A modified discrete sine-Gordon Model

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ABSTRACT

We modify the recently proposed model of Speight and Ward to make it possess time dependent solutions. We find that for each lattice spacing and for each velocity of the sine Gordon kink we can find a modification of the model for which this kink is a solution. We find that this model has really 3 “kink-like” solutions; the original kink, the static kink and a further kink moving with velocity \( v \sim 0.97 \). We discuss various properties of the model, from the point of view of its usefulness for numerical simulations.

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1. Introduction

Recently Speight and Ward, in a very interesting paper,\textsuperscript{[4]} presented a discrete version of the Sine-Gordon model which had topologically stable kink-like solutions. Their idea was to introduce a discrete version of the Bogomolnyi bound\textsuperscript{[3]} and then use it for deriving a bound on the energy of a kink in the model. The field configuration which saturates this bound has the lowest energy and is topologically stable.

Their work starts from the observation that in the continuum

\[
\frac{1}{4} \int_{-\infty}^{\infty} (\psi_x^2 + \sin^2 \psi) \, dx = \frac{1}{4} \int_{-\infty}^{\infty} (\psi_x - \sin \psi)^2 \, dx + \frac{1}{2} \int_{-\infty}^{\infty} (\psi_x \sin \psi) \, dx 
\]  

(1.1)

where \( \psi_x = \frac{\partial \psi}{\partial x} \), \( \psi = \frac{1}{2} \phi \) and where \( \phi \) is the familiar field of the sine-Gordon model\textsuperscript{[3]} i.e. the model whose time-dependent equation of motion is

\[
\phi_{tt} - \phi_{xx} = \sin \phi. 
\]  

(1.2)

Hence as

\[
\int_{-\infty}^{\infty} (\psi_x \sin \psi) \, dx = \int_{-\infty}^{\infty} -(\cos \psi)_{x} \, dx = \cos (\psi(-\infty)) - \cos (\psi(+\infty)) 
\]  

(1.3)

we see that in the class of field configurations which have “kink-like” boundary conditions, \( i.e., \) which satisfy \( \psi(-\infty) = 0, \psi(+\infty) = \pi \) the lowest energy field must solve

\[
\psi_x = \sin \psi. 
\]  

(1.4)

When the continuum is replaced by a spatial lattice we have to decide how to replace the derivatives \( \psi_x \) \( \text{etc.} \); the most naive idea would correspond to using
the forward differences \( i.e. \) putting \( \psi_x = \frac{\psi(x+h) - \psi(x)}{h} \), where \( h \) is the lattice spacing; however, this choice is very nonunique and it is this nonuniqueness that was exploited by Speight and Ward. Namely, they observed that the key line in the argument given above corresponds to

\[
\frac{\partial (\cos \psi)}{\partial x} = -\sin \psi \frac{\partial \psi}{\partial x}.
\] (1.5)

They noted that if the model is discretised in such a way that the factorisation implied by (1.5) holds and \( \frac{\partial \cos \psi}{\partial x} \) is replaced by the finite difference of \( \cos \psi \) the discretised model will possess the topologically stable kinks. As

\[
\cos(\psi(y)) - \cos(\psi(x)) = -2\sin \left( \frac{\psi(y) - \psi(x)}{2} \right) \sin \left( \frac{\psi(y) + \psi(x)}{2} \right),
\] (1.6)

this has led them to replace

\[
sin(\psi(x)) \to \sin \left( \frac{\psi(x + h) + \psi(x)}{2} \right)
\] (1.7)

and

\[
\frac{\partial \psi}{\partial x} \to \frac{2}{h} \sin \left( \frac{\psi(x + h) - \psi(x)}{2} \right).
\] (1.8)

Of course, both terms have the correct \( h \to 0 \) limit.

With this replacement the model possesses stable kink like solutions. They are given by the solutions of the discrete analogue of (1.4); namely,

\[
\sin \left( \frac{\psi(x + h) + \psi(x)}{2} \right) = \frac{2}{h} \sin \left( \frac{\psi(x + h) - \psi(x)}{2} \right).
\] (1.9)

In fact, Speight and Ward have managed to find the analytical form of this discrete kink-like solution which allowed them to discuss its various properties. In particular, they pointed out that as the solution possesses a translational zero mode (despite the lack of translational symmetry); this mode can then be used to study the dynamics of the kinks in the collective coordinate approximation.
To do this they had to introduce the time dependence into the model; for that they added the usual term \( i.e. (\frac{\partial \psi}{\partial t})^2 \) to the Lagrangian density and then calculated the corresponding equations of motion. In their case this equation took the form of

\[
\psi_H = \frac{1}{h^2} \left[ \sin(\psi_+ - \psi) - \sin(\psi - \psi_-) \right] - \frac{1}{4} \left[ \sin(\psi_+ + \psi) + \sin(\psi + \psi_-) \right],
\]

where \( \psi = \psi(x), \psi_\pm = \psi(x \pm h) \).

Speight and Ward studied the dynamics of their kinks using this equation. The time evolution was simulated by the fourth order Kutta-Runge method. Moreover, the results were compared with the results obtained in the collective variable approximation.

First of all, the results showed that the kink in the model behaved very much like the kink in the continuum; the Peierls Nabarro barrier of the usual discrete models \(^{11}\) was eliminated by the novel discretisation and hence the model was a much better approximation of the model in the continuum. Moreover, the results showed that the collective approximation worked very well for small velocities and for small lattice sizes \( i.e. \) for small \( h \). For larger velocities the kink radiated and slowed down. The results were more pronounced for larger values of \( h \); in fact there was a natural lattice-size “cut-off” \( (h = 2) \) above which the whole model was unstable. Although the “cut-off” value of \( h \) was ridiculously high \( (and \ so \ irrelevant \ from \ the \ point \ of \ view \ of \ any \ “physical” \ applications) \) its existence can be traced to Speight and Ward’s choice of their discretisation (1.7) and (1.8). However, their choice was one of many \( (albeit \ perhaps \ the \ most \ “natural” \ one) \); other choices change this value and can even eliminate it altogether.

In this paper we discuss these choices and point out that the kinetic term can also be altered. We then present a model which not only possesses static kinks but also moving kinks of the continuum sine-Gordon type moving at one velocity which is related to the choice of the parameters of the model. We discuss their dynamics and find that the model preserves all the good points of the original model of
Speight and Ward, has considerably reduced radiation effects and so could be used for the numerical study of the dynamics of kinks in the continuum had we not been in the lucky situation of having analytical solutions or for other numerical investigations.

2. Modified Models

2.1 Static case

The choice of Speight and Ward (1.7) and (1.8) is clearly very nonunique. The idea of using (1.6) to factorise it into the right hand terms of (1.7) and (1.8) can clearly be modified; e.g. by the inclusion an arbitrary factor $f(x, h)$ in one term and its inverse in the other; i.e. we can put

$$\sin(\psi(x)) \to f \sin\left(\frac{\psi(x + h) + \psi(x)}{2}\right)$$

(2.1)

and

$$\frac{\partial \psi}{\partial x} \to \frac{1}{f} \frac{2}{h} \sin\left(\frac{\psi(x + h) - \psi(x)}{2}\right).$$

(2.2)

What can $f(x, h)$ be? Clearly, we need $\lim_{h \to 0} f = 1$ but otherwise we have complete freedom. On the other hand, any explicit $x$ dependence in $f$ would be somewhat undesirable since it would break the translational symmetry of the lattice.

The Speight and Ward choice corresponds to $f = 1$ but we not have to be so drastic. Of course, any choice of $f$ will have implications for the static kink solution as the Bogomolnyi bound equation now becomes

$$\sin\left(\frac{\psi(x + h) + \psi(x)}{2}\right) f^2 = \frac{2}{h} \sin\left(\frac{\psi(x + h) - \psi(x)}{2}\right).$$

(2.3)

If we now take

$$f^2 = \frac{h(1 + e^h)}{2(e^h - 1)}$$

(2.4)
then, as is easy to check, (2.3) has
\[
\psi(x) = 2 \arctan\{\exp(x - a)\},
\]
\hspace{1cm} (2.5)
i.e. the usual kink, as its solution. Here, as in the continuum, \(a\) is arbitrary and can be thought of the “position” of the kink (which, in general, falls between the lattice sites). Hence, like in the case of Speight and Ward, the kink solution possesses a continuous translational zero mode.

Of course (2.4) has a correct \(h \to 0\) limit, is monotonic and for large \(hf \approx \sqrt{\frac{h}{2}}\). Hence, as we compare the models with different lattice spacing \(h\) we see that the effect of \(f(h)\) is to decrease the importance of the \(\sin(\psi)^2\) term in the energy density at large \(h\). Of course, the total energy of (2.5) is still the same as in the continuum, i.e. 1 in our units. The kink is stable, and any other field configuration (with the same boundary conditions) has larger energy.

What about the dynamics of the kink? Will it propagate and will it radiate? The work of Speight and Ward has partially answered this question. Using (1.10) they studied the dynamics of their kink and found that it radiated (and the amount of radiation depended on the velocity and on the lattice spacing). However, the choice of the time dependent term in (1.10), although “natural” is again very nonunique; one can consider taking other terms except that in the limit \(h \to 0\) we should recover \(\psi_H\). Any such modification will have implication on the kinetic term in the Lagrangian density and so also on the total energy of a given field configuration.

So what can we choose? Clearly, we should not change the degree of the equation of motion; so all modifications should involve only the additional dependence on \(\psi_t\), \(\psi\) and \(h\). As there is very little guidance as to what to take we have looked first at the case of Speight and Ward and then considered modifications involving the multiplications of \(\psi_H\) by a function \(g(\psi_t, \psi, h)\). However, the simplest modifications would involve \(g\) independent of \(\psi\), as then we do not modify too dramatically
the Lagrangian density - so, in this paper we restrict ourselves to considering only \( g = g(\psi_t, h) \).

### 2.2 Time Dependent Case

The simplest modification, which is not too drastic would involve the replacement

\[
\psi_H \rightarrow \frac{\psi_H}{(1 + \alpha \psi_t^2)},
\]

where \( \alpha \) is a function of \( h \), such that \( \alpha(0) = 0 \). Such a modification leads to the Lagrangian density which consists of two terms (the kinetic energy term (which involves only \( \frac{\partial \psi(x,t)}{\partial t} \)) and the potential term which involves \( \psi(x, t), \psi(x + h, t) \) and \( \psi(x - h, t) \).

Hence, the equation of motion which we want to study, and which is a generalisation of (1.10) is

\[
\frac{\psi_H}{(1 + \alpha \psi_t^2)} = \frac{f^2}{h^2} \left[ \sin(\psi_+ - \psi) - \sin(\psi - \psi_-) \right] - \frac{1}{4f^2} \left[ \sin(\psi_+ + \psi) + \sin(\psi + \psi_-) \right],
\]

where \( \alpha \) and \( f \) are functions of \( h \), such that \( \alpha(0) = 0 \) and \( f(0) = 1 \) and, as before, \( \psi_\pm = \psi(x \pm h) \).

As it is easy to check (2.7) is the equation of motion which can be derived from the following form of the Lagrangian \( L = L_k - L_p \), where

\[
L_k = \hbar \sum_i \left\{ \frac{\psi_H(i)}{2\sqrt{\alpha}} \arctan(\sqrt{\alpha} \psi_H(i)) - \frac{1}{4\alpha} \ln(1 + \alpha \psi_H(i)^2) \right\}
\]

and

\[
L_p = \hbar \sum_i \left\{ \frac{f^2}{h^2} \sin^2 \left( \frac{\psi(i + 1) - \psi(i)}{2} \right) + \frac{1}{4f^2} \sin^2 \left( \frac{\psi(i + 1) + \psi(i)}{2} \right) \right\},
\]

where we have now put \( \psi(i) = \psi(x, t), \psi(i \pm 1) = \psi(x \pm h, t) \) and \( \psi_H(i) = \frac{\partial \psi(x,t)}{\partial t} \).
Our expression for $L_k$ has a very unusual look but, as can be checked, is definitive positive. As $\arctan(x) \sim x - \frac{x^3}{3} + ..$ and $\ln(1 + x) \sim x - \frac{1}{2}x^2 + ..$ for small $x$ we see that for small values of $\sqrt{\alpha} \psi_t$

$$\frac{\psi_t(i)}{2\sqrt{\alpha}} \arctan(\sqrt{\alpha} \psi_t(i)) - \frac{1}{4\alpha} \ln(1 + \alpha \psi_t^2(i)) \sim \psi_t^2(i) - \frac{1}{2} \psi_t^2(i) = \frac{1}{2} \psi_t^2(i) \quad (2.10)$$

and so we recover the conventional behaviour. The differences do show up as $\sqrt{\alpha} \psi_t$ increases. In fig. 1, we present plots of the two expressions of $L_k$ (i.e. the summand in (2.8), and the right hand side of (2.10)) as a function of $\psi_t$ (for $\alpha = 3.7768$). We see that for $\psi_t$ up to about 0.2 the two expressions agree and for larger values of $\psi_t$ (2.8) grows with $\psi_t$ less fast than (2.10). In fact it is easy to see that for larger values of $\psi_t$ the first term in (2.8) dominates more and more and so, for large values of $\psi_t$ the dependence on $\psi_t$ becomes essentially linear. However, as we will discuss in the next section, in practical applications $\psi(t)$ never gets very large and so our values of $L_k$ will not be that different from their usual values.

In the discussion above we have not identified $L_k$ with the kinetic energy of the system. One may be tempted to do so, but this would be incorrect as $L_k$ is not quadratic in $\psi_t$. To find the correct expression we need to calculate the Hamiltonian and then take its part which involves $\psi_t$. Then we find that the kinetic energy $E_{kin}$ is given by

$$E_{kin} = \hbar \sum_i \left( \psi_t(i) \frac{\partial L_k}{\partial \psi_t(i)} - L_k \right) = \hbar \sum_i \left( \frac{1}{4\alpha} \ln(1 + \alpha \psi_t^2(i)^2) \right), \quad (2.11)$$

i.e. by only the log term in (2.8) taken with the opposite sign. Hence in fig.1 we also plot the log term. We see that its growth with $\psi_t$ is even slower than the other two terms.
3. A Propagating Kink

Can we find any solution of (2.7)? Of course this depends on our choices of the functions $f(h)$ and $\alpha(h)$.

However, let us observe that the usual (i.e. continuum) expression for the Sine-Gordon kink

$$\psi(x,t) = 2\arctan \left\{ \exp \left( \frac{x - vt - x_0}{\sqrt{1 - v^2}} \right) \right\}$$  \hspace{1cm} (3.1)

has a chance of being a solution as the time derivatives of (3.1) and the terms on the right hand side of (2.7) tend to give relatively similar expressions. In fact, a couple of pages of algebra shows that (3.1) is a solution of (2.7) if $\alpha$ is chosen to be given by

$$\alpha(h) = \frac{(1 - v^2)}{v^2} \sinh^2(2\beta)$$  \hspace{1cm} (3.2)

and $f(h)$ is given by

$$f^2 = \frac{v^2}{1 - v^2} + \sqrt{\left( \frac{v^2}{1 - v^2} \right)^2 + \frac{4\sinh^2(2\beta)}{h^2}} \frac{h^2}{8\sinh^2\beta},$$  \hspace{1cm} (3.3)

where $\beta = \frac{h}{2\sqrt{1 - v^2}}$. Thus we see that we can have a moving kink but its velocity is fixed by the model; namely is determined by $\alpha$ and $f$. Note that our expressions for $\alpha$ and $f$ have the correct $h \to 0$ limit ($\alpha \to 0$ and $f \to 1$). On the other hand if we keep $h$ fixed and take $v \to 0$

$$f^2 \to \frac{h}{2} \coth(h),$$  \hspace{1cm} (3.4)

i.e. we obtain the expression we have mentioned before (2.4). At the same time, however, $\alpha \to \infty$; so we cannot use (3.3) for $v$ too small. In practice, this does not matter as we will argue below, and we will use $v = 0.7$. 
What is the behaviour of $f^2$ as we vary $h$ and $v$? It is easy to check that for small $h$ and for small $v$ $f^2 \sim 1$. If we keep $v$ not too large (say below (0.8)) $f^2$ grows with $h$ but this growth is very small until $h \sim 2$. For larger values of $v$ (say above 0.85) $f^2$ is close to 1 for small $h$ then decreases and reaches a minimum around $h = 1.5$, and then becomes very similar to $f^2$ for smaller $v$. This is presented in fig. 2 where we have plotted the dependence of $f^2$ on $v$ for some selected values of $h$. Hence, for almost any value of $h$ the dependence on $v$, as long as $v < 0.9$, is very small and so we may expect that motions with different values of velocity will not be qualitatively very different. We discuss this in the next section.

However, before we do that, let us observe that our model, in addition to having moving a kink-like solution, possesses also a static solution which satisfies the Bogomolnyi bound.

Static solutions are solutions of (2.3) for $f^2$ given by (3.3). They can be easily found by observing that (2.3) is equivalent to

$$\tan\left[\frac{\psi(x + h)}{2}\right] = \tan\left[\frac{\psi(x)}{2}\right] \frac{c + 1}{c - 1},$$

(3.5)

where $c = \frac{2f^2}{h}$. Hence putting $\psi(x) = 2\arctan\{u(x)\}$ we see that (3.5) reduces to

$$u(x + h) = u(x) \frac{c + 1}{c - 1},$$

(3.6)

which has as its solution $u(x) = \alpha \exp\{\beta(x - x_0)\}$ where $\beta$ is given by

$$\beta = \frac{1}{h} \ln\left\{\frac{1 + c}{c - 1}\right\},$$

(3.7)

and where $x_0$ is arbitrary. Hence, the static solution is again in the form of a moving kink with its profile determined by $\beta$. To see the values of $\beta$ (3.7) gives let us put $h = 1$ and $v = 0.7$ in (3.3) (as these are the values we will use in the next sections). Then we find $\beta = 1.02257$, which in turn corresponds to $v = 0.208938$. Hence a static kink, in our model with $h = 1$ and $v = 0.7$ in (3.3), is given by a kink
whose profile has been deformed as if it were moving with velocity $v = 0.208938$. It is interesting that this velocity is so low. It is easy to check that the potential energy of such a static kink is indeed 1 (in fact, numerically, we find 1.0000000004).

4. Numerical Simulations

Before we present and discuss our results let us point out that we have really two “obvious” modifications of the original model of Speight and Ward. The first one, for the static case, is the one in which we can take $f^2 = \frac{h}{2} \coth(h)$ and then the time dependent one with $f^2$ given by (3.3). Of course, we could also consider fixing $f^2$ at its value given by (3.3) and then study kinks whose profile is given by $\beta$ close to $\beta \sim 1.02257$ (i.e. corresponding to the static kink mentioned above) and which move at some other (small) velocity. However, we have not studied such motions in much detail as we expect them to be not very dissimilar from the case when $f^2 = \frac{h}{2} \coth(h)$.

Hence, restricting our attention first to the case when $f^2 = \frac{h}{2} \coth(h)$ we see that this modification can be used for the study of static or slowly moving kinks in the cases when the lattice spacing $h$ has to be large. In the static case we have the topological bound, i.e. the potential energy is always bounded from below by 1 and equal to 1 for the static field which solves the Bogomolnyi bound (2.3). When we consider the slowly moving kinks, and introduce the motion by a nonrelativistic boost the potential still satisfies the Bogomolnyi bound but now the field starts radiating readjusting itself to its new shape. The readjustment is related to the velocity of the kink; it grows with the velocity. This is the radiation observed by Speight and Ward.

On the other hand we can start the kink with a modified profile (by a Lorentz factor); then the kink needs less readjustment and so it radiates less. However, as soon as we start considering moving kinks it makes more sense to use $f^2$ given by (3.3). In this case we can perform the simulation with the initial velocity of the kink being close to the value of the velocity used in the definition of $f^2$ or
being quite different. We would expect our results to depend on the value of this velocity; the outgoing radiation is expected to increase with the mismatch of velocities. Moreover, as soon as we use \( f^2 \) with \( v \neq 0 \) the field configuration does not saturate (2.3) and so the potential energy is different from 1.

Also, even when we take the same value of the velocity as in the definition of \( f^2 \) we expect some radiation; this is due to the fact that our kink solves (2.7) in which time is continuous while the numerical simulation (fourth order Runge-Kutta) uses a finite (albeit very small) \( dt \).

Next we present some results of our simulations. Most of the simulations refer to the case when \( f^2 \) was given by (3.3) with the value of \( v \) taken as \( v = v_1 = 0.7 \). Later we looked at other values too but as there was very little difference all our results in this paper refer to this value of \( v_1 \).

We have performed many numerical simulations (on grids of different sizes) varying both \( h, v \) (of the actual kink) and using both expressions for the kinetic energy (conventional and (2.11).) In all simulations involving one kink we used (3.1) to calculate from it \( \psi(0) \) and \( \psi_t(0) \) and then used these quantities as our initial value data.

Of course, for small values of \( h \) all the results of our simulations were essentially indistinguishable from each other (for the same value of \( v \)); hence let us concentrate our discussion on the case when \( h \) was sizeable, say \( h = 1 \).

First we look at the motion of the kink for which \( v = v_1 \). In fig. 3a we present plots of the field at \( t=0, \ 400, \ 800, \ 1200, \ 1600 \) and 2000. To put all the fields into one picture the fields at \( t \neq 0 \) have been displaced by subtracting \( 0.2 * \frac{t}{\text{min}} \) from their values. Our results were obtained on the grid stretching from -2000 to +2000; the time step of our simulation was 0.001. In fig. 3b we present the plot of our field at \( t = 1200 \) for \( 800 < x < 880 \) as it is only in this region that the field is substantially different from its asymptotic values (0, and \( \pi \)). Note that our kink is described by essentially 6-10 lattice points, the others are very little different from their asymptotic values. We see essentially no radiation.
However, although we see virtually no radiation, strictly speaking, the radiation is nonzero (due to the $t$ discretisation); this is exhibited in fig. 3c where we plot the field at $t = 800$ for $-600 < x < -100$. Note the vertical scale which demonstrates the smallness of the radiation.

We also looked at the velocity with which the kink was propagating along the lattice. Of course, as the kink is described by a few points on the lattice it is difficult to decide what is meant by its velocity and how to calculate it. We have used two methods to calculate it. The first method involved finding the best approximate position of the kink $x(t)$ at each value of $t$ (by a linear interpolation between the largest value of $x$ for which the field was still negative and the smallest $x$ for which it was positive); the velocity was then approximated by the rate of change of this position function $x(t)$. As the extrapolation is not very exact the resultant velocity has a tendency to oscillate a little and so, to be more precise, we should average the obtained results.

The other method involved calculating the “average relevant” $X(t)$ by defining

$$X(t) = \frac{\sum_i x(i) \, eng(i, t)}{\sum_i eng(i, t)}$$

(4.1)

where $eng(i, t)$ was the total energy of the lattice point $i$. Then we could use the rate of change of this $X(t)$ to calculate the velocity. This approach does not involve any extrapolation and is more “natural”, but it involves not only the kink but the whole field so it takes into account also radiation effects. Thus as long as only few lattice points are involved in the dynamics the two expressions give the same value for the velocity and the approximation based on $X(t)$ is smoother; when the radiation effects become appreciable $X(t)$ is pulled down by the radiation waves moving to the left and the resultant velocity does not measure the velocity of the kink but a kind of “overall” velocity of the field.

In fig. 4 we present the time dependence of the velocity calculated by the second method (the result of the first method is essentially indistinguishable except
for some initial oscillations). We see that, as expected, the kink propagates with \( v = 0.7 \) with no slowing down.

Next we looked at the kinks started of with different values of \( v \); i.e. we used (3.1) with different values of \( v \) to calculate our initial data. The results can be summarised as follows; for most velocities below \( v = v_1 \) there is very little radiation; the kink propagates with a velocity a little lower than \( v \) and radiating slows down. The rate of the slowing down is very low. For \( v > v_1 \) the kink starts propagating with velocity larger than \( v \), also radiates and slows down. This time the radiation effects are more pronounced but still very small; hence even now the slowing down is not very pronounced. Thus we see that we have a model in which kinks radiate little. We may wonder why the initial velocity of the kink is not \( v \) (except for \( v = v_1 \)). This is due to the readjustment of the kink; the initial configuration does not fit with the lattice distribution and so the kink readjusts itself and travels with velocity which corresponds to the speed of this readjusted field. Of course we do not expect these effects to be too large and this is what we have observed in our simulations.

In fig. 5 we present plots of the time dependence of the velocity of the kinks initiated with different values of \( v \). We see that all kinks have radiated very little, in agreement with our expectations. This should be contrasted with the results obtained by Speight and Ward. They only studied kinks moving at small velocities, but even then their kinks radiated more. The results in fig. 5 present the results for \( t \) up to 200; one may wonder what happens when the kinks propagate longer. Speight and Ward, in their paper, present the time dependence of the velocity of a kink initially moving at \( v = 0.28 \). Their plot shows relatively rapid slowing down which later decreases slightly. To see what happens in our case we rerun some of our simulations for much longer times. To avoid any boundary condition problems we extended the lattice sufficiently far; this has extended the time needed for the completion of each simulation. In fig. 6a we present the plots of the fields at \( t = 0, 800, 1600, 2400, 3200 \) and \( 4000 \) of the kink started off with \( v = 0.25 \). As before they have been successively displaced down by 0.2. In fig. 6b we present the plot of
the time dependence of the kink’s velocity seen in this simulation. The plot shows
two curves corresponding to our two ways of evaluating the velocity of the kink. As
expected, there is very little difference between the two curves; the slowing down
is not very substantial over the whole period of our simulation. Of course, there is
some radiation and so the curve of the time dependence of the velocity evaluated
with $X(t)$ lies lower. However, the two curves differ very little; the fact that there
is little radiation is confirmed by looking at fig. 6c where we exhibit the details of
the field at $t = 4000$ for $900 < x < 970$.

Our results should be contrasted with the results obtained using our lattice
and the standard expression for the kinetic energy and the model of Speight and
Ward. In fig. 6d we present the time dependence of the velocity of the kink as seen
in all these types of simulations. We find that both other schemes lead to much
more radiation and so in both of them the kink slows down much more.

Looking at the velocities shown in fig. 5 we see that the simulation started
at $v = 0.8$ involved more radiation; hence we looked at it in detail. In fig. 7 we
present the fields at 4 values of time ($t = 0, 800, 1600$ and $2400$) for this case
(again the fields are displaced vertically). We see that this time the radiation is
more visible.

For the simulations with $v$ around 0.9 we find, once again, very little slowing
down of the kink, hence in fig. 8a we present the fields at some values of time for
the simulation with $v = 0.9$. In fig. 8b we present the plot of the time dependence
of the kink. We see that the radiation effects are very small and that the kink
propagates with velocity $v \approx 0.978$.

For larger values of $v$ (say $v = 0.95$ or larger) we again have bigger radiation
effects - fig. 9a shows the fields at 3 values of time for $v = 0.95$. For $v = 0.97$,
given the coarseness of the effective lattice (so that initially effectively only 3
lattice points are involved), the radiation effects are so large that the kink becomes
unstable and it develops a jump of $\pi$ at $x \approx 0$. Given the coarseness of the lattice
this is easy to achieve but the upshot of this is that, unexpectedly, the kink begins
to move in the opposite direction. In fig. 9b we present the fields at 3 values of time
(0, 80 and 160). One can calculate the velocity of the kink and find that it is about
-0.3. Hence we see that for large velocities and for coarse lattice cases the system
becomes unstable; not surprisingly, as initially only few lattice points are involved,
they can move up or down by $\pi$ and so the dynamics of the kink could become
very different from what may be expected from its continuum configuration.

5. Other field configurations

Given our lattice we can check whether it allows the propagation of other
kink-like configurations. In particular, had we not known the analytic form of the
two-kink and other field configurations we could have used it for such numerical
studies. To answer this question, at least partially, we decided to look at the field
configurations describing two moving kinks and a breather.

5.1 TWO KINKS

First we looked at the configuration involving two kinks. As is well known$^4$ the
analytic form of such a solution is given by

$$\psi(x,t) = 2 \arctan \left\{ \frac{\sqrt{1-v^2} x}{\sqrt{1-v^2}} \right\},$$

(5.1)

where $\pm v$ is the velocity of each kink relative to $x = 0$.

We have looked at the time evolution of such field configurations; as before, we
used (5.1) to calculate the initial condition $\psi(0)$ and $\psi_t(0)$ which were then used in
our simulations. In fig. 10 we plot the field configurations at some chosen values
of time obtained in a simulation with $v = 0.3$. We note some radiation, but again
not very much of it; in fig. 11 we plot the velocity of the kinks which confirms that
the kinks do not slow down too much. We have performed simulations with kinks
started off at other velocities ($v=0.1$, and 0.9); the resultant simulations were very
similar in each case; in each simulation the kinks first accelerated to reach their maximum speed (which was always a little lower than \( v \)), then slowed down a little, although this slowing down was hardly perceptible. The two effects, the slowing down and not reaching \( v \) were partly due to the radiation.

5.2 Breather

We have also looked at the behaviour of the breather in our model. The initial condition was taken from the analytic form of the breather, namely:

\[
\psi(x, t) = 2 \arctan \left[ \frac{\sqrt{1 - \lambda^2}}{\lambda} \sin \left\{ \lambda (t - t_0) \right\} \right],
\]

where \( \lambda \) describes the frequency of the breather. \( t_0 \) was chosen to be \( \frac{\pi}{2 \lambda} \) so that at \( t = 0 \) the field was temporarily at rest. Several simulations were performed with different values of \( \lambda \). This time, by comparison with the case of one and two kinks, the radiation effects were more important but still not that important. The amount of the radiation depended on the period of the breather, which is determined by \( \lambda \). For small values of \( \lambda \) (say \( \lambda = 0.25 \)) the breather radiated with the amplitude of the field at the origin gradually decreasing. However, this decrease was very pronounced only at the very beginning of the simulation but later became hardly noticeable. In fig. 12a we plot the time dependence of the value of the field at \( x = 0 \) as seen in our simulation. In fig. 12b we present our fields at 6 values of time (0, 400, 800, 1200, 1600 and 2000) successively displaced in the vertical direction down by 2 units. We see that the breather radiates, and this radiation then propagates out. Gradually the breather radiates less and less and this is consistent with the slow decrease of its amplitude. In fig. 12c we replot 12b by restricting \( x \) to \(-50 < x < 50\). This shows the decrease of radiation in the immediate neighbourhood of the breather itself.

In fig. 13 we present similar results for \( \lambda = 0.9 \). This time the radiation effects are much smaller; the amplitude hardly decreases and we see very little radiation.
in the fields themselves (this time displaced up). In fig. 13c, this time, we show the details of the field at \( t = 2000 \) for this case from which we see that, although weakly, the breather radiates all the time.

Similar results were seen in other simulations.

6. Some general comments and conclusions

The model we have used in our study is based on the model of Speight and Ward. It retains many nice features of their model and, when used in simulations on very coarse lattices, it leads to considerably reduced radiation effects.

In fact, studying various properties of the model in a one kink sector we see that the model has really three stable "kink-like" solutions; a moving kink (in our case moving at \( v = 0.7 \)), a static kink (the usual kink whose profile has been modified so that it looks as a kink moving with \( v \sim 0.209 \)) and a kink moving with \( v \sim 0.97 \).

Thus, as such, the model may be used as a tool for studying the interactions of kinks. In particular, it may be used to study the interactions of fast moving kinks, which in the more conventional models are plagued with big radiation effects. In our simulations we have set \( v = v_1 = 0.7 \). There was nothing special about this value; we have also performed simulations at other values of \( v_1 \); the results were similar; although too low a value of \( v_1 \) (like \( v_1 \sim 0.1 \)) led to increased radiation effects due to the higher value of \( \alpha \) and hence increased nonlinearity in \( \psi_4 \). For \( v \sim v_1 \) the radiation effects were negligible and the kink propagated along the lattice with virtually no slowing down.

The price we have paid for having a discrete model with moving kinks was, of course, the somewhat unusual form of the kinetic energy of the kink; however, its properties were not that different from the usual energy and the distinction was noticeable only at higher speeds.
When we compare energies of the kinks which were started off at different speeds the effects due to the unconventional form of the kinetic energy became significant for larger velocities. In fig 14 we present the plots of the total energy in simulations for different values of \( v \) (the two curves represent the usual expression for the kinetic energy, and ours). As they include the energy of the radiation they cannot be thought of as the energy of the kink, but at least initially, they are close to it. We see the same trend of both expressions; however, the energy of our kink grows with the increase of velocity slower than in the conventional case.

One of the main advantages of the model of Speight and Ward was that the fields in this model satisfied the Bogomolnyi bound. In our case, because of the explicit dependence on \( v \) in the expression for \( f \), this was no longer true; however, in practice, the potential energy, given by \( L_p \) in (2.9), was almost constant in time and varied very little with \( v \). This is shown in fig. 15. As before the curves have been vertically displaced by 0.2 units upwards. We see very little variation of \( L_p \) with time. This is, of course, in agreement with our expectations. The exact moving kink solution describes a kink which displaces itself along the lattice. The kink is not on the flat bottom of Bogomolnyi solutions and so the orbit of the kink is not an equipotential curve, but instead, the potential varies periodically along this orbit. The variation is very small indeed. It is reassuring to see that the total energy was extremely well conserved (to within 9 decimal points); hence the kinetic energy varied periodically too, in precise antiphase with the potential. In addition \( L_k \), the time dependent term in the Lagrangian, also varied periodically with \( t \), but its amplitude was greater.

Hence we see that we have a model which can be used for numerical investigations of kink-like structures and their interactions on coarse lattices. As such it reproduces the continuum behaviour very well; although at very large velocity and for very coarse lattices the radiation effects can become significant and can alter the behaviour of the kinks. The model is also reasonably successful in reproducing the behaviour of periodic structures, like breathers.
At the same time, the model may become interesting in itself for studying
the dynamics of discrete systems. In this case we have to decide whether we
should take seriously our choice of the kinetic term. The term is unusual but, as
such, it does not violate any basic principles. However, its unusual form suggests
cautions. Thus if we cannot justify its use then, perhaps, we should return to the
static form of $f$ and the conventional form of the kinetic term. However, then we
would expect the radiation effects to become more pronounced. But whether this
is physically justified or not can only be determined by looking in detail at the
concrete applications.

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FIGURE CAPTIONS

Fig1 : Comparison of our time dependent term in the Lagrangian ($L_k$) (a) (2.8)
with the conventional energy (2.10) (b) and with the kinetic energy ($E_k$) (c)
for different values of $\psi$.

Fig2 : $v$ dependence of $f^2$ given by (3.3) for $h = 0.1, 0.3, 0.5, 0.7, 1, 2, 3$ and $4$. For
small $v$, $f^2$ monotonically increases with $h$. 

20
**Fig 3**: Fields as seen in the simulation with \( v = 0.7 \), a) Fields at \( t = 0, 400, 800, 1200, 1600 \) and \( 2000 \) displaced successively down by 0.2. b) Field at \( t = 1200 \) for \( 800 < x < 880 \), c) Field at \( t = 800 \) for \( -600 < x < -100 \).

**Fig 4**: Time dependence of the velocity of the kink as seen in the simulation of Fig. 3.

**Fig 5**: Time dependence of the velocity of the kink as seen in the simulations initiated with \( v = 0.01, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.85 \) and 0.9.

**Fig 6**: Simulation initiated with \( v = 0.25 \). a) Fields at \( t = 0, 800, 1600, 2400, 3200 \) and \( 4000 \) displaced successively down by 0.2. b) Time dependence of the velocity of the kink; two method of evaluation a) by extrapolation \( \beta \) in terms of \( X(t) \), c) Field at \( t = 4000 \) for \( 900 < x < 970 \), d) Time dependence of the velocity of the kink as seen in our simulation (\( \alpha \)), simulation in our model but with the conventional expression for the kinetic energy (\( \beta \)), and the model of Speight and Ward (\( \gamma \)).

**Fig 7**: Fields at \( t = 0, 800, 1600 \) and 2400 as seen in the simulation with \( v = 0.8 \). The fields are successively displaced down by 0.4.

**Fig 8**: Fields (a), at \( t = 0, 400, 800, 1200, 1600 \) and \( 2000 \) as seen in the simulation with \( v = 0.9 \). The fields are successively displaced down by 0.2. b) Time dependence of the velocity of the kink.

**Fig 9**: Fields at \( t = 0, 80 \) and 160 as seen in simulations with larger \( v \), a) \( v = 0.95 \), b) \( v = 0.97 \). Again, fields have been displaced vertically.

**Fig 10**: Simulations of two kinks. Fields at \( t = 0, 400, 800 \) and 1200 successively displaced up by 0.4 for a simulation initiated with \( v = 0.3 \).

**Fig 11**: Time dependence of each kink as seen in the simulation shown in Fig 10.

**Fig 12**: Simulation of the breather with \( \lambda = 0.25 \). a) Time dependence of the field at \( x = 0 \), b) Fields at \( t = 0, 400, 800, 1200, 1600 \) and 2000, successively displaced down by 2. c) As b) but restricted to \( -50 < x < 50 \).
Fig13 : Simulation of the breather with $\lambda = 0.9$. a) Time dependence of the field at $x = 0$, b) Fields at $t = 0, 400, 800, 1200, 1600$ and $2000$, successively displaced up by 2. c) Detail of the field at $t = 2000$.

Fig14 : Comparison of the velocity dependence of the energies of one kink; a) with the conventional term, b) with our term i.e. (2.8).

Fig15 : Time dependence of potential energies as seen in our simulations of one kink initiated with $v = 0.1 \times i$ for $i = 1, \ldots, 9$ again displaced successively up by 0.2.

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