8 in [17]. $U_1$ is bounded if there exists $K$ such that $\|P_1\psi\| < \epsilon$ only if $\|M_1\psi\| + \|N_1X\psi\| < K\epsilon$ for all sufficiently small $\epsilon > 0$. Similarly, $U_2$ is bounded if $\|P_2\psi\| < \epsilon$ only if $\|M_2Y\psi\| + \|N_2\psi\| < K\epsilon$ for all sufficiently small $\epsilon > 0$. Physically, this equates to the reasonable condition that the combined evolution can be ‘almost unitary’ on a given state only if the individual evolutions are also ‘almost unitary’.

As a particular instance of the above, we consider the case where $Y$ is unitary. The $N_i$ therefore vanish and the $N_i$ are trivial; in addition, $P_1 = M_1$ and $P_2 = Y^*M_2Y$. The operator $\tilde{Y}$ is

$$\tilde{Y} = \begin{pmatrix} Y & 0 \\ 0 & H_{\mathcal{M}_1} \end{pmatrix}$$

and $\tilde{X} = \tilde{X}$. The combined evolution is thus

$$\tilde{X}\tilde{Y} = \tilde{X} \begin{pmatrix} Y & 0 \\ 0 & H_{\mathcal{M}_1} \end{pmatrix}$$

which is unitarily equivalent to $\tilde{X}\tilde{Y}$ in the sense that

$$\tilde{X} \begin{pmatrix} Y & 0 \\ 0 & H_{\mathcal{M}_1} \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & Y \end{pmatrix} \tilde{X}\tilde{Y}.$$ (5.9)

We emphasise that the first factor on the RHS has no physical significance and is merely concerned with mapping the auxiliary spaces $\mathcal{P}_2$ to $\mathcal{M}_2$ in a natural way.

To conclude this section, we note that if $A$ belongs to the class of observables which avoid the Jacobson ambiguity for each CTC region individually, then it also avoids this ambiguity for the combined evolution; for if $A = X^*AX = Y^*AY$, then certainly $A = X^*AXY$.

6 Conclusion

We have examined Anderson’s proposal for restoring unitarity to quantum evolution in CTC spacetimes, and noted that it entails that the natural class of observables must commute with the non-unitary part of the time evolution. In addition, we have proposed a new method by which unitarity can be restored, based on the mathematical theory of unitary dilations.

Our philosophy here has been to regard the non-unitarity of $X$ as a signal that the full physics (and a unitary evolution) is being played out on a larger state space than is observed. This bears some resemblance to the situation in special relativity, where
time dilation signals that one must pass to spacetime (and an indefinite metric) in order to restore an orthogonal transformation between reference frames. (Indeed, the Lorentz boost in two dimensional Minkowski space is precisely an orthogonal dilation of the time dilation effect).

For our case of interest, the physical picture is that the \( M \) correspond to degrees of freedom within the CTC region. Non-unitarity of the evolution signals that a particle cannot pass through the CTC region unscathed; part of the initial state becomes trapped in the auxiliary space corresponding to the CTCs.

Degrees of freedom within the CTC region may have positive or negative norm-squared, depending on the inner products of the \( \mathcal{K} \). To determine this, one must compute the operator norm of the evolution operator for a concrete CTC model. If, as is possible, this norm exceeds unity, then the \( \mathcal{K} \) become indefinite Krein spaces. Perturbative calculations in \( \lambda \phi^4 \) theory by Boulware [1] suggest that this may well occur. One is then faced with a choice:

1. One could seek a natural positive definite subspace of the initial and final Krein spaces. The obvious choice would be to take the initial Hilbert space to be \( \mathcal{H} \) and the final Hilbert space to be the image of \( \mathcal{H} \) under \( \hat{X} \). However, this may lead to some problems in defining observables on the final Hilbert space.

2. Alternatively, one could decide that CTCs are incompatible with the twin requirements of unitarity and a Hilbert space structure. The initial and final state spaces would naturally be Krein spaces. This would not be entirely unexpected: studies of quantum mechanics on the ‘spinning cone’ spacetime [19] have concluded that the inner product becomes indefinite precisely inside the region of CTCs. It is possible that this may signal instability of the CTC spacetime (cf. Hawking’s Chronology Protection Conjecture [20]).

Detailed statements concerning our proposal must await concrete calculations; however, we have seen that our proposal removes Jacobson’s ambiguity and is not time reversible. Finally, our treatment has been entirely in terms of states and operators; it would be interesting to see how it translates into density matrices and the language of generalised quantum mechanics [6].

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Appendix

Here, we consider another possible method for the restoration of unitarity which suffers from similar problems to that of Anderson. Instead of focusing on direct sums of Hilbert spaces, this proposal uses tensor products and always maintains a positive definite inner product. We start with $X : \mathcal{H} \to \mathcal{H}$, bounded with bounded inverse and non-unitary as before, and define a new Hilbert space $\mathcal{H}_X = (II \otimes X)\Sigma$, where $\Sigma \subset \mathcal{H} \otimes \mathcal{H}$ is the closure of the space of finite linear combinations of terms of form $\psi \otimes \phi$ for $\phi \in \mathcal{H}$. Similarly, we define $\mathcal{H}_{X^{-1}} = (II \otimes X^{-1})\Sigma$. Now define the operator $\hat{X} = X \otimes X^{-1}$ restricted to $\mathcal{H}_X$.

Clearly, $\hat{X}(\psi \otimes X\psi) = \varphi \otimes X^{-1}\varphi$ where $\varphi = X\psi$, and so $\hat{X} : \mathcal{H}_X \to \mathcal{H}_{X^{-1}}$. Moreover,

$$\langle \hat{X}(\psi \otimes X\psi) | \hat{X}(\varphi \otimes X\varphi) \rangle = \langle X\psi \otimes \psi | X\varphi \otimes \varphi \rangle = \langle X\psi | X\varphi \rangle \langle \psi | \varphi \rangle = \langle \psi \otimes X\psi | \varphi \otimes X\varphi \rangle$$

and therefore $\hat{X}$ is a unitary operator from $\mathcal{H}_X$ to $\mathcal{H}_{X^{-1}}$.

Let us examine the structure of this proposal in more detail. First, there is a natural transposition operation $T$ on $\mathcal{H} \otimes \mathcal{H}$: $T(\varphi \otimes \psi) = \psi \otimes \varphi$. It is easy to see that $\hat{X}$ is the restriction of $T$ to $\mathcal{H}_X$: hence all the information about $X$ is encoded into the definition of $\mathcal{H}_X$. Have we lost any information in this process? Suppose $\mathcal{H}_X = \mathcal{H}_Y$ for two distinct operators $X$ and $Y$. Then $II \otimes Z$ is a bounded invertible linear map (though not necessarily unitary) of $\Sigma$ onto itself, where $Z = X^{-1}Y$. Because $T$ restricts to the identity on $\Sigma$, we require $\psi \otimes Z\psi = Z\psi \otimes \psi$ for each $\psi \in \mathcal{H}$. Taking an inner product with $\varphi \otimes \psi$ for some $\varphi$, we obtain

$$\langle \phi | \psi \rangle \langle \psi | Z\psi \rangle = \langle \phi | Z\psi \rangle \langle \psi | \psi \rangle.$$  \hspace{1cm} (A.2)

Because $\phi$ is arbitrary, $\psi$ is therefore an eigenvector of $Z$. But $\psi$ was also arbitrary and therefore $Z = \lambda II$ for some constant $\lambda \in \mathbb{C} \setminus \{0\}$. Thus $Y = \lambda X$, so this construction loses exactly one scalar degree of freedom. Effectively, we have lost the (scalar) operator norm $\|X\|$ of $X$, but no other information. This is much less information than is lost by ignoring the operator $(XX^*)^{1/2}$ as is apparently the case in the Anderson proposal.

We have therefore restored unitarity at the price of introducing a second Hilbert space and correlations between the two. The evolution on the large space is unitary. This fits well with the picture of acausal interaction between the initial space and the CTC region in its future. The physical interpretation is as follows: the ‘time machine’ contains a copy of the external universe, which evolves backwards in time, starting with the final state of the quantum fields and ending with their initial state. It is impossible to prepare the initial state of the CTC region independently from the initial state of the exterior quantum fields.

However, problems arise when observables are defined. Here, observables on the initial space are naturally defined to be self-adjoint operators on $\mathcal{H} \otimes \mathcal{H}$ with $\mathcal{H}_X$ as an invariant subspace (observables on the final space would have $\mathcal{H}_{X^{-1}}$ invariant). An operator of form $A \otimes B$ maps $\mathcal{H}_X$ to itself only if $B = XAX^{-1}$; combining this with
the requirement of self-adjointness, one finds that $A$ must commute with $X^*X$ and its powers. Thus this proposal suffers from a similar objection to that of Anderson, although we note the curious fact that one requires commutation with $X^*X$, and the other with $XX^*$, suggesting that the two proposals are in some sense dual to each other.

The requirement that $\mathcal{H}_X$ be an invariant subspace for all observables was adopted so that our space of initial states is invariant under the unitary groups generated by observables (e.g. translations). If we relax this, and define observables to be self-adjoint operators on $\mathcal{H} \otimes \mathcal{H}$, it appears that $A \otimes I$ corresponds naturally to the operator $A$ on $\mathcal{H}$. However, this suffers from the ambiguity pointed out by Jacobson [5].
References