I Introduction.

This paper is a continuation of the study, begun in earlier papers [1], [2], of the Racah-Wigner calculus for quantum superalgebra $U_q(osp(1|2))$. The quantum superalgebra $U_q(osp(1|2))$ can be considered either as the super-analog of the $U_q(su(2))$ quantum algebra, for which the Racah-Wigner calculus has been developed in several papers [3], [4], [5], [6] or as the quantum deformation of the super algebra $osp(1|2)$, the Racah-Wigner calculus of which has also been constructed in Refs.[7], [8], [9], [10].

The quantum superalgebra $U_q(osp(1|2))$ has been defined and studied by Kulish and Reshetikhin [11], [12]. Its Clebsch-Gordan coefficients were derived by Kulish [13] using a recursion relation and in Ref.[1] using the projection operator method. A particular case of Clebsch-Gordan coefficients was also given by Saleur [14].

The general case of reduction of tensor product of two arbitrary irreducible representations of the quantum superalgebra $U_q(osp(1|2))$, where the bilinear Hermitian form is not necessarily positive definite, has been studied in Ref.[2]. It has been shown that the Clebsch-Gordan coefficients for the quantum superalgebra $U_q(osp(1|2))$ do not depend either on the signatures of the bilinear Hermitian form or on the class of the representations but they depend on the parities of the representation spaces. Several properties of Clebsch-Gordan coefficients for $U_q(osp(1|2))$, such as orthogonality relations and symmetry properties of these coefficients have been derived in Ref.[2]. In particular, it was shown that under some condition, Clebsch-Gordan coefficients satisfy Regge symmetry. The study of the symmetry properties of the Clebsch-Gordan coefficients allowed one to define $3-j$ symbols which possess good symmetry properties which depend on the parity of bases in graded representation spaces. It has been shown that the dependence on parities in this symbol can be factored out, so that one can define parity independent $3-j$ symbols, which are super-analogs of $3-j$ symbols for the quantum algebra $U_q(su(2))$ [6]. In this article, Clebsch-Gordan coefficients and $3-j$ symbols will be used to define Racah coefficients and $6-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$.

The paper has the following structure: Section II contains the definition of the quantum superalgebra $U_q(osp(1|2))$ and the basic properties of its irreducible representations. In section III, we recall the definition and some properties of Clebsch-Gordan coefficients for the quantum superalgebra $U_q(osp(1|2))$. Section IV is devoted to the definition and basic properties of parity dependent and parity independent $3-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$. In section V, we consider the tensor product of three irreducible representations of $U_q(osp(1|2))$ with arbitrary Hermitian forms. We define Racah coefficients for the quantum superalgebra $U_q(osp(1|2))$ and we derive the basic properties of these coefficients: symmetry properties, pseudo-orthogonality relations and some particular values. In Section VI, we consider $6-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$. In the first subsection, we define in a conventional way using an invariant metric and parity dependent $3-j$ symbols, parity dependent $6-j$ symbols and we derive their basic properties. In the second subsection, it is shown that that it is possible to factor out the dependence on parities in the parity dependent $6-j$ symbols, so one can define parity independent $6-j$ symbols whose properties are quite similar to those of the $6-j$ symbols for the quantum algebra $U_q(su(2))$. 

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II The irreducible representations of the quantum superalgebra $U_q(osp(1|2))$.

The quantum superalgebra $U_q(osp(1|2))$ is generated by three elements: $H$ (even) and $v_{\pm}$ (odd) with the following (anti)commutation relations

$$[H,v_{\pm}] = \pm \frac{1}{2} v_{\pm}, \quad [v_+,v_-] = -\frac{\text{sh} (\eta H)}{\text{sh} (2\eta)},$$

(2.1)

where the deformation parameter $\eta$ is real and $q = e^{\frac{\eta}{2}}$. We choose $\eta > 0$ so that $\eta < 1$.

The following formulas for coproduct $\Delta$, antipode $S$, and counit $\epsilon$ define $U_q(osp(1|2))$ the structure of a Hopf algebra

$$\Delta(v_{\pm}) = v_{\pm} \otimes q^H + q^{-H} \otimes v_{\pm},$$

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(1) = 1 \otimes 1,$$

$$S(H) = -H, \quad S(v_{\pm}) = -q^{\pm \frac{1}{2}} v_{\pm},$$

$$\epsilon(H) = \epsilon(v_{\pm}) = 0, \quad \epsilon(1) = 1,$$

(2.2, 2.3, 2.4, 2.5)

where the coproduct $\Delta$ is the homomorphism

$$\Delta : U_q(osp(1|2)) \rightarrow U_q(osp(1|2)) \otimes U_q(osp(1|2)).$$

(2.6)

A representation of a quantum superalgebra $U_q(osp(1|2))$ in a finite dimensional graded space $V$ is a homomorphism $T : U_q(osp(1|2)) \rightarrow L(V,V)$, of the associative graded algebra $U_q(osp(1|2))$ into the associative graded algebra $L(V,V)$ of linear operators on $V$, such that

$$[T(H),T(v_{\pm})] = \pm \frac{1}{2} T(v_{\pm}), \quad [T(v_+),T(v_-)] = -\frac{\text{sh} (\eta T(H))}{\text{sh} (2\eta)},$$

(2.8)

It has been shown in Ref.[2], that any finite dimensional grade star representation of $U_q(osp(1|2))$ is characterized by four parameters: the superspin $l$ (a non negative integer), the parity $\lambda = 0,1$ of the highest weight vector in the representation space and by $\varphi, \psi = 0,1$ the signature parameters of the Hermitian form in the representation space $V$. The parity $\lambda$ and the signature $\varphi$ define the class $\epsilon = \lambda + \varphi + 1 \mod(2)$.

(2.9)

The representation space $V = V^H(\lambda)$ is a graded vector space of dimension $2l + 1$ with basis $e^{(l)}_{m}(\lambda)$, where $-l \leq m \leq l$. The parity of the basis vectors $e^{(l)}_{m}(\lambda)$ is determined by the values of $l, m$ and $\lambda$,

$$\text{deg}(e^{(l)}_{m}(\lambda)) = -l - m + \lambda \mod(2).$$

(2.10)

The vectors $e^{(l)}_{m}(\lambda)$ are orthogonal with respect to the Hermitian form and their normalization is determined by the signature parameters $\varphi, \psi$,

$$\langle e^{(l)}_{m}(\lambda), e^{(l)}_{m'}(\lambda) \rangle = (-1)^{\varphi(l-m)} \delta_{m,m'},$$

(2.11)

where $(\cdot, \cdot)$ denotes the Hermitian form in the representation space. The operators $T(v_{\pm})$ and $T(H)$ act on the basis $e^{(l)}_{m}(\lambda)$ in the following way:

$$T(H)e^{(l)}_{m}(\lambda) = \frac{m}{2} e^{(l)}_{m}(\lambda),$$

(2.12)

$$T(v_{+})e^{(l)}_{m}(\lambda) = (-1)^{\varphi l - m} \sqrt{l + m + 1} \gamma e^{(l+1)}_{m+1}(\lambda),$$

(2.13)

$$T(v_{-})e^{(l)}_{m}(\lambda) = \sqrt{l + m + 1} \gamma e^{(l-1)}_{m-1}(\lambda),$$

(2.14)

where the symbol $[n]$ is the Kulkus graded quantum symbol [13] defined by

$$[n] = \frac{q^{-\frac{n^2}{2}} - (-1)^{\varphi} q^{\frac{n^2}{2}}}{q^{\frac{n}{2}} - 1}.$$

(2.15)

Note that the action of the operators $T(v_{\pm})$ and $T(H)$ does not depend on the parameters $\lambda, \varphi, \psi$. The representation $T$ of class $\epsilon$ which acts in the representation space $V^H(\lambda)$ with an Hermitian form characterized by the signature parameters $\varphi$ and $\psi$ is denoted by $T_{\epsilon,\varphi,\psi}$. However, for simplicity, the indices $\epsilon, \varphi, \psi$ will sometimes be omitted in the following. Note that the parity $\lambda$ is expressed in terms of the other parameters by relation (2.9), i.e.,

$$\lambda = \epsilon + \varphi + 1 \mod(2).$$

(2.16)

In the limit $q \rightarrow 1$, the grade star representation $T_{\epsilon,\varphi,\psi}$ becomes a grade star representation of superalgebra $osp(1|2)$. For more details on the irreducible grade star representations of $U_q(osp(1|2))$, see Ref.[2].

III Clebsch-Gordan coefficients for the quantum superalgebra $U_q(osp(1|2))$.

A Tensor product of two irreducible representations.

In this section, we will recall basic properties of the tensor product of two irreducible representations $T_{\epsilon,\varphi,\psi}$ and $T_{\epsilon',\varphi',\psi'}$ of the same class $\epsilon$. The bilinear Hermitian form in the tensor product space $V^H(\lambda_1) \otimes V^H(\lambda_2)$ is defined in the following way:

$$((X_1 \otimes X_2, Y_1 \otimes Y_2)) = (-1)^{\text{deg}(X_2) \text{deg}(Y_1)}(X_1, Y_1)(X_2, Y_2),$$

(3.1)

where $X_1, X_2 \in V^H(\lambda_1)$, $Y_1, Y_2 \in V^H(\lambda_2)$. The space $V^H(\lambda_1) \otimes V^H(\lambda_2)$ is a representation space for the tensor product of two representations $T_{\epsilon,\varphi,\psi} \otimes T_{\epsilon',\varphi',\psi'}$. The generators $v_{\pm}$ and $H$ are represented in this case by the following operators acting in the space $V^H(\lambda_1) \otimes V^H(\lambda_2)$

$$v_{\pm}^{\epsilon_1,\varphi_1,\psi_1}(1,2) = (T^{\epsilon_1}_H \otimes T^{\epsilon_2}_H)(\Delta(v_{\pm})),$$

(3.2)

$$H^{\epsilon_1,\varphi_1,\psi_1}(1,2) = (T^{\epsilon_1}_H \otimes T^{\epsilon_2}_H)(\Delta(H)) = T^{\epsilon_1}_H(1) \otimes T^{\epsilon_2}_H(2) + T^{\epsilon_2}_H(1) \otimes T^{\epsilon_1}_H(2).$$

(3.3)

It has been shown in [2] and [1] that the tensor product of representations $T_{\epsilon,\varphi,\psi} \otimes T_{\epsilon',\varphi',\psi'}$ is a representation of class $\epsilon$ with respect to the Hermitian form (3.1) and that the tensor product of representation spaces $V^H(\lambda_1) \otimes V^H(\lambda_2)$ can be decomposed into a direct sum of subspaces $V^H(\lambda)$

$$V^H(\lambda_1) \otimes V^H(\lambda_2) = \oplus \gamma V^H(\lambda).$$

(3.4)
where \( l \) is an integer satisfying the conditions
\[
|l_1 - l_2| \leq l \leq l_1 + l_2.
\]
In this decomposition, each subspace appears only once, i.e., the tensor product of two representations of the same class is simply reducible.

**B U_q(\text{osp}(1|2)) Clebsch-Gordan coefficients.**

By definition, the Clebsch-Gordan coefficients \((l_1m_1, \lambda_1, l_2m_2, \lambda_2; l_m, \lambda_m)\), relate the pseudo-normalized basis \(e_{m_1}^{(l_1)}(\lambda_1) \otimes e_{m_2}^{(l_2)}(\lambda_2)\) to the reduced pseudo-normalized basis \(e_{m}^{(l)}(\lambda)\) in the following way:
\[
e_{m}^{(l)}(l_1, l_2, \lambda) = \sum_{m_1, m_2} (l_1m_1, \lambda_1, l_2m_2, \lambda_2; l_m, \lambda_m) e_{m_1}^{(l_1)}(\lambda_1) \otimes e_{m_2}^{(l_2)}(\lambda_2),
\]
where \( m = m_1 + m_2\). The bases \(e_{m}^{(l)}(l_1, l_2, \lambda)\) in the reduced tensor product space are orthogonal and satisfy the normalization condition:
\[
(e_{m}^{(l_1, l_2)}(l_1, l_2, \lambda), e_{m'}^{(l_1, l_2)}(l_1, l_2, \lambda)) = (-1)^{\varphi(l_1m_1 + l_2m_2)} \delta_{m m'},
\]
where
\[
\varphi = l_1 + l_2 + l + \lambda_1 + \lambda_2 \mod (2),
\]
\[
\psi = (l_1 + l_2 + l + \lambda_1 + \lambda_2) + (l_1 + l_2 + l + \lambda_1 + \lambda_2) \mod (2),
\]
\[
\lambda = l_1 + l_2 + l + \lambda_1 + \lambda_2 \mod (2).
\]
It should be noticed that even if the bases \(e_{m_1}^{(l_1)}(\lambda_1)\) and \(e_{m_2}^{(l_2)}(\lambda_2)\) are positive definite, i.e., \(\varphi, \psi = 0\), \(\lambda = 1\), the reduced basis \(e_{m}^{(l)}(l_1, l_2, \lambda)\) need not be positive definite.

The operators \(H^{*}(1, 2)\) and \(u_{0}(1, 2)\) act on the vectors \(e_{m}^{(l)}(l_1, l_2, \lambda)\) in a standard way given by formulae (2.12), (2.14). Recently, Clebsch-Gordan coefficients for the quantum superalgebra \(U_q(\text{osp}(1|2))\) have been derived by use of various methods [13, 1, 2]. In the following we will use the results obtained in Refs.[1, 2] by application of the projection the operator method. In the limit \(q \rightarrow 1\), the quantum Clebsch-Gordan coefficients become \(\text{osp}(1|2)\) Clebsch-Gordan coefficients
\[
\lim_{q \rightarrow 1} (l_1m_1, l_2m_2, l_3; m_3) = (l_1m_1, l_2m_2, l_3),
\]
Let us recall some properties of the Clebsch-Gordan coefficients for the quantum superalgebra \(U_q(\text{osp}(1|2))\) [2]. They satisfy the pseudo-orthogonality relations
\[
\sum_{m=m_3} (1)^{(l_1m_1+m_2)+m(m)-m} (l_1m_1, l_2m_2, l_3; m_3) (l_1m_1, l_2m_2, l_3; m_3) = (-1)^{(l_1m_1+m_2)+m(m)-m} \delta_{m_1 m_2} \delta_{m_3 m_4},
\]
\[
\sum_{l} (1)^{(l_1m_1+m_2)+l(m)-m} (l_1m_1, l_2m_2, l_3, m_3) (l_1m_1, l_2m_2, l_3; m_3) = (-1)^{(l_1m_1+m_2)+l(m)-m} \delta_{m_1 m_2} \delta_{m_3 m_4}.
\]

where \( L = l_1 + l_2 + l \). These pseudo-orthogonality relations do not depend on the parameters \(\varphi, \psi, \lambda, l, \) and \(c\).

Considering the action of operators \(u_q(1, 2)\) on the defining relations for Clebsch-Gordan coefficients, one can derive the following recursion relations for Clebsch-Gordan coefficients, cf. Ref.[13]
\[
\sqrt{l_1 + m_1([l_1 - m_1 + 1])} (l_1m_1, l_2m_2, l_3; l_4) = q^{\frac{\varphi}{2}} \sqrt{l_1 + m_1([l_1 + m_1 + 1])} (l_1m_1, l_2m_2, l_3; l_4) \nonumber
\]
\[
+ (-1)^{c(k-m_1)} q^{-\frac{\varphi}{2}} \sqrt{l_1 - m_1([l_1 + m_1 + 1])} (l_1m_1, l_2m_2, l_3; l_4),
\]
\[
\sqrt{l_1 - m_1([l_1 - m_1 + 1])} (l_1m_1, l_2m_2, l_3; l_4) = (-1)^{c(k-m_1)} q^{\frac{\varphi}{2}} \sqrt{l_1 + m_1([l_1 + m_1 + 1])} (l_1m_1, l_2m_2, l_3; l_4) \nonumber
\]
\[
+ q^{-\frac{\varphi}{2}} \sqrt{l_1 + m_1([l_1 - m_1 + 1])} (l_1m_1, l_2m_2, l_3; l_4).
\]
In the classical theory of Racah-Wigner calculus, a very important role is played by the Clebsch-Gordan coefficients \((j_m, j_0, j_0)\), which defines an invariant metric. In the case of the quantum superalgebra \(U_q(\text{osp}(1|2))\), the corresponding metrics also define an invariant metric. It has the form
\[
C_{m}^{(l)}(\lambda) = \sqrt{[2l + 1](l_m, l_m)[00]} = (-1)^{(l_1m_1 - l_2m_2)} (-1)^{(l_2m_2 - l_1m_1)} q^{\frac{\varphi}{2}} \delta_{m_1, m_2},
\]
and it satisfies the properties
\[
C_{m}^{(l)}(\lambda) C_{m'}^{(l')}^{\dagger} = (-1)^{(l_1m_1 - l_2m_2)} C_{m'}^{(l')}^{\dagger}(\lambda), \quad C_{m}^{(l)}(\lambda) C_{m'}^{(l')}^{\dagger} = \delta_{m_1, m_2}.
\]
In the following, we will use this invariant metric to construct parity dependent 6−j symbols for the quantum superalgebra \(U_q(\text{osp}(1|2))\).

One easily checks that in the limit \(q = 1\), the invariant metric (3.16) becomes the invariant metric for the classical superalgebra \(\text{osp}(1|2)\) defined in Ref.[10].

It has been shown in Ref.[2], that Clebsch-Gordan coefficients have the following symmetries
\[
(l_1m_1, l_2m_2, l_3; l_4) = (-1)^{(l_1m_1 + l_2m_2)} (-1)^{(l_1m_1 + l_2m_2)} (l_1m_1, l_2m_2, l_3; l_4),
\]
\[
(l_1m_1, l_2m_2, l_3; l_4) = (-1)^{(l_1m_1 + l_2m_2)} (l_1m_1, l_2m_2, l_3; l_4),
\]
\[
(l_1m_1, l_2m_2, l_3; l_4) = (-1)^{(l_1m_1 + l_2m_2)} (l_1m_1, l_2m_2, l_3; l_4),
\]
\[
(l_1m_1, l_2m_2, l_3; l_4) = (-1)^{(l_1m_1 + l_2m_2)} (l_1m_1, l_2m_2, l_3; l_4),
\]
\[
(l_1m_1, l_2m_2, l_3; l_4) = (-1)^{(l_1m_1 + l_2m_2)} (l_1m_1, l_2m_2, l_3; l_4),
\]
\[
(l_1m_1, l_2m_2, l_3; l_4) = (-1)^{(l_1m_1 + l_2m_2)} (l_1m_1, l_2m_2, l_3; l_4),
\]
\[
(l_1m_1, l_2m_2, l_3; l_4) = (-1)^{(l_1m_1 + l_2m_2)} (l_1m_1, l_2m_2, l_3; l_4),
\]
The Clebsch-Gordan coefficients satisfy also the "mirror" symmetry
\[
(l_1, m_1, \lambda_1, l_2, m_2, \lambda_2 | l_3, m_3, \lambda_3)_q = (-1)^{(l_1+m_1+\lambda_1)}(-1)^{(l_2+m_2+\lambda_2)}(-1)^{(l_3+m_3+\lambda_3)} (l_1 - m_1, \lambda_1, l_2 - m_2, \lambda_2 | l_3 - m_3, \lambda_3)_{q^{-1}},
\]
where \( L = l_1 + l_2 + l_3 \). All these symmetries of Clebsch-Gordan coefficients present the same structure as the symmetries of Clebsch-Gordan coefficients for quantum algebra \( U_q(\text{su}(2)) \) [6], except that the phases are non linear in \( l_i, m_i \), and that they depend on the parities \( \lambda_i \), \( i = 1, 2, 3 \).

Besides, it has been shown that these Clebsch-Gordan coefficients for the quantum superalgebra \( U_q(\text{osp}(1|2)) \) satisfy also Regge symmetry. However in this case, Regge symmetry is realized only under some condition on the arguments, cf. Ref.[18].

## IV Symmetric 3–j symbols for quantum superalgebra \( U_q(\text{osp}(1|2)) \)

### A Parity dependent 3–j \( \lambda \) symbols.

For the quantum superalgebra \( U_q(\text{osp}(1|2)) \), the parity dependent 3–j symbols, denoted \( s_{q^3-J} - \lambda \), have been defined in Ref.[2] as follows
\[
\left( \begin{array}{ccc}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3 \\
 \end{array} \right)_{q^3-J} = (-1)^{(l_1+m_1+\lambda_1)}(-1)^{(l_2+m_2+\lambda_2)}(-1)^{(l_3+m_3+\lambda_3)}
\]
\[
\times q^{-\frac{1}{2}(m_1-m_2)} \delta_{\lambda_1,\lambda_2} \delta_{m_1,m_2} \delta_{\lambda_3,\lambda_4} \delta_{m_3,m_4}.
\]

Their arguments satisfy the same constraints as for Clebsch-Gordan coefficients
\[
|l_1 - l_2| < l_3 < l_1 + l_2,
\]
\[
m_1 + m_2 + m_3 = 0,
\]
\[
l_1 + l_2 + l_3 = \lambda_1 + \lambda_2 + \lambda_3 \mod(2),
\]
and the symbols have symmetry properties similar to the symmetry properties of 3–j symbols for the quantum algebra \( U_q(\text{su}(2)) \). Under an even permutation of columns one has
\[
\left( \begin{array}{ccc}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3 \\
 \end{array} \right)_{q^3-J} = \left( \begin{array}{ccc}
  l_3 & l_1 & l_2 \\
  m_3 & m_1 & m_2 \\
 \end{array} \right)_{q^3-J} = \left( \begin{array}{ccc}
  l_2 & l_3 & l_1 \\
  m_2 & m_3 & m_1 \\
 \end{array} \right)_{q^3-J},
\]
and under an odd permutation they transform as
\[
\left( \begin{array}{ccc}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3 \\
 \end{array} \right)_{q^3-J} = \beta \left( \begin{array}{ccc}
  l_2 & l_3 & l_1 \\
  m_2 & m_3 & m_1 \\
 \end{array} \right)_{q^{-1}}
\]
\[
= \beta \left( \begin{array}{ccc}
  l_1 & l_3 & l_2 \\
  m_1 & m_3 & m_2 \\
 \end{array} \right)_{q^{-1}},
\]
where \( \beta \) is the phase factor
\[
\beta = (-1)^{\sum_{i=1}^3 \theta(l_i-m_i, \lambda_i)}.
\]

Under "mirror" symmetry, the \( s_{q^3-J} - \lambda \) symbol transform in a slightly different way
\[
\left( \begin{array}{ccc}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3 \\
 \end{array} \right)_{q^{-1}} = (-1)^{\sum_{i=1}^3 \theta(l_i-m_i, \lambda_i)} \left( \begin{array}{ccc}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3 \\
 \end{array} \right)_{q^{-1}}.
\]

Note also that the \( s_{q^3-J} \) symbols, as well as the Clebsch-Gordan coefficients, satisfy a conditional Regge symmetry, cf. Ref.[2], and that they satisfy modified pseudo-orthogonality relations, namely
\[
\sum_{m_1,m_2} (-1)^{(m_1-m_2)}q^{\frac{1}{2}(m_1-m_2)} \left( \begin{array}{ccc}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3 \\
 \end{array} \right)_{q^3-J} \left( \begin{array}{ccc}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3 \\
 \end{array} \right)_{q^{-1}} = (-1)^{(l_1-m_1)} \frac{\delta_{\lambda_1,\lambda_2} \delta_{m_1,m_2} \delta_{\lambda_3,\lambda_4} \delta_{m_3,m_4}}{[2l_3+1]},
\]
\[
\sum_{m_2,m_3} (-1)^{(l_1-m_2)}q^{\frac{1}{2}(m_2-m_3)} \left( \begin{array}{ccc}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3 \\
 \end{array} \right)_{q^3-J} \left( \begin{array}{ccc}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3 \\
 \end{array} \right)_{q^{-1}} = (-1)^{(l_1-m_1)} \frac{\delta_{\lambda_1,\lambda_2} \delta_{m_1,m_2} \delta_{\lambda_3,\lambda_4} \delta_{m_3,m_4}}{[2l_3+1]}.
\]

In the limit \( q \to 1 \), the \( s_{q^3-J} - \lambda \) symbols become identical to the \( s_{3-J} \) symbols for superalgebra \( \text{osp}(1|2) \) defined in Ref.[10].

### B Parity independent 3–j symbols.

In Ref.[2], it has been shown that for \( s_{q^3-J} - \lambda \) symbols, the dependence on \( \lambda \) can be factored out, so that for the quantum superalgebra \( U_q(\text{osp}(1|2)) \) one can define parity independent 3–j symbols that will be called \( s_{q^3-J} - \lambda \) symbols
\[
\left( \begin{array}{ccc}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3 \\
 \end{array} \right)_{q^3-J} = (-1)^{l_1+m_1}(-1)^{l_2+m_2}(-1)^{l_3+m_3} \frac{\delta_{\lambda_1,\lambda_2} \delta_{m_1,m_2} \delta_{\lambda_3,\lambda_4} \delta_{m_3,m_4}}{[2l_3+1]}
\times \frac{\delta_{\lambda_1,\lambda_2} \delta_{m_1,m_2} \delta_{\lambda_3,\lambda_4} \delta_{m_3,m_4}}{[2l_3+1]}.
\]

The parity independent \( s_{q^3-J} \) symbols satisfy constraints similar to those of the \( s_{q^3-J} - \lambda \) symbols: for an even permutation of columns
\[
\left( \begin{array}{ccc}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3 \\
 \end{array} \right)_{q^3-J} = \left( \begin{array}{ccc}
  l_3 & l_1 & l_2 \\
  m_3 & m_1 & m_2 \\
 \end{array} \right)_{q^3-J} = \left( \begin{array}{ccc}
  l_2 & l_3 & l_1 \\
  m_2 & m_3 & m_1 \\
 \end{array} \right)_{q^3-J},
\]
and for an odd permutation of columns
\[
\left( \begin{array}{ccc}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3 \\
 \end{array} \right)_{q^{-1}} = \alpha \left( \begin{array}{ccc}
  l_2 & l_3 & l_1 \\
  m_2 & m_3 & m_1 \\
 \end{array} \right)_{q^{-1}},
\]
\[
= \alpha \left( \begin{array}{ccc}
  l_1 & l_3 & l_2 \\
  m_1 & m_3 & m_2 \\
 \end{array} \right)_{q^{-1}} = \alpha \left( \begin{array}{ccc}
  l_3 & l_1 & l_2 \\
  m_3 & m_1 & m_2 \\
 \end{array} \right)_{q^{-1}}.
\]
where \( \alpha \) is the phase factor

\[
\alpha = (-1)^{\sum_{i=1}^{n-1} l_{i} - m_{i} + l_{n} - m_{n}} (-1)^{\sum_{i=1}^{n-1} m_{i} - n_{i}}.
\]  

(4.14)

As \( sq3-j \) symbols, these \( sq3-j \) symbols satisfy also a "mirror" symmetry and a conditional Regge symmetry. But for the latter property, the \( sq3-j \) symbols are not invariant, they are multiplied by a phase factor, cf. Ref. [2]. Moreover, the \( sq3-j \) symbols satisfy the same pseudo-orthogonality relations (4.9) (4.10) as \( sq3-j \) symbols.

Using the parity independent \( sq3-j \) symbol, one can define a parity independent invariant metric

\[
C_{mn}^{q} = q_{lm}^{m-n/2} (l_{m} \quad m \quad 0) = (-1)^{l_{m}-m_{n}} q_{mn}^{l_{m}+m_{n}} \delta_{m-n}.
\]  

(4.15)

It is related to the parity dependent invariant metric \( C_{mn}^{p}(\lambda) \) defined in relation (3.16) by

\[
C_{mn}^{q} = (-1)^{l_{m}+m_{n}} C_{mn}^{p}(\lambda).
\]  

(4.16)

The parity independent \( sq3-j \) symbols and invariant metric will be used in the following to construct the parity independent symmetric \( s-j \) symbols for the quantum superalgebra \( U_{q}(osp(1|2)) \).

In the limit \( q \rightarrow 1 \), \( sq3-j \) symbols become identical to the parity independent \( s3-j \) symbols for the superalgebra \( U_{q}(1|2) \) defined in Ref.[10].

V Racah Coefficients for the quantum superalgebra \( U_{q}(osp(1|2)) \).

A Tensor product of three irreducible representations

Let us consider the tensor product of three irreducible representations of the quantum superalgebra \( U_{q}(osp(1|2)) \) of the same class \( \epsilon \)

\[
T^{i_{1}}_{\nu_{1}} \otimes T^{i_{2}}_{\nu_{2}} \otimes T^{i_{3}}_{\nu_{3}}.
\]  

(5.1)

The representation space for such a tensor product is the tensor product of the representation spaces

\[
V^{i_{1}}(\lambda_{1}) \otimes V^{i_{2}}(\lambda_{2}) \otimes V^{i_{3}}(\lambda_{3}),
\]  

(5.2)

with a basis spanned by the vectors

\[
e_{m_{1}}^{i_{1}}(\lambda_{1}) \otimes e_{m_{2}}^{i_{2}}(\lambda_{2}) \otimes e_{m_{3}}^{i_{3}}(\lambda_{3}).
\]  

(5.3)

The bilinear Hermitian form in the graded space (5.2) is defined with the bilinear Hermitian forms in each space \( V^{i_{1}}(\lambda_{1}) \), \( i = 1, 2, 3 \), in the following way

\[
\langle e_{m_{1}}^{i_{1}}(\lambda_{1}) \otimes e_{m_{2}}^{i_{2}}(\lambda_{2}) \otimes e_{m_{3}}^{i_{3}}(\lambda_{3}) | e_{n_{1}}^{i_{1}}(\lambda_{1}) \otimes e_{n_{2}}^{i_{2}}(\lambda_{2}) \otimes e_{n_{3}}^{i_{3}}(\lambda_{3}) \rangle = (-1)^{\sum_{i=1}^{n-1} l_{i} - m_{i} + 1} (-1)^{\sum_{i=1}^{n-1} m_{i} + n_{i} + 1} \delta_{m_{1} n_{1}} \delta_{m_{2} n_{2}} \delta_{m_{3} n_{3}}.
\]  

(5.4)

The reduction of the tensor product of representations (5.1) can be done, as in the classical case, in two different schemes. In the first scheme, one couples first the representations \( T^{i_{1}}, \) and \( T^{i_{2}} \) and then the result is coupled to the representation \( T^{i_{3}} \) in order to give as a final result the representation \( T^{i} \). This scheme can be expressed in the short way

\[
T^{i} \subset (T^{i_{1}} \otimes T^{i_{2}}) \otimes T^{i_{3}}.
\]  

(5.5)

The reduced basis corresponding to this scheme is given by the expression

\[
(e_{m}^{i_{1}}(1, 2, 3, \lambda) = \sum_{m_{1}}(l_{m_{1}} \lambda_{1}, l_{m_{2}} \lambda_{2}, l_{m_{3}} \lambda_{3}) l_{m_{1}} \lambda_{1} e_{m_{1}}^{i_{1}}(\lambda_{1}) \otimes e_{m_{2}}^{i_{2}}(\lambda_{2}) \otimes e_{m_{3}}^{i_{3}}(\lambda_{3}),
\]  

(5.6)

where \( i = 1, 2, 3 \) and we have

\[
m = m_{1} + m_{2} + m_{3},
\]  

(5.7)

\[
\lambda = \sum_{i=1}^{3} (\lambda_{i} + l_{i}) + l.
\]  

(5.8)

The operators \( H, v_{\lambda} \) are represented in this reduced representation space by

\[
v_{\lambda}^{q}(12, 3) = (T^{\lambda} \otimes T^{\lambda} \otimes T^{\lambda})((\Delta \otimes id)\Delta(\lambda_{g})),
\]  

(5.9)

\[
H^{q}(12, 3) = (T^{\lambda} \otimes T^{\lambda} \otimes T^{\lambda})((\Delta \otimes id)\Delta(H)).
\]  

(5.10)

In the second scheme, one couples first \( T^{i_{1}} \) with result \( T^{i_{2}} \) of the coupling of representations \( T^{i_{3}} \) and \( T^{i_{3}} \) in order to yield \( T^{i} \). That is we consider

\[
T^{i} \subset (T^{i_{1}} \otimes T^{i_{2}} \otimes T^{i_{3}}),
\]  

(5.11)

In this case, the reduced basis vectors are of the form

\[
e_{m}^{i_{1}}(1, 1, 2, 3) = \sum_{m_{1}}(l_{m_{1}} \lambda_{1}, l_{m_{2}} \lambda_{2}, l_{m_{3}} \lambda_{3})(123) \otimes \Delta(\lambda_{g}) e_{m_{1}}^{i_{1}}(\lambda_{1}) \otimes e_{m_{2}}^{i_{2}}(\lambda_{2}) \otimes e_{m_{3}}^{i_{3}}(\lambda_{3}),
\]  

(5.12)

here \( i = 1, 2, 3, 23 \) and \( m \) and \( \lambda \) satisfy the same condition as in the previous reduction scheme.

The operators \( H, v_{\lambda} \) are represented in this case by

\[
v_{\lambda}^{q}(12, 3) = (T^{\lambda} \otimes T^{\lambda} \otimes T^{\lambda})((id \otimes \Delta)\Delta(\lambda_{g})),
\]  

(5.13)

\[
H^{q}(12, 3) = (T^{\lambda} \otimes T^{\lambda} \otimes T^{\lambda})((id \otimes \Delta)\Delta(H)).
\]  

(5.14)

Due to the coassociativity of the coproduct, we have

\[
v_{\lambda}^{q}(12, 23) = v_{\lambda}^{q}(12, 3) = v_{\lambda}^{q}(123),
\]  

(5.15)

\[
H^{q}(12, 23) = H^{q}(12, 3) = H^{q}(123).
\]  

(5.16)

and

\[
v_{\lambda}^{q}(123) = T^{\lambda}(v_{\lambda}) \otimes q^{T^{\lambda}(H)} \otimes q^{T^{\lambda}(H)} + q^{T^{\lambda}(H)} \otimes T^{\lambda}(v_{\lambda}) \otimes q^{T^{\lambda}(H)},
\]  

(5.17)
\[ H^{(123)} = T^{(1)} \otimes T^{(2)} \otimes T^{(3)} + T^{(3)} \otimes T^{(1)} \otimes T^{(2)} + T^{(1)} \otimes T^{(3)} \otimes T^{(2)}. \]  

(5.18)

Using the recursion formulae (3.14) and (3.15), one can check that the operators \( s^{(123)} \) and \( H^{(123)} \) act on the bases \( e^{(12)}_{\lambda}(l, l_2, \lambda) \) and \( e^{(12)}_{\lambda}(l, l_3, \lambda) \) in the standard way given by formulæ (2.12), (2.14). The bases \( e^{(12)}_{\lambda}(l, l_2, \lambda) \) and \( e^{(12)}_{\lambda}(l, l_3, \lambda) \) are orthogonal and normalized in the following way

\[
\langle e^{(12)}_{\lambda}(l, l_2, \lambda), e^{(12)}_{\lambda}(l, l_2, \lambda) \rangle = (-1)^{\psi_{12}1(l, l_2, \lambda) + \psi_{12}2(l, l_2, \lambda)},
\]

(5.19)

\[
\langle e^{(12)}_{\lambda}(l, l_3, \lambda), e^{(12)}_{\lambda}(l, l_3, \lambda) \rangle = (-1)^{\psi_{12}1(l, l_3, \lambda) + \psi_{12}2(l, l_3, \lambda)},
\]

(5.20)

where

\[
\psi_{12}1 = \psi_{12}2 = \psi_{12}1 + \psi_{12}2 \mod(2),
\]

(5.21)

\[
\psi_{12} = \psi_{12}1 + \psi_{12}2 \mod(2).
\]

(5.22)

and

\[
L = l_1 + l_2 + l_3 + l.
\]

(5.23)

\[
L = l_1 + l_2 + l_3 + l.
\]

(5.24)

**B. Racah Coefficients**

Racah coefficients are defined for the quantum superalgebra \( U_q(\text{so}(2n)) \), and denoted by \( U^{(1)}(l_1, l_2, l_3, l_4, l_5, l_6, l_7, q) \), are defined in the standard way as the coefficients that relate two reduced bases in two different reduction schemes

\[
e^{(1)}_{\lambda}(l, l_2, \lambda, \lambda) = \sum_{l_3} (-1)^{(l_3+1)(l_3+3)} U^{(1)}(l_1, l_2, l_3, l_4, l_5, l_6, l_7, q) e^{(1)}_{\lambda}(l, l_2, \lambda).
\]

(5.25)

From the defining relations relations (5.6), (5.12) and from the orthogonality relations for Clebsch-Gordan coefficients (3.12), (3.13) the Racah coefficients are related to the Clebsch-Gordan coefficients as follows

\[
U^{(1)}(l_1, l_2, l_3, l, l_2, l_3, l_2, l_3, q) = \sum_{l_4} (-1)^{(l_3+1)(l_3+3)} U^{(1)}(l_1, l_2, l_3, l, l_4, l_2, l_3, q)
\]

(5.26)

where \( i = 1, 2, 3, 12, 23, 1.2.3. \) Similarly to Racah coefficients for \( U_q(\text{so}(2n)) \), Racah coefficients depend on the parities \( l_1 \) but they depend neither on \( \epsilon \) nor on the signature parameters \( \psi, \psi_1 = 1, 2, 3 \).

Considering the action of the operators \( s^{(123)} \) and \( H^{(123)} \) on the defining relation (5.25) for Racah coefficients and using the relation \( \psi_{12,3} = \psi_{12,3} \), one can show that the coefficients \( U^{(1)}(l_1, l_2, l_3, l_2, l_3, l_2, q) \) do not depend on \( m \). Besides, Racah coefficients satisfy the following orthogonality relations

\[
\sum_{l_4} (-1)^{(l_4+1)(l_4+3)} U^{(1)}(l_1, l_2, l, l_4, l_2, l_3, l_2, q) = (-1)^{(l_4+1)(l_4+3)} U^{(1)}(l_1, l_2, l, l_2, l_3, l_2, q).
\]

(5.27)

Using these properties, one can easily inverse relation (5.25) :

\[
e^{(1)}_{\lambda}(l, l_2, \lambda, \lambda) = \sum_{l_3} (-1)^{(l_3+1)(l_3+3)} U^{(1)}(l_1, l_2, l_3, l_4, l_2, l_3, q) e^{(1)}_{\lambda}(l, l_2, \lambda, \lambda).
\]

(5.29)

Let us now derive some special values of Racah coefficients. If any one of the superspins \( l_1, l_2, l_3 \) is zero, then by definition the Racah coefficients is equal to unity

\[
U^{(1)}(l_2, l_3, l_4, l_5, l_6, q) = U^{(1)}(l_2, l_3, l_4, l_5, l_6, l_7, q) = U^{(1)}(l_2, l_3, l_4, l_5, l_6, l_7, q) = 1.
\]

(5.30)

The case \( l = 0 \) can be calculated directly by means of expression (5.26), which gives

\[
U^{(1)}(l_1, l_2, l_3, l_4, l_5, l_6, l_7, q) = (-1)^{(l_1+1)(l_2+1)(l_3+1)}.
\]

(5.31)

Similarly, it can be proved that

\[
U^{(1)}(l_1, l_2, l_3, l_4, l_5, l_6, l_7, q) = (-1)^{(l_1+1)(l_2+1)(l_3+1)} \left( \frac{[2l_1 + 1]}{[2l_1 + 1][2l_1 + 1]} \right)^{1/2}
\]

(5.32)

\[
U^{(1)}(l_1, l_2, l_3, l_4, l_5, l_6, l_7, q) = (-1)^{(l_1+1)(l_2+1)(l_3+1)} \left( \frac{[2l_1 + 1]}{[2l_1 + 1][2l_1 + 1]} \right)^{1/2}
\]

(5.33)

These formulæ are very similar to the corresponding ones for quantum algebra \( U_q(\text{so}(2n)) \). Ref. [8].

Taking into account the fact that the coefficients \( U^{(1)}(l_1, l_2, l_3, l_4, l_5, l_6, l_7, q) \) do not depend on \( m \) and making use of the identity

\[
\sum_{m=-i}^{m=i} (-1)^{(l-m)} q^m = [2l+1],
\]

(5.34)

one can express the coefficient \( U^{(1)}(l_1, l_2, l_3, l_4, l_5, l_6, l_7, q) \) in terms of Clebsch-Gordan coefficients in a more symmetric way

\[
U^{(1)}(l_1, l_2, l_3, l_4, l_5, l_6, l_7, q) = \left( \frac{[2l_1 + 1]}{[2l_1 + 1][2l_1 + 1]} \right)^{1/2} \sum_{i=1}^{2l+1} (-1)^{(l-m)} \left( \frac{[2l_1 + 1]}{[2l_1 + 1][2l_1 + 1]} \right)^{1/2} \sum_{m=-i}^{m=i} (-1)^{(l-m)} q^m \times \left( \frac{(l_m, l_1, l_2, l_3, l_4, l_5, l_6, l_7)}{[l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_m]} \right)^{1/2}
\]

(5.35)

Using this formula and the "mirror" symmetry of Clebsch-Gordan coefficients (3.21), one can prove the following property of the Racah coefficients

\[
U^{(1)}(l_1, l_2, l_3, l_4, l_5, l_6, l_7, q) = U^{(1)}(l_1, l_2, l_3, l_4, l_5, l_6, l_7, q^{-1}).
\]

(5.36)

Comparison of the two coupling schemes

\[ (T^i \otimes T^j)_{k-1} \otimes (T^k)^{(i+1)}_{k-1} \]

and

\[ (T^j \otimes (T^i \otimes T^k)_{k-1})^{(i+1)}_{k-1} \]

(5.37)
and use of the first symmetry (3.18) of Clebsch-Gordan coefficients yield the following symmetry of $su(3)$ Racah coefficients

$$U^*(l_1, l_2, l_3, l_1, l_2, l_3, q) = U^*(l_3, l_2, l_1, l_2, l_3, l_1, q).$$

(5.38)

Using symmetry properties of the Clebsch-Gordan coefficients (3.18-3.20), one can write the expression (5.26) in the form

$$U^*(l_1, l_2, l_3, l_1, l_2, l_3, q) = \frac{\rho}{[2l_1+1]} \sum_{a,m} (-1)^{l_1-m_1} C^{l_1}_{m_1, l_1-m_1} C^{l_2}_{m_2, l_2-m_2} C^{l_3}_{m_3, l_3-m_3} q^{m_1} \times \left( l_2 m_2 l_3 m_3 \right) \left( l_1 m_1 l_1 m_1 \right) \left( l_2 m_2 l_3 m_3 \right) \left( l_1 m_1 l_1 m_1 \right) \times \left( l_2 m_2 l_3 m_3 \right) \left( l_1 m_1 l_1 m_1 \right) \left( l_2 m_2 l_3 m_3 \right) \left( l_1 m_1 l_1 m_1 \right),$$

(5.39)

where the phase $\rho$ is

$$\rho = (-1)^{C(l_1, l_2, l_3) + C(l_2, l_3, l_1) + C(l_3, l_1, l_2) + C(l_1, l_2, l_3) + C(l_2, l_3, l_1) + C(l_3, l_1, l_2)}. (5.40)$$

This relation, together with relation (5.35), leads to the symmetry property

$$U^*(l_1, l_2, l_3, l_1, l_2, l_3, q) = \rho U^*(l_3, l_2, l_1, l_2, l_3, q).$$

(5.41)

In the next section, we will show that similarly to the non-deformed case of $osp(1|2)$ [10], $su(3)$ Racah coefficients are related to $6-j$ symbols with better symmetry properties.

VI The $6-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$

A Parity dependent $6-j$ symbols.

By analogy with the corresponding construction of $su(2)$ $6-j$ symbols, the parity dependent $s6-j$ symbols for the quantum superalgebra $U_q(osp(1|2))$ are defined by contraction of a product of four $s6-j$ symbols with use of the parity dependent invariant metric $C^\perp_{\perp}(\lambda) :$

$$\left\{ l_1l_2l_3l_4l_5l_6 \right\} = \sum_{m=m'} \left( \prod_{a=1}^{4} C_{m_a, m'_a}(\lambda_a) \right) \left( l_1l_2l_3l_4l_5l_6 \right) \times \left( m_1m'_1 m_2m'_2 m_3m'_3 m_4m'_4 m_5m'_5 m_6m'_6 \right).$$

(6.1)

Using the explicit form (3.16) of the invariant metric, the symbol may be written

$$\left\{ l_1l_2l_3l_4l_5l_6 \right\} = \sum_{m=m'} (-1)^{\Sigma_{a=1}^{4} l_a(l_a-m_a)(l_a-m_a-1)} \left( \prod_{a=1}^{4} C_{m_a, m'_a}(\lambda_a) \right) \times \left( l_1l_2l_3l_4l_5l_6 \right) \times \left( m_1m'_1 m_2m'_2 m_3m'_3 m_4m'_4 m_5m'_5 m_6m'_6 \right).$$

(6.2)

In order for the $s6-j$ symbols to exist, the four superspin-parity triplets

$$\left\{ l_1l_2l_3l_4l_5l_6 \right\}, \left\{ l_1l_2l_3l_4l_5l_6 \right\}, \left\{ l_1l_2l_3l_4l_5l_6 \right\}, \left\{ l_1l_2l_3l_4l_5l_6 \right\},$$

must satisfy triangular constraints of the form (4.2), (4.3) and relation (4.4) for the parities. Comparison of definition (6.2) with the expression (5.35) for the $su(3)$ Racah coefficients shows that both quantities are proportional. Explicitly, taking into account a change of notation, this relation reads

$$U^*(l_1, l_2, l_3, l_1, l_2, l_3, q) = (-1)^{l_1+l_2+l_3+l_4} \frac{\left( l_1m_1 l_2m_2 l_3m_3 \right)}{[2l_1+1][2l_2+1][2l_3+1]} \times \left\{ l_1l_2l_3l_4l_5l_6 \right\} \times \left\{ l_1l_2l_3l_4l_5l_6 \right\} \times \left\{ l_1l_2l_3l_4l_5l_6 \right\} \times \left\{ l_1l_2l_3l_4l_5l_6 \right\}.$$
Note that the phase \( \Psi \) has the same symmetry as \( s_q6 - j \lambda \) symbol. The difference between the two symbols in Eq.(6.9) follows from the fact that in Ref.[10] a different basis in the representation space has been used. For more details on the bases in representation spaces, see Ref.[2].

In the next subsection it will be shown that the parity dependence of \( s_q6 - j \lambda \) symbols can be factored out so that, exactly as in the case of \( 3 - j \) symbols, it is possible to define parity independent \( 6 - j \) symbols.

**B Parity independent \( 6 - j \) symbols**

Along the same lines as in the previous subsection, we define parity independent \( s_q6 - j \) symbols for the quantum superalgebra \( U_q(sl(12)) \) by contraction of four parity independent \( s_q6 - j \) symbols with use of the parity independent metric \( C^\lambda_{\mu \nu} \):

\[
\begin{align*}
\left\{ \frac{l_1 l_2 l_3}{l_4 l_5 l_6} \right\}_q &= \sum_{m} \left( \prod_{i=1}^{6} C^\lambda_{m_i,m_i} \right) \left\{ \frac{l_1 l_2 l_3 l_4}{l_5 l_6 l_m} \right\}_q \\
&\quad \times \left( \prod_{i=1}^{6} C^\lambda_{m_i,m_i} \right) \left\{ \frac{l_4 l_5 l_6}{l_1 l_2 l_3} \right\}_q \\
&\quad \times \left( \prod_{i=1}^{6} C^\lambda_{m_i,m_i} \right) \left\{ \frac{l_1 l_2 l_3}{l_4 l_5 l_6} \right\}_q.
\end{align*}
\]

(6.11)

Using the explicit form (4.16) of the invariant metric, the symbol may be written

\[
\begin{align*}
\left\{ \frac{l_1 l_2 l_3}{l_4 l_5 l_6} \right\}_q &= \sum_{m} \left( \prod_{i=1}^{6} C^\lambda_{m_i,m_i} \right) \left\{ \frac{l_1 l_2 l_3}{l_4 l_5 l_6} \right\}_q \\
&\quad \times \left( \prod_{i=1}^{6} C^\lambda_{m_i,m_i} \right) \left\{ \frac{l_4 l_5 l_6}{l_1 l_2 l_3} \right\}_q \\
&\quad \times \left( \prod_{i=1}^{6} C^\lambda_{m_i,m_i} \right) \left\{ \frac{l_1 l_2 l_3}{l_4 l_5 l_6} \right\}_q.
\end{align*}
\]

(6.12)

The \( s_q6 - j \) symbols satisfy the same triangular conditions as the \( s_q6 - j \lambda \) symbols. Using the definition (6.12) of the \( s_q6 - j \lambda \) symbol and the analytical formula of the \( s_q6 - j \lambda \) symbols, one can derive the following proportionality relation between the \( s_q6 - j \lambda \) and \( s_q6 - j \) symbols

\[
\left\{ \frac{l_{4,5,6}}{l_{1,2,3}} \right\} = (-1)^{\Phi(l_4, l_5, l_6)} \left\{ \frac{l_{1,2,3}}{l_{4,5,6}} \right\}_q,
\]

(6.13)

where the phase \( \Phi(l_4, l_5, l_6) \) is given by

\[
\Phi(l_4, l_5, l_6) = \sum_{i=1}^{6} \lambda_i - 3 \sum_{i=1}^{6} l_i.
\]

(6.14)

Thus, all the dependence on \( \lambda_i \) in \( s_q6 - j \lambda \) symbols is contained in the phase factor. It is noticeable that \( \Phi(l_4, l_5, l_6) \) is a completely symmetric function of the six indices and hence, the \( s_q6 - j \) symbols have all the symmetry properties of the \( s_q6 - j \lambda \) symbols. Furthermore, it is easy to check that the \( s_q6 - j \) symbols satisfy pseudo-orthogonality relations similar to relations (6.5), (6.7) satisfied by the \( s_q6 - j \lambda \) symbols.

Let us remark that

\[
\Phi(l_i) = 0 \mod(2),
\]

(6.15)

therefore, the \( s_q6 - j \) symbol coincides with the \( s_q6 - j \lambda \) symbol for the fixed parity convention \( \lambda_i = l_i \mod(2) \). Note also that when one argument vanishes, the value of the \( s_q6 - j \) symbols is

\[
\left\{ \frac{l_1 l_2 l_3}{l_4 l_5 l_6} \right\}_q = \left\{ \frac{l_1 l_2 l_3}{l_4 l_5 l_6} \right\}_q = \frac{(-1)^3 s_{l_1, l_2, l_3, l_4, l_5, l_6}}{\sqrt{2l_1 + 1}[2l_2 + 1]}
\]

(6.16)

thus, in this particular case, parity dependent and parity independent \( 6 - j \) symbols are equal.

In the limit \( q \to 1 \), the \( s_q6 - j \) symbols become identical, up to the phase factor \( \Psi \), to the \( s6 - j \) symbols for the superalgebra \( U_q(12) \) defined in Ref.[10]

\[
\left\{ \frac{l_1 l_2 l_3}{l_4 l_5 l_6} \right\}_q = \left\{ \frac{l_1 l_2 l_3}{l_4 l_5 l_6} \right\}_q
\]

(6.17)

with \( l_i = 2j_i \).

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**References**