On the Hopf structure of $U_{p,q}(gl(1|1))$ and the universal $T$-matrix of $Fun_{p,q}(GL(1|1))$

R. Chakrabarti$^{1,}$, R. Jagannathan$^{2,3,4}$

$^1$Department of Theoretical Physics, University of Madras,
Guindy Campus, Madras - 600025, INDIA
$^2$International Centre for Theoretical Physics, 34100 Trieste, ITALY
$^3$The Institute of Mathematical Sciences
C.I.T Campus, Tharamani, Madras - 600113, INDIA

Abstract

Using the technique developed by Fronsdal and Galindo (Lett. Math. Phys. 27 (1993) 57) for studying the Hopf duality between the quantum algebras $Fun_{p,q}(GL(2))$ and $U_{p,q}(gl(2))$, the Hopf structure of $U_{p,q}(gl(1|1))$, dual to $Fun_{p,q}(GL(1|1))$, is derived and the corresponding universal $T$-matrix of $Fun_{p,q}(GL(1|1))$, embodying the suitably modified exponential relationship $U_{p,q}(gl(1|1)) \rightarrow Fun_{p,q}(GL(1|1))$, is obtained.

---

$^1$E-mail: ranabi@unimad.ernet.in
$^1$E-mail: jagan@imsc.ernet.in
$^{3,4}$Permanent address
1. Introduction

Recently Fronsdal and Galindo [1] have studied the duality structure relating $\text{Fun}_{p,q}(GL(2))$ and $\text{U}_{p,q}(gl(2))$ (see also [2,3] for some aspects of a similar approach to $\text{Fun}_q(GL(2))$ and $\text{Fun}_q(SL(2))$) wherein the transfer matrix $T$ gets replaced by the representation-independent universal $T$-matrix. The universal $T$-matrix, identified with the dual form

$$T = \sum_A x^A \otimes X_A,$$

where $\{x^A\}$ and $\{X_A\}$ are respectively the basis elements of a pair of dually conjugate Hopf algebras $\mathcal{A}$ and $\mathcal{U}$, expresses the generalization of the familiar exponential mapping ($g \to G$) in the case of Lie algebras. In the case of $\text{Fun}_{p,q}(GL(2))$ the corresponding universal $T$-matrix has been explicitly shown [1] to have the all the required algebraic properties [4] of the $T$-matrix. Further, in the universal $T$-matrix one has all the representations of a quantum matrix pseudo-group $\text{Fun}_\ldots(G)$ expressed in terms of the matrix representations of its dually paired (in the Hopf sense) quantized universal enveloping algebra $U_\ldots(g)$ (see [2] for the case of $\text{Fun}_q(GL(2))$). Apart from considering the universal $T$-matrix of $\text{Fun}_q(SL(2))$ in detail, following [1], the universal $T$-matrices of some inhomogeneous quantum groups have also been obtained in [5] using suitable contraction techniques. With the aim of providing a new explicit example of the universal $T$-matrix we consider here the two-parametric ($p,q$) Hopf superalgebras corresponding to the classical $GL(1|1)$ and $gl(1|1)$. We study the duality structure relating $\text{Fun}_{p,q}(GL(1|1))$ [6,7] and $\text{U}_{p,q}(gl(1|1))$ [7], following [1], and construct the universal $T$-matrix of $\text{Fun}_{p,q}(GL(1|1))$. When $p = q$ the analysis will reduce to the single-parameter case (in this context, see [8,9] for some recent studies on the finite-dimensional irreducible representations of $U_q(gl(n|m)))$. 
2. Hopf structure of $\text{Fun}_{p,q}(GL(1|1))$

The algebra $\text{Fun}_{p,q}(GL(1|1))$ is generated by the elements of the $2 \times 2$ matrix

$$T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix},$$

(2.1)

obeying the braiding relations

$$
\begin{align*}
a\beta &= p^{-1}\beta a, \quad a\gamma = q^{-1}\gamma a, \quad d\beta = p^{-1}\beta d, \quad d\gamma = q^{-1}\gamma d, \\
p^{-1}\beta\gamma + q^{-1}\gamma\beta &= 0, \quad ad - da = (q - p^{-1})\beta\gamma, \\
\beta^2 &= 0, \quad \gamma^2 = 0.
\end{align*}
$$

(2.2)

The even elements $a$ and $d$ are invertible. The coproduct ($\Delta$), counit ($\epsilon$) and the antipode ($S$) maps are given respectively by

$$
\begin{align*}
\Delta(T) &= T \otimes T, \\
\Delta(a^{-1}) &= a^{-1} \otimes a^{-1} - a^{-1}\beta a^{-1} \otimes a^{-1}\gamma a^{-1}, \\
\Delta(d^{-1}) &= d^{-1} \otimes d^{-1} - d^{-1}\gamma d^{-1} \otimes d^{-1}\beta d^{-1},
\end{align*}
$$

(2.3)

$$\epsilon(T) = 1l,$$

(2.4)

$$S(T) = T^{-1}, \quad S(a^{-1}) = a - \beta d^{-1}\gamma, \quad S(d^{-1}) = d - \gamma a^{-1}\beta,$$

(2.5)

where

$$T^{-1} = \begin{pmatrix} a^{-1} + a^{-1}\beta d^{-1}\gamma a^{-1} & -a^{-1}\beta d^{-1} \\
-d^{-1}\gamma a^{-1} & d^{-1} + d^{-1}\gamma a^{-1}\beta d^{-1} \end{pmatrix}$$

(2.6)
and \( \otimes \) denotes the tensor product combined with the matrix multiplication. Here, and throughout the paper, the tensor product \((\otimes)\) is a graded one corresponding to superalgebras such that

\[
(\Gamma_1 \otimes \Gamma_2)(\Gamma_3 \otimes \Gamma_4) = (-1)^{\text{deg}(\Gamma_2)\text{deg}(\Gamma_3)}\Gamma_1\Gamma_3 \otimes \Gamma_2\Gamma_4,
\]

(2.7)

where \( \text{deg}(\Gamma) = 0 (1) \) for an even (odd) \( \Gamma \). The invertible quantum superdeterminant

\[
D = ad^{-1} - \beta d^{-1} \gamma d^{-1}
\]

(2.8)

is a central element of the algebra and follows a grouplike coproduct rule

\[
\Delta(D) = D \otimes D.
\]

(2.9)

The counit and antipode maps read

\[
\epsilon(D) = 1, \quad S(D) = D^{-1},
\]

(2.10)

where

\[
D^{-1} = da^{-1} + \beta a^\gamma a^\gamma.
\]

(2.11)

A factorization of the \( T \)-matrix (2.1) as

\[
T = \left( \begin{array}{cc} 1 & 0 \\ \zeta & 1 \end{array} \right) \left( \begin{array}{cc} a & 0 \\ 0 & d \end{array} \right) \left( \begin{array}{cc} 1 & \xi \\ 0 & 1 \end{array} \right)
\]

(2.12)

introduces new variables related to the old ones by

\[
\beta = a \xi, \quad \gamma = \zeta a, \quad d = \zeta a \xi + \hat{d},
\]

(2.13)

where \( \hat{d} \) is an invertible element. The quantum superdeterminant and its inverse now read
\[ D = ad^{-1}, \quad D^{-1} = \hat{a} \hat{d}^{-1}. \quad (2.14) \]

The algebra (2.2) assumes the form

\[
\begin{align*}
  a\xi &= p^{-1}a\xi, \quad a\zeta = q^{-1}\zeta a, \quad \dot{a}\xi = p^{-1}\xi \dot{a}, \quad \dot{a}\zeta = q^{-1}\zeta \dot{a}, \\
  \{\xi, \zeta\} &= 0, \quad [a, \ddot{d}] = 0, \quad \xi^2 = 0, \quad \zeta^2 = 0.
\end{align*}
\quad (2.15) \]

The coalgebra maps are rewritten as

\[
\begin{align*}
  \Delta(a) &= (a \otimes I)(I \otimes I + \xi \otimes \zeta)(I \otimes a), \\
  \Delta(\xi) &= I \otimes \xi + \xi \otimes D^{-1}, \quad \Delta(\zeta) = \zeta \otimes I + D^{-1} \otimes \zeta, \\
  \Delta(\dot{d}) &= (\dot{d} \otimes I)(I \otimes I + \xi \otimes \zeta)(I \otimes \dot{d}).
\end{align*}
\quad (2.16) \]

\[
\begin{align*}
  e(a) &= 1, \quad e(\xi) = 0, \quad e(\zeta) = 0, \quad e(\dot{d}) = 1.
\end{align*}
\quad (2.17) \]

\[
\begin{align*}
  S(a) &= a^{-1} + \xi a^{-1}D\zeta, \quad S(\xi) = -\xi D, \\
  S(\zeta) &= -D\zeta, \quad S(\dot{d}) = \dot{d}^{-1} + \xi \dot{d}^{-1}D\zeta.
\end{align*}
\quad (2.18) \]

A further map of the even generators

\[ a = e^x, \quad \dot{d} = e^{\dot{x}} \quad (2.19) \]

and a reparametrization

\[ p = e^{-\omega}, \quad q = e^{-\nu}, \quad (2.20) \]

give the algebra (2.15) a Lie superstructure
\[ [x, \xi] = \omega \xi, \quad [x, \zeta] = \nu \zeta, \quad [x, \xi] = \omega \xi, \quad [\dot{x}, \zeta] = \nu \zeta, \]
\[ [x, \dot{x}] = 0, \quad \{\xi, \zeta\} = 0, \quad \xi^2 = 0, \quad \zeta^2 = 0, \quad (2.21) \]

with the nontrivial coalgebra maps

\[ \Delta (x) = x \otimes 1 + 1 \otimes x + \Omega \xi \otimes \xi, \quad \Delta (\dot{x}) = \dot{x} \otimes 1 + 1 \otimes \dot{x} + \Omega \xi \otimes \xi, \]
\[ \Delta (\xi) = 1 \otimes \xi + \xi \otimes e^{\hat{x}-x}, \quad \Delta (\zeta) = \zeta \otimes 1 + e^{\hat{x}-x} \otimes \xi, \]
\[ \text{with} \quad \Omega = \frac{\nu + \omega}{e^\nu - e^{-\omega}}, \quad (2.22) \]
\[ e(x) = e(\dot{x}) = e(\xi) = e(\zeta) = 0, \quad (2.23) \]
\[ S(x) = -x + \Omega \xi e^{x-\hat{x}} \zeta, \quad S (\dot{x}) = -\dot{x} + \Omega \xi e^{\hat{x}-x}, \]
\[ S(\xi) = -\xi e^{x-\hat{x}}, \quad S(\zeta) = -e^{x-\hat{x}} \zeta. \quad (2.24) \]

It is now evident that \( \text{Fun}_{p,q}(GL(1|1)) \) may be embedded in the enveloping algebra of a Lie superalgebra with a noncocommutative coproduct structure. Next, we show that this enveloping algebra is dual to the Hopf superalgebra \( U_{p,q}(gl(1|1)) \).

3. \( \text{Fun}_{p,q}(GL(1|1)) - U_{p,q}(gl(1|1)) \) duality and the universal \( \mathcal{T} \)-matrix of \( \text{Fun}_{p,q}(GL(1|1)) \)

Two Hopf algebras \( \mathcal{A} \) and \( \mathcal{U} \) are in duality (see, e.g., [10] for details) if there exists a doubly nondegenerate bilinear form

\[ \langle \, , \, \rangle : (a, u) \to \langle a, u \rangle \quad \forall a \in \mathcal{A}, \quad \forall u \in \mathcal{U}, \quad (3.1) \]
such that, for \((a, b) \in \mathcal{A}, (u, v) \in \mathcal{U}\),

\[
\langle a, uv \rangle = \langle \Delta_{\mathcal{A}}(a), u \otimes v \rangle, \quad \langle ab, u \rangle = \langle a \otimes b, \Delta_{\mathcal{U}}(u) \rangle,
\]

(3.2)

\[
\langle a, 1_{\mathcal{U}} \rangle = \epsilon_{\mathcal{A}}(a), \quad \langle 1_{\mathcal{A}}, u \rangle = \epsilon_{\mathcal{U}}(u),
\]

(3.3)

\[
\langle a, S_{\mathcal{U}}(u) \rangle = \langle S_{\mathcal{A}}(a), u \rangle.
\]

(3.4)

Let \(\{\epsilon^A | \epsilon^A = \zeta^{a_1} x^{a_2} \hat{x}^{a_3} \xi^{a_4}, \ A = (a_1, a_2, a_3, a_4), (a_1, a_4) = (0, 1), (a_2, a_3) \in \mathbb{Z}_+\}\) be a basis of monomials for \(\text{Fun}_{\mathbb{Q}(\mathbb{Q})}(GL(1|1))\) obeying the multiplication and the induced coproduct rules

\[
\epsilon^A \epsilon^B = \sum_C f^{AB}_{C} \epsilon^C,
\]

(3.5)

\[
\Delta (\epsilon^A) = \sum_{BC} h^A_{BC} \epsilon^B \otimes \epsilon^C.
\]

(3.6)

The unit element is obtained by choosing \(A = \underline{0}\), where \(\underline{0} = (0, 0, 0, 0)\). The dual basis elements \(\{E_A\}\) are defined by

\[
\langle \epsilon^A, E_B \rangle = \delta^A_B, \quad \delta^A_B = \Pi_{i=1}^4 \delta^{a_i}_{b_i}.
\]

(3.7)

Then, we obtain the following multiplication and coproduct structures for the basis set \(\{E_A\}\):

\[
E_A E_B = \sum_C h^C_{AB} E_C,
\]

(3.8)

\[
\Delta (E_A) = \sum_{BC} f^{BC}_{A} E_B \otimes E_C.
\]

(3.9)

Using the algebra (2.2) the structure tensor \(f^{AB}_C\) is obtained:
\[ f^{AB}_{C} = (-1)^{a_1 b_1} \delta^{a_1 b_1} \delta^{a_1 b_1} \theta^{a_2 b_2 + b_2 \theta^{c_3 + b_3} \delta^{a_4 b_4}} \]
\[
\sum_{k} \left( \begin{array}{l} a_2 \\ k \end{array} \right) \left( \begin{array}{l} b_2 \\ c_2 - k \end{array} \right) \left( \begin{array}{l} a_3 \\ l \end{array} \right) \left( \begin{array}{l} b_3 \\ c_3 - l \end{array} \right) (\nu b_1)^{a_2 + a_3 - k - l},
\]
\[ (-\omega a_4)^{b_2 + b_3 - c_2 - c_3 + k + l}, \quad (3.10) \]

where \( \delta^{ab} = \delta^{a_0 b_0} + \delta^{a_1 b_1} + \delta^{a_2 b_2} \) and \( \theta^{b_0} = 1 \) if \( a \geq b \) \((< b)\). Some special cases relevant for later use are:

\[ f_{1000}^{AB} = \left( \delta^{a_0 a_1 b_1} + \delta^{a_1 a_0 b_1} \right) \delta^{b_2} \delta^{b_3} \delta^{c_4} \delta^{b_4} \left( \nu b_1 \right)^{a_2 + a_3}, \quad (3.11a) \]
\[ f_{0001}^{AB} = \delta^{a_0 a_1 b_2} \delta^{a_2 a_3} \delta^{b_3} \left( -\omega a_4 \right)^{b_2 + b_3}, \quad (3.11b) \]
\[ f_{0100}^{AB} = \delta^{a_0 a_1 a_4 b_4} \delta^{a_2} \delta^{b_3} + \delta^{a_0 a_1 a_4 b_4} \delta^{a_2} \delta^{b_3} \delta^{b_4} \delta^{b_4}, \quad (3.11c) \]
\[ f_{0010}^{AB} = \delta^{a_0 a_1 a_3 b_3} \delta^{a_4} \delta^{b_3} \delta^{b_4} + \delta^{a_0 a_1 a_4 b_4} \delta^{a_2} \delta^{b_3} \delta^{b_4} \delta^{b_4}. \quad (3.11d) \]

The tensor \( h^A_{BC} \) is determined by the induced coproduct rule for the basis set \( \{ e^A \} \)

\[ \Delta \left( e^A \right) = \Delta \left( \xi^A \right) \Delta \left( x^A \right) \Delta \left( \hat{x}^A \right) \Delta \left( \xi^A \right). \quad (3.12) \]

The following special cases, necessary for determining the dual algebraic structure, may be directly read from (3.12):

\[ h^A_{B0} = \delta^A_B, \quad h^A_{0B} = \delta^A_B, \quad (3.13a) \]
\[ h^A_{b_2 b_0 c_2 c_0} = \delta^{a_1 b_0} \delta^{a_2 b_2 + c_2} \delta^{a_3 c_3 + c_3} \delta^{a_4 b_4} \left( \begin{array}{l} a_2 \\ b_2 \end{array} \right), \quad (3.13b) \]
\[ h^A_{1000B} = \delta^A_{b_0} \Pi^A_{i=2} \delta^{a_i b_i}, \quad h^A_{B0001} = \left( \Pi^A_{i=1} \delta^{a_i b_i} \right) \delta^{a_1 b_1} \delta^{a_0 b_0}, \quad (3.13c) \]
\[ h^A_{01000100} = \delta^A_{b_1} \delta^{a_1 b_1} \delta^{a_2 c_2} \delta^{a_3 c_3} \delta^{a_4 c_4} - \delta^A_{b_1} \Pi^A_{i=2} \delta^{a_i}, \quad (3.13d) \]
\[ h^A_{00101000} = \delta^A_{b_2} \delta^{a_1 c_1} \delta^{a_2 c_2} \delta^{a_3 c_3} \delta^{a_4 c_4} + \delta^A_{b_2} \Pi^A_{i=2} \delta^{a_i}, \quad (3.13e) \]

\[ 8 \]
\[
\begin{align*}
\hat{h}_{00010100}^A &= \delta_0^{a_1} \delta_1^{a_2} \delta_0^{a_3} \delta_1^{a_4} - \left( \prod_{i=1}^3 \delta_0^{a_i} \right) \delta_1^{a_4} \hat{\epsilon}, \\
\hat{h}_{00010001}^A &= \delta_0^{a_1} \delta_1^{a_2} \delta_0^{a_3} \delta_1^{a_4} + \left( \prod_{i=1}^3 \delta_0^{a_i} \right) \delta_1^{a_4} \\
\hat{h}_{00011000}^A &= -\delta_1^{a_1} \delta_0^{a_2} \delta_0^{a_3} \delta_1^{a_4} + \delta_0^{a_1} \delta_0^{a_4} \sum_{n=1}^{\infty} \Omega_n \delta_n^{a_2+a_3},
\end{align*}
\]

where \( \Omega_n = \frac{\omega^n - (\omega^{-1})^n}{e^{\omega - e^{-\omega}}} \). The counit and the antipode maps for the basis elements \( \{ e^A \} \) are obtained from (2.23) and (2.24) respectively:

\[
\epsilon \left( e^A \right) = \delta^n_0 .
\]

\[
S \left( e^A \right) = (-1)^{a_1 a_4} S(\xi)^{a_4} S(\hat{x})^{a_3} S(x)^{a_2} S(\zeta)^{a_1} \\
= (-1)^{\sum_{i=1}^4 a_i} \zeta^{a_1} \left( \hat{x} + \nu a_1 - \omega a_4 + \Omega \zeta e^{x-\hat{x}} \right)^{a_3} \\
\epsilon^{(a_1+a_4)}(x-\hat{x}) \left( x + \nu a_1 - \omega a_4 + \Omega \zeta e^{x-\hat{x}} \right)^{a_2} \zeta^{a_4} .
\]

The second equality in (3.15) is obtained by using the commutation relations (2.21) and will be later used to compute the antipode maps for the dual basis elements.

Employing the duality property, we now extract the multiplication relations for the dual basis \( \{ E_A \} \). From (3.8) and (3.13a) we obtain the unit element:

\[
E_A E_\emptyset = E_A , \quad E_\emptyset E_A = E_A \quad \Rightarrow \quad E_\emptyset = \I .
\]

The generators of the dual algebra are chosen as

\[
E_- = E_{1000}, \quad H = E_{0100}, \quad \hat{H} = E_{0010}, \quad E_+ = E_{0001} .
\]

By repeated use of the relations (3.13b) and (3.13c) we express an arbitrary dual basis element as
\[ E_A = (a_2!a_3!)^{-1} E_{a_1}^a \tilde{H}_{a_2}^a \tilde{H}_{a_3}^a E_{a_4}^a, \]  

(3.18)

where \((a_1, a_4) = (0, 1)\) and \((a_2, a_3) \in \mathbb{Z}_+^2\). Further use of the special values of the structure tensor \(h_{BC}^A\) in (3.13) now yields the dual algebra \(U_{p,q}(gl(1|1))\):

\[
[H, E_\pm] = \pm E_\pm, \quad [\tilde{H}, E_\pm] = \mp E_\pm, \quad [H, \tilde{H}] = 0, \\
\{E_+, E_-\} = \frac{e^\nu (H + \tilde{H}) - e^{-\omega (H + \tilde{H})}}{e^\nu - e^{-\omega}}, \quad E_{\pm}^2 = 0. \tag{3.19}
\]

These relations, in turn, allow us to compute the general expression for the structure tensor \(h_{BC}^A\). From (3.8), (3.18) and (3.19), we get

\[
h_{BC}^A = \begin{aligned}
&(-1)^{b_2+c_2-a_2+c_1} b_1 \delta_{a_1+c_1} \delta_{b_1+c} \delta_{b_2+c_2+c_3} \delta_{b_3+c_4} a_2!a_3!(b_2!b_3!c_2!c_3!)^{-1} \sum_{kl} \left( \begin{array}{c}
b_2 \\
k
\end{array} \right) \left( \begin{array}{c}
c_2 \\
a_2 - k
\end{array} \right) \left( \begin{array}{c}
b_3 \\
l
\end{array} \right) \left( \begin{array}{c}
c_3 \\
a_3 - l
\end{array} \right) \\
&c_1^{b_1+b_2-k-l} b_4^{c_2+c_3-a_2-a_3+k+l} + \delta_{a_1} \delta_{b_1} \delta_{b_2} \delta_{b_3} \delta_{a_2} \delta_{a_3} a_2!a_3! (b_2!b_3!c_2!c_3! (a_2 - b_2 - c_2)! (a_3 - b_3 - c_3)!)^{-1} \\
&\sum_{a} \Omega_{a} \delta_{b_2+b_3+c_2+c_3+a}^{a_2+a_3}.
\end{aligned} \tag{3.20}
\]

The coproduct rules for the generators of the dual algebra \(U_{p,q}(gl(1|1))\) are obtained from (3.9) and (3.11):

\[
\Delta(H) = H \otimes \mathbb{L} + \mathbb{L} \otimes H, \quad \Delta(\tilde{H}) = \tilde{H} \otimes \mathbb{L} + \mathbb{L} \otimes \tilde{H}, \\
\Delta(E_+) = E_+ \otimes e^{-\omega (H + \tilde{H})} + \mathbb{L} \otimes E_+, \\
\Delta(E_-) = E_- \otimes \mathbb{L} + e^{\nu (H + \tilde{H})} \otimes E_. \tag{3.21}
\]

The counit maps for the dual generators are read from (3.3) and (3.14):

\[
\epsilon(X) = 0, \quad \forall X \in (H, \tilde{H}, E_\pm). \tag{3.22}
\]
To determine the antipode maps for the dual generators, we compute, using (3.7) and (3.15), the following elements of the bilinear form:

\[
\begin{align*}
\langle S(e^A), H \rangle &= -\delta^a_0 \delta^a_1 \delta^a_2 \delta^a_3,  \\
\langle S(e^A), \hat{H} \rangle &= -\delta^a_0 \delta^a_1 \delta^a_2 \delta^a_3,  \\
\langle S(e^A), E_+ \rangle &= -\delta^a_0 \delta^a_1 \omega^{a_2+a_3},  \\
\langle S(e^A), E_- \rangle &= -\delta^a_1 \delta^a_0 (-\nu)^{a_2+a_3}.
\end{align*}
\]

(3.23)

The duality relation (3.4) now immediately yields the antipode maps:

\[
\begin{align*}
S(H) &= -H,  \\
S(\hat{H}) &= -\hat{H},  \\
S(E_+) &= -e^{u(H+\hat{H})}E_+,  \\
S(E_-) &= -E_- e^{-\nu(H+\hat{H})}.
\end{align*}
\]

(3.24)

A map of the dual generators

\[
Z = \frac{1}{2} (H + \hat{H}),  \\
J = \frac{1}{2} (H - \hat{H}),  \\
\chi_\pm = E_\pm Q^Z \chi^{-Z+\frac{1}{2}},
\]

(3.25)

with \( Q = \sqrt{p/q} \) and \( \lambda = \sqrt{p/q} \), now reexpresses the Hopf structure of \( U_{p,q}(gl(1|1)) \) ((3.19),(3.21),(3.22) and (3.24)) in the standard form:

\[
\begin{align*}
[J, \chi_\pm] &= \pm \chi_\pm,  \\
\{\chi_+, \chi_-\} &= \frac{Q^2 Z - Q^{-2} Z}{Q - Q^{-1}},  \\
\chi_+^2 &= 0,  \\
[Z, X] &= 0,  \\
\forall X \in (J, \chi_\pm),
\end{align*}
\]

(3.26)

\[
\begin{align*}
\Delta(Z) &= Z \otimes I + I \otimes Z,  \\
\Delta(J) &= J \otimes I + I \otimes J,  \\
\Delta(\chi_{\pm}) &= \chi_{\pm} \otimes Q^Z \chi_{\pm}^Z + Q^{-Z} \chi_{\mp}^Z \otimes \chi_{\pm},
\end{align*}
\]

(3.27)

\[
\epsilon(X) = 0,  \\
S(X) = -X,  \\
\forall X \in (Z, J, \chi_\pm).
\]

(3.28)
Finally, following the prescription (1.1) for the universal $T$-matrix, we obtain the universal $T$-matrix of $Fun_{p,q}(GL(1|1))$ explicitly as

\[ T = \sum_A e^A \otimes E_A \]

\[ = \sum_{a_1, a_4 = 0}^1 (-1)^{a_1 a_4} \left( \zeta^{a_1} \otimes E_{-}^{a_4} \right) e^{x \otimes H} e^{\hat{x} \otimes \hat{H}} \left( \zeta^{a_1} \otimes E_{+}^{a_4} \right). \]  \hspace{1cm} (3.29)

It is seen that corresponding to the two-dimensional irreducible representation of the generators of $U_{p,q}(gl(1|1))$ given by

\[ E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{H} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \]

the above universal $T$-matrix (3.29) reduces, as required, to the $T$-matrix (2.1), read with (2.13) and (2.19):

\[ T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \begin{pmatrix} e^x & e^x \xi \\ \zeta e^x & e^{\hat{x}} + \zeta e^x \xi \end{pmatrix}. \]  \hspace{1cm} (3.31)

4. Conclusion

To conclude, let us summarize. Using the method developed by Fronsdal and Galindo [1] for analysing the duality between the Hopf algebras $Fun_{p,q}(GL(2))$ and $U_{p,q}(gl(2))$, we have extracted the Hopf structure of the quantum superalgebra $U_{p,q}(gl(1|1))$ [7] from its duality relationship with $Fun_{p,q}(GL(1|1))$ [6,7] obtained by a two-parametric ($p, q$) quantization of the algebra of functions on the supergroup $GL(1|1)$. The universal $T$-matrix of $Fun_{p,q}(GL(1|1))$, identified with the corresponding dual form, is seen to exhibit the suitably modified exponential relationship $U_{p,q}(gl(1|1)) \rightarrow Fun_{p,q}(GL(1|1))$. 

12
Acknowledgements

One of us (R.J) thanks Prof. Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, where part of this work was done while visiting the High Energy Section during August 1994; in this regard, thanks are also due, in particular, to Prof. S. Randjabar-Daemi and Prof. K.S. Narain. He wishes also to thank Prof. G. Baskaran for kind encouragement. We dedicate this paper to the memory of our beloved Prof. R. Vasudevan who was to us much more than a senior colleague.
References


