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Exactly Solvable Quantum Macroscopic Motions and Decoherence
Mechanism based on Many Quantum-Mechanical Space Theory

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ABSTRACT

Exactly solvable macroscopic systems composed of harmonic oscillators are investigated on the basis of nonstandard analysis. Wave-functions for the motions of the center of mass coordinate of the systems, which are described as ultra-eigenfunctions representing a classical ensemble of the macroscopic objects in nonstandard extension of quantum mechanics, are explicitly obtained. Many quantum-mechanical spaces are naturally introduced as the physical background of the ultra-eigenfunctions. The so-called wave-function collapse described by the transition from pure states to mixed states, that is, decoherence among different ultra-eigenfunctions, is derived from integrations with respect to the freedom of the many quantum-mechanical spaces contained in the ultra-eigenfunctions. It is also shown that this decoherent mechanism is quite different from that of coherent states reproducing classical trajectories.

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1 Introduction

How to describe macroscopic systems in quantum mechanics is an important theme in the present physics, especially, in the descriptions of physics for intermediate regions between microscopic(quantum) and macroscopic(classical) objects such as phenomena in mesoscopic regions. A difficulty in the description of macroscopic systems is how to introduce errors, contained in real measurements of physical observables, in wave-functions. We know that such errors can be treated as small as possible in classical mechanics. However, we also know the fact that almost infinite numbers of different quantum-mechanical(QM) states are contained behind such errors. Macroscopic physical observables with errors are not enough to specify a QM(quantum-mechanical) state for a macroscopic object. In order to specify it, we have to observe all the internal quantum numbers also. In realistic measurement processes for macroscopic objects the observations of all the internal quantum numbers are impossible. Therefore, we may understand that one of important points for describing macroscopic objects is how to describe the ambiguity due to unobserved internal motions which will be an origin of the errors in real measurement processes for macroscopic objects.

In general macroscopic systems are composed of N =more than 10^{23} ($\sim$ Avogadro's number) constituents and then the uncertainty of the total energy of the systems is estimated as

$$\Delta E \sim N \Delta \epsilon, \quad (1)$$

where the energy uncertainty for one constituent ($\Delta \epsilon$) is, of course, QM(quantum-mechanical) order, i.e.

$$\Delta \epsilon \sim O(\hbar). \quad (2)$$

Taking account of the fact that N is of the order of Avogadro's number, the energy uncertainty may possibly be estimated as

$$\Delta E \sim N \Delta \epsilon \sim O_N(1) \text{ or } O(\hbar), \quad (3)$$

where $O_N(1)$ means

$$\lim_{\hbar \rightarrow 0, N\hbar = \text{fixed}} O_N(1) \neq 0 \text{ but } < \infty \quad (4)$$

instead that

$$\lim_{\hbar \rightarrow 0} O(\hbar) = 0 \quad (5)$$

even in the limit with $N\hbar$ fixed. The error of $O_N(1)$ can be observed even in the so-called classical limit $\hbar \rightarrow 0$, while that of $O(\hbar)$ cannot be observed in the limit. Of course, both of them are observable in QM measurements. We may understand that ambiguities for describing macroscopic objects are due to unobserved errors with the order of $O(\hbar)$, which cannot be determined in the classical limit $\hbar \rightarrow 0$. Description of macroscopic systems having energy uncertainty of the order of $O(\hbar)$ in terms of quantum mechanics is the main theme of this paper.

On the other hand the author has shown that classical mechanics can be described by the quantum mechanics on the basis of nonstandard analysis[1], that is, on nonstandard Hilbert space[2] with $\hbar \approx 0$. [3,4] In the papers[3,4] ensembles of classical objects have been represented by ultra-eigenfunctions on the Hilbert space extended by the non-standard analysis (${}^*\mathcal{H}$). It has been shown that the ultra-eigenfunctions are solutions satisfying the approximate equality $H\psi \approx E\psi$, while the eigenfunctions satisfy the exact equality $H\psi = E\psi$. [2] Taking into account that macroscopic objects always have classical nature, such an uncertainty contained in the approximate equality, which disappears in the classical limit, seems to have certain connections with the unobserved errors in the the measurements of macroscopic objects. It is also pointed out that many QM spaces appear in the physical background of the ultra-eigenfunctions.[3,4] Since classical objects are always described as macroscopic systems, we may consider that the physical background such as many QM spaces have close connections with the macroscopic nature of the classical objects. Provided that we can explicitly write wave-functions corresponding to the ultra-eigenfunctions for macroscopic systems, we will be able more clearly to understand the reason why the unobserved errors in the macroscopic limit are introduced

in the wave-functions for macroscopic objects and also the reason why many QM spaces appear in the QM description of macroscopic objects. We may furthermore expect that the connection between the unobserved errors and the many QM spaces will be clarified. We will also find out other interesting background that the macroscopic objects should be described in terms of the nonstandard analysis. For instance, the realization in terms of ultra-products in nonstandard analysis is quite similar to the representation of states for macroscopic systems written by direct-products of constituent-particle states. We will be able to try to derive the ultra-eigenfunctions from the direct-product representations. In this paper we shall study harmonic oscillator motions of macroscopic objects composed of oscillator constituents because of exact calculability and then we shall explicitly write down wave-functions with energy uncertainty ΔE for macroscopic objects. The meaning of many QM spaces will be clarified through those discussions. The transition from pure states to mixed states(so-called wave-function collapse) which is due to the freedom of many QM spaces will be also derived in an exactly calculable scheme.

In §2 we shall study one-dimensional harmonic oscillator motions of macroscopic objects composed of arbitrary number of oscillator constituents and explicitly write down their wave-functions. In §3 the wave-functions are extended to the ultra-product space and ultra-eigenfunctions on nonstandard extension of Hilbert space(${}^*\mathcal{H}$) are derived. Relations between equivalence relation for the ultra-products based on ultra-filters and many QM physical spaces as the background of the ultra-eigenfunctions are discussed in §4 and decoherence mechanism among the ultra-eigenfunctions is studied in §4 and in §5. Similar discussions will be performed for coherent states derived from the eigenfunctions in §6. Discussions and remarks on the differences of decoherence mechanisms in comparison with other models are presented in §7.

2 Harmonic oscillator motions of many body systems

In this section we shall construct an exactly solvable model for quantum motions of many body systems in one dimensional case. Let us study a many body system composed of N equivalent particles which are each other bounded by harmonic oscillator potentials $(\frac{k}{2N} \sum_{i=1}^N \sum_{j=1}^i (x_i - x_j)^2)$ and also trapped around a fixed point X_0 by other harmonic oscillator potentials $(\frac{K}{2} \sum_{i=1}^N (x_i - X_0)^2)$. That is, the Hamiltonian of the system is given as follows;

$$H^N = \frac{1}{2m} \sum_{i=1}^N (p_i)^2 + \frac{k}{2N} \sum_{i=1}^N \sum_{j=1}^i (x_i - x_j)^2 + \frac{K}{2} \sum_{i=1}^N (x_i - X_0)^2, \quad (6)$$

where m the mass of the constituents, k and K the oscillator constants, p_i and x_i , respectively, stand for the momentum and position operators of i -th constituent. This Hamiltonian describes the bounded N -oscillator system moving in the harmonic oscillator potential of which center is at X_0 . Our interest is focused on the relative motion between the fixed point X_0 and the center of mass(CM) of the N -oscillator system, because the motion will become the observable as the motions of the macroscopic system in the macroscopic limit $N \rightarrow \infty$. We may, of course, put on a particle with the mass M at X_0 and add its kinetic term $\frac{1}{2M} P_{X_0}^2$ to the Hamiltonian (6). Such a model, however, has no essential difference, if we study the relative motion between the particle being at X_0 and the CM of the oscillator system. Actually the model given by (6) can be considered to represent the case of the limit $M \rightarrow \infty$. In the right hand side of (6) the reason why the oscillator constant of the second term is divided by the number of the constituents N such as k/N is not clear here. We may consider that this effect stands for shadow effects which appear in many body system. It is also noted that this choice is necessary to keep the finiteness of the oscillator constant for one relative coordinate in the macroscopic limit of $N \rightarrow \infty$. This fact will be seen soon later.

The Hamiltonian is separable in terms of the following choice of coordinates;

$$\begin{aligned} R^N &= X_G^N - X_0, \\ \rho_n^N &= \frac{1}{\sqrt{n(n+1)}} [nx_{n+1} - (\sum_{i=1}^n x_i)], \end{aligned} \quad (7)$$

for $n = 1, 2, \dots, N-1$, where $X_G^N = \frac{1}{N} \sum_{i=1}^N x_i$ is the CM(center of mass) coordinate of the N -oscillator system. We can rewrite (6) as

$$H^N = \frac{1}{2mN} (P^N)^2 + \frac{NK}{2} (R^N)^2 + \sum_{n=1}^{N-1} H_n, \quad (8)$$

where

$$H_n = \frac{1}{2m} p_n^2 + \frac{1}{2} (k + K) \rho_n^2. \quad (9)$$

If k/N is replaced by k in (6), the oscillator constant k in the term $k\rho^2/2$ of (9) must be kN and then it becomes infinity in the macroscopic limit $N \rightarrow \infty$. It is important that in (8) the diverging property in the limit $N \rightarrow \infty$ remains only in the total mass $M_N = mN$ and the oscillator constant $K_N \equiv NK$ for the center of mass coordinate R^N . Since the mass of the total system M_N and the oscillator constant K_N have to be physical observables in the limit, that is, we should impose the relations

$$\lim_{N \rightarrow \infty} mN = M \in \mathcal{R}, \quad (10)$$

$$\lim_{N \rightarrow \infty} NK = \hat{K} \in \mathcal{R}, \quad (11)$$

where $\mathcal{R}$ is the set of all real numbers. Furthermore we have to take into account that we are interested in the CM(center of mass) motions of the macroscopic object, which are observed as classical motions in the macroscopic limit. This means that the macroscopic limit now interested in should simultaneously realize the classical limit $\hbar \rightarrow 0$. Then we may take the right macroscopic limit (hereafter we represent it by $\lim_{macro}$) as the limit of $N \rightarrow \infty$ and $\hbar \rightarrow 0$. From (10) and (11) we know that m , k and K are required to be of the order of $1/N$ ($O(1/N)$). Note that the angular frequency $\omega \equiv \sqrt{\frac{k+K}{m}}$ for ρ_n is of the

order of $O(1)$, that is, $\lim_{macro} O(1) \neq 0$ and $\in \mathcal{R}$. We shall present the exact expression of the macroscopic limit based on the nonstandard analysis in the next section.

The eigenfunctions for (8) are obtained as follows;

$$H^N \Psi^N = (E_R + \sum_{n=1}^{N-1} \epsilon_n) \Psi^N, \quad (12)$$

where

$$\Psi^N(R^N, [\rho_n]) = \Phi_R(R^N) \prod_{n=1}^{N-1} \phi_{l_n}(\rho_n) \quad (13)$$

with $[\rho_n] \equiv [\rho_1, \rho_2, \dots, \rho_{N-1}]$, which satisfy

$$\begin{aligned} H_R \Phi_R &= E_R \Phi_R, \\ H_n \phi_{l_n} &= \epsilon_n \phi_{l_n} \end{aligned} \quad (14)$$

with $E_R = (n_R + 1/2)\omega_R \hbar$ ($\omega_R = \sqrt{K_N/M_N} = \sqrt{K/m}$) and $\epsilon_n = (l_n + 1/2)\omega \hbar$ ($\omega = \sqrt{(k+K)/m}$). Note that the eigenvalues of H^N and $\sum_n H_n$ are, respectively, given by

$$E^N = E_R + \epsilon_L^N \quad (15)$$

and

$$\epsilon_L^N \equiv \sum_{n=1}^{N-1} \epsilon_n = (L + \frac{1}{2}(N-1))\omega \hbar \quad (16)$$

with $L = \sum_{n=1}^{N-1} l_n$ and all the energies are of the order of $O(\hbar)$, i.e.

$$E^N \sim E_R \sim \epsilon_L^N \sim O(\hbar). \quad (17)$$

We see that E_R and ϵ_L^N are not enough to specify the state given in (13) uniquely. That is, there are many different states having a fixed value of ϵ_L^N , of which multiplicity is evaluated as

$$W(N, L) = \frac{(L+N-2)!}{L!(N-2)!} \simeq \frac{(L+N)^{L+N-3/2}}{L^{L+1/2} N^{N-3/2}}, \quad (18)$$

where the Stirling's formula $N! \simeq \sqrt{2\pi} e^{-N} N^{N+1/2}$ is used to derive the last equation. In general we should write eigenfunctions specified by E_N and ϵ_L^N as

$$\Psi_L^N(E^N; R^N, [\rho_n]) = \Phi_{E_R}(R^N) \sum_{l_1=0}^L \cdots \sum_{l_{N-1}=0}^L \delta_{\sum_{n=1}^{N-1} l_n, L} a([l_n]) \prod_{n=1}^{N-1} \phi_{l_n}(\rho_n), \quad (19)$$

with $[l_n] \equiv [l_1, l_2, \dots, l_{N-1}]$, where $a([l_n])$ are the coefficients satisfying the constraint required from the normalization

$$\sum_{l_1=0}^L \cdots \sum_{l_{N-1}=0}^L \delta_{\sum_{n=1}^{N-1} l_n, L} |a([l_n])|^2 = 1. \quad (20)$$

When we are interested only in macroscopic quantities of the object, which are described in terms of the CM coordinate R^N and the momentum P^N , all the eigenfunctions (19) which have a fixed value of E_R but different numbers of L s and the different coefficients $a([l_n])$ give the same expectation value for a macroscopic quantity. That is to say, all the effects due to the internal motions characterized by L and $a([l_n])$ disappear by the trace operations in the evaluations of the expectation values of the macroscopic quantities $\mathcal{O}(R^N, P^N)$ such that

$$(\Psi_L^N, \mathcal{O}(R^N, P^N) \Psi_L^N) = (\Phi_{E_R}, \mathcal{O}(R^N, P^N) \Phi_{E_R}). \quad (21)$$

Since all the coordinate variables are completely independent of each other in the present model, the macroscopic quantities do not depend on the states of the internal oscillators at all. In other words one macroscopic state represented by Φ_{E_R} has the infinitely different representations for the internal states given by (19) in the macroscopic limit where the multiplicity given in (18) becomes infinity. Note here that the difference arises from two different origins. One is due to the different combinations of $[l_n]$ counted in (18) and the other is the freedom of the choice of the coefficients $a([l_n])$ represented by $W(N, L)$ number of complex numbers.

The equation (21) also tells us that the expectation value does not depend on ϵ_L at all. This means that the most general expression for the state determined only from the CM motions should be described as

$$\Psi^N(E_R, [c_L]; R^N, [\rho_n]) = \Phi_{E_R}^N \sum_{L=0}^{L_f} c_L \phi_L([\rho_n]), \quad (22)$$

where

$$\phi_L([\rho_n]) \equiv \sum_{l_1=0}^L \cdots \sum_{l_{N-1}=0}^L \delta_{\sum_{n=1}^{N-1} l_n, L} a([l_n]) \prod_{n=1}^{N-1} \phi_{l_n}(\rho_n), \quad (23)$$

$[c_L] \equiv [c_0, c_1, \dots, c_{L_f}]$, the normalization condition is expressed as

$$\sum_{L=0}^{L_f} |c_L|^2 = 1 \quad (24)$$

and L_f may be taken as an arbitrary natural number ($L_f \in \mathcal{N}$ (the set of all natural numbers)). It is trivial that all the wave-functions with different L_f described by (22) give the same expectation value given in (21) for a macroscopic quantity, whereas the expectation values of the total energy for them have the dependence on the internal variables such as

$$\langle E \rangle \equiv (\Psi^N(E_R, [c_L]; R^N, [\rho_n]), H^N \Psi^N(E_R, [c_L]; R^N, [\rho_n])) = E_R + \sum_{L=0}^{L_f} |c_L|^2 \epsilon_L. \quad (25)$$

Note here that since the coefficients c_L can be arbitrary complex numbers except the normalization condition, the freedoms arising from the coefficients are continuous.

The density matrix for (22) is obtained as

$$\begin{aligned} \rho^N &= \Psi^N(E_R, [c_L]; R^N, [\rho_n]) \Psi^N(E_R, [c_L]; R^N, [\rho_n])^\dagger \\ &= \rho_{E_R}^N \sum_{L=0}^{L_f} \sum_{L'=0}^{L_f} c_L \bar{c}_{L'} \rho_{\epsilon_L, \epsilon_{L'}}, \end{aligned} \quad (26)$$

where

$$\rho_{E_R}^N = \Phi_{E_R}^N(R) \Phi_{E_R}^N(R)^\dagger \quad (27)$$

and

$$\rho_{\epsilon_L, \epsilon_{L'}} = \phi_L^N([\rho_n]) \phi_{L'}^N([\rho_n])^\dagger. \quad (28)$$

It is interesting that in ρ^N the density matrix for the CM variable ($\rho_{E_R}^N$) and that for the internal variables ($\rho_{\epsilon_L, \epsilon_{L'}}$) are completely factorized.

3 Eigenfunctions and ultra-eigenfunctions for the CM motions on nonstandard Hilbert space

Now let us study the limit represented by $N \rightarrow \infty$. The state given in (13) becomes an infinite direct-product

$$\Psi_{E_R, L}(R; [\rho_n]) = \phi_{E_R}(R) \prod_{n=1} \phi_{l_n}(\rho_n). \quad (29)$$

The Hamiltonian (8) is modified as

$$\hat{H} = H_R + \sum_{n=1} \hat{H}_n, \quad (30)$$

where in order to evade the divergence arising from the sum of zero point oscillations $\sum_{n=1}^{N-1} \frac{1}{2} \hbar \omega$ in the limit of $N \rightarrow \infty$ $\hat{H}_n$ is taken as

$$\hat{H}_n = H_n - \frac{1}{2} \hbar \omega. \quad (31)$$

We have

$$\hat{H} \Psi_{E_R, L}(R, [\rho_n]) = (E_R + \epsilon_L) \Psi_{E_R, L}(R, [\rho_n]), \quad (32)$$

where

$$\begin{aligned} H_R \Phi_{E_R}(R) &= E_R \Phi_{E_R}(R), \\ \hat{H}_n \phi_{l_n}(\rho_n) &= l_n \hbar \omega \phi_{l_n}(\rho_n) \end{aligned} \quad (33)$$

and $\epsilon_L = L \hbar \omega$ with

$$L = \sum_{n=1} l_n. \quad (34)$$

Note that, since $l_n \in \mathcal{N}$ for $\forall n \in \mathcal{N}$, then

$$L \in \mathcal{N}. \quad (35)$$

Following the expression of (19), we can write the most general wave-functions for the macroscopic object characterized by the CM(center of mass) energy eigenvalue E_R as

$$\Psi(E_R, [C_L([l_n]); R, [\rho_n]) = \Phi_{E_R}(R) \phi([C_L([l_n]); [\rho_n]), \quad (36)$$

where

$$\phi([C_L([l_n]); [\rho_n]) = \sum_{L=0}^{L_f} \sum_{l_1=0}^L \sum_{l_2=0}^L \cdots \delta_{\sum_{n=1} l_n, L} C_L([l_n]) \prod_{n=1} \phi_{l_n}(\rho_n), \quad (37)$$

L_f is an arbitrary natural number ($L_f \in \mathcal{N}$) and the normalization condition is given by

$$\sum_{L=0}^{L_f} \sum_{l_1=0}^L \sum_{l_2=0}^L \cdots \delta_{\sum_{n=1} l_n, L} |C_L([l_n])|^2 = 1. \quad (38)$$

We can consider that

$$C_L([l_n]) \equiv c_L a([l_n]),$$

where c_L and $a([l_n])$ are given in (22) and (23). The expectation value of the total energy is obtained as

$$\langle E \rangle = E_R + \Delta E([C_L]), \quad (39)$$

where

$$\Delta E([C_L]) \equiv \sum_{L=0}^{L_f} \sum_{l_1=0}^L \sum_{l_2=0}^L \cdots \delta_{\sum_{n=1} l_n, L} |C_L([l_n])|^2 \epsilon_L. \quad (40)$$

Now let us consider measurements of the CM energy. When we try to observe it by using a photon as a probe, we have to measure it through the interaction of the photon with the constituents. This means that we cannot measure the CM energy directly and then we have to take account of the internal motions of the macroscopic object even if the CM variables are completely factorized in the density matrix as was shown in (26). That is to say, we have to separate the CM energy and the internal energy from the observed energy of the constituent which directly interacts with the probe photon.

In realistic measurement processes for macroscopic objects, which will be carried out by using a photon flux composed of many photons (comparable with or more than the number of constituents), we should consider that direct observable is the total energy rather than the CM energy. In those measurements the total of the internal energy $\Delta E([C_L]) \equiv \langle E \rangle - \langle E_R \rangle$ may be understood to be the errors for the CM energy. Note that the errors should not be confused with those arising from inefficiencies of detectors. We may conclude that we have always to take account of the existence of these errors in the

observed CM energies when we discuss the CM motions of macroscopic objects which are studied in the classical mechanics.

In order to clarify this situation in terms of nonstandard analysis, let us go back to the expression written by the infinite direct-product given in (29). As was noted in Ref.[1], such direct-products can be bases of ultra-powers in nonstandard analysis by introducing the equivalence relation based on ultra-filters. On the nonstandard extension (${}^*\mathcal{H}$) of the Hilbert space ($\mathcal{H}$; the Hilbert space based on $\hat{H}$) in the nonstandard number space (${}^*\mathcal{R}$; the nonstandard extension of the set of real numbers $\mathcal{R}$)[2] the wave-functions derived from (29) contain not only usual eigenfunctions satisfying the exact equality

$$\hat{H}\Psi = E\Psi, \quad (41)$$

but also ultra-eigenfunctions satisfying only the approximate equality[2,3,4]

$$\hat{H}\Psi \approx_q E\Psi. \quad (42)$$

In (42) the approximate equality $\approx_q$ indicates that the norm

$$\|(\hat{H} - E)\Psi\| \approx O(q) \quad (\approx_q 0) \quad (43)$$

is infinitesimal when q is taken as an infinitesimal in ${}^*\mathcal{R}$. [3] Hereafter $O(q)(\approx_q 0)$ will represent

$$st_q(O(q)) = 0, \quad (44)$$

where the st_q -operation stands for the operation taking the standard part with respect to the infinitesimal $q \approx 0$, which is the well-defined projection of nonstandard numbers in ${}^*\mathcal{R}$ to real numbers in $\mathcal{R}$ in nonstandard analysis and corresponds to taking the limit of $q \rightarrow 0$ in $\mathcal{R}$.

Now we have to determine what are taken as infinitesimals or infinities on ${}^*\mathcal{H}$ for the observations of the CM motions of macroscopic objects. Note here that the CM motions should represent the classical motions of the macroscopic object. Taking into account

that undetectable small number and undetectable large number in the measurements of the CM motions of macroscopic objects are, respectively, Planck's constant/ 2π , $\hbar$, and the number of constituents, N , we may choose as infinitesimals in the present model,

$$\frac{1}{N} \approx 0 \quad \text{and} \quad \hbar \approx 0. \quad (45)$$

in the macroscopic limit. Then we may define the macroscopic limit in nonstandard analysis by using the standard part operation following the choice of $1/N \sim \hbar \approx 0$, i.e.

$$st_{macro} \equiv st_{(1/N, \hbar)}.$$

Since the oscillator constant of the CM motions, $\hat{K} = NK$ and the mass, $M = Nm$, should be observed in the classical CM motions, i.e. $st_{macro}(\hat{K})$ and $st_{macro}(M)$ are nonzero and finite in $\mathcal{R}$, then we shall take

$$K \sim k \sim m \sim O(1/N) \approx 0. \quad (46)$$

These relations indicate that

$$\omega = \sqrt{\frac{k}{m}} \sim \omega_R = \sqrt{\frac{\hat{K}}{M}} \sim O(1), \quad (47)$$

where $O(1)$ means that $st_{macro}(\omega)$ and $st_{macro}(\omega_R)$ are nonzero and finite in $\mathcal{R}$, i.e.

$$st_{macro}(\omega), \quad st_{macro}(\omega_R) \in \mathcal{R}/[0]. \quad (48)$$

Here we have to note that there are two different ultra-eigenfunctions derived from the expressions of (29). One is the ultra-eigenfunctions for the CM motions which were studied in Ref.s[3,4], where the physical constants, i.e. the mass M and the oscillator constant $\hat{K}$, are taken as macroscopic quantities. The other is those for the internal motions described by the infinite direct-product, even if each internal motion is described by a microscopic motion represented by the eigenfunctions with the energy eigenvalue

$$\epsilon_n = l_n \hbar \omega \sim O(\hbar) \approx 0. \quad (49)$$

It will be clear that we are now interested in the former, that is, the classical CM motions. We should, however, expect that the latter will be also important because such representations have close connection with the introduction of unobserved errors in macroscopic wave-functions. Although the latter is also important to describe the thermodynamical limit (quantum statistical mechanics).[4] we shall not discuss about it in this paper.

From now on we shall discuss only the ultra-eigenfunctions of which the energy eigenvalues E_R are of the order of $O(1)$, i.e.

$$st_{macro}(E_R) \in \mathcal{R}_+ \quad (50)$$

instead that $st_{macro}(\epsilon_n) = 0$. The fact that the total internal energy $\epsilon_L = L\hbar\omega \sim O(\hbar)$ is an infinitesimal and then the errors in the CM energy measurements represented by $\Delta E([C_L])$ is also an infinitesimal indicates that the errors can be made as small as possible in the present macroscopic limit defined by $1/N \sim \hbar \approx 0$. This is a desirable property to describe the mechanics representing the classical motions.

Let us discuss the CM motions on $\mathcal{H}$. The expectation value of the total energy of the object $E = \langle E \rangle$ is a direct observable in experiments. In the macroscopic limit it coincides with E_R as

$$st_{macro}(E) = st_{macro}(E_R) \quad (51)$$

because of $st_{macro}(\epsilon_L) = 0$. We start from the equation

$$H_R \Psi_{E_R,L} \approx_{macro} E_R \Psi_{E_R,L}, \quad (52)$$

where $E_R = E - \epsilon_L$ and $\approx_{macro}$ means

$$st_{macro}(\|(H_R - E_R)\Psi\|) = 0.$$

As was discussed in Ref.s[3,4], we have to different types of eigenfunctions for the CM motions. One is represented by the eigen-pairs $(E_R, \Psi_{E_R,L})$ which satisfy the exact equality

$$(H_R - E_R)\Psi_{E_R,L} = 0 \quad (53)$$

and the other is called ultra eigen-pairs $(E_R, \Psi_{E_R,L})$ which satisfy the approximate equality

$$(H_R - E_R)\Psi_{E_R,L} \approx_{macro} 0. \quad (54)$$

It should be noted that in the macroscopic limit the last equation for the ultra eigen-pairs satisfies the exact equality.

(1) Eigenfunctions

We can easily derive eigenfunctions satisfying the exact equality (53) as

$$H_R \Psi_{E_R,L} = (n_R + 1/2)\hbar\omega_R \Psi_{E_R,L}, \quad (55)$$

where $n_R \in {}^*\mathcal{N}$ (nonstandard extension of $\mathcal{N}$) and

$$\Psi_{E_R,L} = \Phi_{n_R}(R)\phi_L([\rho_n]) \quad (56)$$

and ϕ_L are given in (23). The wave-function $\Phi_{n_R}(R)$ can be written by

$$\Phi_{n_R}(R) = N_{n_R} H_{n_R}(\alpha R) e^{-\frac{1}{2}\alpha^2 R^2}, \quad (57)$$

where N_{n_R} stands for the normalization constant, $\alpha = \sqrt{M\omega_R/\hbar}$ and $H_{n_R}(\alpha R)$ is the Hermite polynomial extended to $n_R \in {}^*\mathcal{N}$. It is noted that the CM motions are completely independent of all the internal motions, that is, both motions can be treated as completely separable motions. Note also that the classical motions having the constraint $st_{macro}(E_R = n_R\hbar\omega_R) \neq 0$ must be expressed by n_R belonging to the nonstandard number

$$n_R \in {}^*\mathcal{N} - \mathcal{N}, \quad (58)$$

that is, $st_{macro}(n_R) = \infty$. [3,4,5] Eigenfunctions with continuous eigenvalues, which are expressed by coherent states and reproduce classical orbits in the classical limit,[3,4] will be discussed in §6.

(2) Ultra-eigenfunctions

Let us start from the most general expression of ultra-eigenfunctions satisfying the approximate equality

$$\hat{H}\Psi_c(R, [\rho_n]) \approx_{macro} E\Psi_c(R, [\rho_n]), \quad (59)$$

where $st_{macro}(E) \in \mathcal{R}$ and $\neq 0$, that is, it is observable in the classical limit. In the above equation Ψ_c is factorized with respect to the CM motion and the internal ones as

$$\Psi_c = \Phi_c(R)\phi([\rho_n]).$$

Since E can have the freedom of the order of $O(\hbar)$, the most general expression for the internal motions is given in (37) as

$$\phi([C_L([l_n])]; [\rho_n]) = \sum_{L=0}^{L_f} \sum_{l_1=0}^L \cdots \sum_{l_n=0}^L \cdots \delta_{\sum_{n=1}^{L_f} l_n, L} C_L([l_n]) \prod_{n=1}^{L_f} \phi_{l_n}(\rho_n), \quad (60)$$

of which energy expectation value is obtained as

$$\Delta E([C_L([l_n])]) \equiv \sum_{L=0}^{L_f} \sum_{l_1=0}^L \cdots \sum_{l_n=0}^L \cdots \delta_{\sum_{n=1}^{L_f} l_n, L} |C_L([l_n])|^2 \epsilon_L \sim O(\hbar). \quad (61)$$

We can derive the equation for the CM motions by operating the internal trace operation represented by the partial trace operation for all the internal variables ($\forall \rho_n$), [6] that is,

$$\begin{aligned} \langle \phi([C_L([l_n])]; [\rho_n]), H\Psi_c(R, [\rho_n]) \rangle_{internal} &= (H_R + \Delta E)\Phi_c(R) \\ &\approx_{macro} H_R\Phi_c(R). \end{aligned} \quad (62)$$

As was shown in Ref.s[3,4], it is required for us to solve the equation only in the classical region satisfying

$$st_{macro}(E - \Delta E - V(R)) \in \mathcal{R}_+.$$

In order to obtain stationary states represented by

$$\Phi_c^{E_R}(R) = \hat{N} e^{iW(R)/\hbar}, \quad (63)$$

where $\hat{N}$ denotes the normalization constant, we can reduce the Schroedinger equation to that for $W(R)$ as

$$i \frac{\hbar}{2M} \frac{d^2 W}{dR^2} \approx_{macro} \frac{1}{2M} \left(\frac{dW}{dR} \right)^2 + \frac{1}{2} \hat{K} R^2 - (E - \Delta E). \quad (64)$$

In the classical region it has been shown that $W(R)$ may be described as[3,4]

$$W(R) \approx W_{cl}^{E, \Delta E}(R) + \frac{1}{2} i \hbar \ln(u_{cl}^{E, \Delta E}(R)), \quad (65)$$

where

$$\begin{aligned} W_{cl}^{E, \Delta E}(R) &= \int^R \sqrt{2M(E - \Delta E - V(R'))} dR', \\ u_{cl}^{E, \Delta E}(R) &= \sqrt{2M(E - \Delta E - V(R))} \end{aligned} \quad (66)$$

and $V(R) = \frac{1}{2} \hat{K} R^2$. (In details for the derivation of $\Phi_c^{E_R}$, see Ref.s[3,4].) It should be stressed that $\rho_c = |\Phi_c^{E_R}|^2$ in the classical limit gives the exact distribution for the ensemble of the particles moving in the potential $V(R)$, which is expected from classical mechanics.

Let us estimate the order of the norm

$$\|\hat{H}\Psi_c^{E, \Delta E}(R, [\rho_n]) - E\Psi_c^{E, \Delta E}(R, [\rho_n])\|, \quad (67)$$

where $\Psi_c^{E, \Delta E}$ is given by

$$\Psi_c^{E, \Delta E} = \frac{\hat{N}_E}{\sqrt{u_{cl}^{E, \Delta E}}} e^{iW_{cl}^{E, \Delta E}(R)/\hbar} \phi([C_L([l_n])]; [\rho_n]) \quad (68)$$

in the classical region. Taking account of the result that we can put

$$\Psi_c = 0$$

in the non-classical region defined by $st_{macro}(E - \Delta E - V(R)) \leq 0$, [3,4]

$$\begin{aligned} \|(\hat{H} - E)\Psi_c^{E, \Delta E}\| &\approx_{macro} \|(H_R - E)\Psi_c^{E, \Delta E} + \Phi_c^{E_R} \sum_{n=1}^{L_f} \hat{H}_n \phi([C_L([l_n])])\| \\ &\approx_{macro} \|\Delta E \Psi_c^{E, \Delta E} + \Phi_c^{E_R} \sum_{n=1}^{L_f} \hat{H}_n \phi([C_L([l_n])])\| \\ &\leq \|\Delta E \Psi_c^{E, \Delta E}\| + \|\Phi_c^{E_R} \sum_{n=1}^{L_f} \hat{H}_n \phi([C_L([l_n])])\| \\ &= \Delta E + \left\{ \sum_{L=0}^{L_f} \sum_{l_1=0}^L \cdots \sum_{l_n=0}^L \cdots \delta_{\sum_{n=1}^{L_f} l_n, L} |C_L([l_n])|^2 \epsilon_L^2 \right\}^{1/2}. \end{aligned} \quad (69)$$

Since ΔE and ϵ_L are of the order of $O(\hbar)$, we can obtain

$$\|H\Psi_c^{E, \Delta E} - E\Psi_c^{E, \Delta E}\| \approx_{macro} 0. \quad (70)$$

Now we may say that $\Psi_c^{E,\Delta E}$ is the ultra-eigenfunction. It is very interesting that $W_{cl}^{E,\Delta E}(R)$ has dependence on the coefficients $[C_L([l_n])]$ through $\Delta E([C_L([l_n])])$ and all the coefficients are free parameters except the constraint due to the normalization. We shall discuss in details about this freedom in the next section.

Here we would like to notice about the orthogonality among the solutions. The orthogonality between two eigenfunctions with different energies $E_R \neq E_{R'}$, i.e. $n_R \neq n_{R'}$, is trivial, because it is reduced to the orthogonality between two extended Hermite polynomials with $n_R \neq n_{R'}$. The orthogonality between two ultra-eigenfunctions with different energies $E \neq E'$ and that between an eigenfunction with the energy E and an ultra-eigenfunction with the energy E' can be directly shown by evaluating

$$\begin{aligned} st_{macro}(\langle \Psi_c^{E,\Delta E}, \Psi_c^{E',\Delta E'} \rangle) &= 0, \\ st_{macro}(\langle \Psi_c^{E,\Delta E}, \Psi_{E_R,L} \rangle) &= 0, \end{aligned} \quad (71)$$

which are due to the diverging phases such as

$$st_{macro}((W_{cl}^{E,\Delta E}(R) - W_{cl}^{E',\Delta E'}(R))/\hbar) = \infty$$

and

$$st_{macro}(W_{cl}^{E,\Delta E}(R)/\hbar) = \infty.$$

In details, see Ref.[3,4].

4 Many QM physical spaces and equivalence relation of ultra-eigenfunctions

In the last section the ultra-eigenfunction $\Psi_c^{E,\Delta E}$ has infinite degrees of freedom with respect to the coefficients $[C_L([l_n])]$ in the macroscopic limit where the number of the constituents goes to infinity. It is noted that the CM(center of mass) wave-function also

has the C_L -dependence in $W_{cl}^{E,\Delta E}(R)$ via ΔE , while the CM wave-function of the eigenfunctions $\Phi_{n_R}(R)$ does not have the dependence. Since ΔE is of the order of $O(\hbar)$ as noted in the last section, we easily see that all the wave-functions with different values of ΔE being the order of $O(\hbar)$ are each other equivalent. The physical space representing the CM motions of the macroscopic objects ($\mathcal{S}_{macro}(\mathcal{H})$) must be defined by the quotient space of $\mathcal{H}$ with respect to these equivalent representations which bring the same expectation value for each macroscopic observable,[2,3,4] that is,

$$\mathcal{S}_{macro}(\mathcal{H}) = \mathcal{H}/[\text{equivalent representations}].$$

Let us study the details of the equivalence relation for the internal states represented by the direct-products(ultra-products). We consider the following wave-function

$$\Psi_c^{E_R,\epsilon_L}(R, [\rho_n]) = \Phi_c^{E_R}(R) \phi_L([\rho_n]), \quad (72)$$

where $\Phi_c^{E_R}$ given in (68) and ϕ_L given in (23), respectively, satisfy

$$\begin{aligned} H_R \Phi_c^{E_R}(R) &\approx_{macro} E_R \Phi_c^{E_R}(R), \\ \hat{H}_n \phi_L([\rho_n]) &= \epsilon_L \phi_L([\rho_n]). \end{aligned} \quad (73)$$

Note that the energy eigenvalue $\epsilon_L = L\hbar\omega$ with $L \in \mathcal{N}$ is of the order of $O(\hbar)$, i.e.

$$st_{macro}(\epsilon_L) = 0.$$

Taking account of $\Delta E \sim O(\hbar)$ and the relation

$$\hat{H} \Psi_c^{E_R,\epsilon_L} \approx_{macro} E \Psi_c^{E_R,\epsilon_L}$$

with $E = E_R + \epsilon_L$, we can derive the following relation;

$$\begin{aligned} \|\hat{H} \Psi_c^{E_R,\epsilon_L} - (E_R + \Delta E) \Psi_c^{E_R,\epsilon_L}\| &= \|(\hat{H} - E) \Psi_c^{E_R,\epsilon_L} - (\epsilon_L - \Delta E) \Psi_c^{E_R,\epsilon_L}\| \\ &\leq \|(\hat{H} - E) \Psi_c^{E_R,\epsilon_L}\| + |\Delta E - \epsilon_L| \\ &\approx_{macro} 0. \end{aligned} \quad (74)$$

This relation means that all the ultra-eigenfunctions $\Psi_c^{E_R, \epsilon_L}$ characterized by a fixed value of E_R but different numbers of L s are equivalent to arbitrary wave-functions $\Psi_c^{E, \Delta E}$ written down in terms of superposition of $\Psi_c^{E_R, \epsilon_L}$ on ${}^*\mathcal{H}$. We can explicitly write this equivalence relation for the macroscopic objects by using the ultra-filter in nonstandard analysis as follows;

$$\phi_L([\rho_n]) = \prod_{n=1} \phi_{l_n}(\rho_n) \quad (75)$$

with $\epsilon_L = \sum_n l_n \hbar \omega$ and

$$\phi_{L'}([\rho_{n'}]) = \prod_{n'=1} \phi_{l'_{n'}}(\rho_{n'}) \quad (76)$$

with $\epsilon_{L'} = \sum_{n'} l'_{n'} \hbar \omega$ are equivalent ($\longleftrightarrow_{macro}$), if and only if the number of $n \in \mathcal{N}$ satisfying $l_n \neq 0$ and that of $n' \in \mathcal{N}$ satisfying $l'_{n'} \neq 0$ are finite numbers. That is, it is represented as

$$\phi_L \longleftrightarrow_{macro} \phi_{L'}.$$

if and only if the sets of numbers defined by

$$(n \in \mathcal{N}; l_n \neq 0) \text{ and } (n' \in \mathcal{N}; l'_{n'} \neq 0) \text{ are finite sets of } \mathcal{N}. \quad [[A]]$$

Now we can represent the physical space for the CM motions of the macroscopic objects by

$$\mathcal{S}_{macro}({}^*\mathcal{H}) = {}^*\mathcal{H} / \longleftrightarrow_{macro}.$$

Since it is known in nonstandard analysis that there exist ultra-filters containing the relation $[[A]]$ which is represented by Fréchet filter described by the set of the complementary sets of all the finite subsets on $\mathcal{N}$, [1] we may say that the relation $[[A]]$ is a right equivalence relation for the ultra-products on ${}^*\mathcal{H}$. It may be said that this equivalence relation for the macroscopic objects is the right equivalence relation on which the classical ensemble is based. We may also understand that the freedom in $\Psi_c^{E, \Delta E}$ introduced by the coefficients $[C_L]$ originates from the existence of the different expressions for the equivalent internal states. It seems that this freedom naturally represents the situation

for the classical states which always contain some differences in the quantum level even if they are equivalent in the classical level.

Let us study this freedom in details. As noted in Ref.s[3,4], the equation with the approximate equality

$$(H_R - (E - \epsilon_L))\Psi_c^{E_R, \epsilon_L} \approx_{macro} 0 \quad (77)$$

can be rewritten by using that with the exact equality as follows;

$$(H_R - (E - \epsilon_L - \delta V_L))\Psi_c^{E_R, \epsilon_L} = 0, \quad (78)$$

where

$$\delta V_L = \frac{1}{\Phi_c^{E_R}(R)}(H_R - E + \epsilon_L)\Phi_c^{E_R}(R) \quad (79)$$

in the classical region. Since $\Phi_c^{E_R}$ is smooth and has no zero point in the classical region, δV_L is a smooth function of R . And δV_L is of the order of $O(\hbar)$ in the region, because $(H_R - E)\Phi_c^{E_R} \approx_{macro} 0$ in the region. Taking into account that δV_L depends on ϵ_L , the Hamiltonians given by $\hat{H}_L = \hat{H} + \delta V_L$ with different L s are each other different in the QM (quantum mechanical) level where $\hbar \neq 0$ and $st(\hbar) \in \mathcal{R}_+$. In other words, in the background of the ultra-eigenfunction $\Psi_c^{E_R, \Delta E}$ there are infinitely many number of different QM physical spaces defined by $\hat{H}_L$ which are each other different in the QM level. Note, however, that there is one to one correspondence between $\hat{H}_L$ and $\Psi_c^{E_R, \epsilon_L}$ which satisfy the exact equality

$$\hat{H}_L \Psi_c^{E_R, \epsilon_L} = (E_R + \epsilon_L) \Psi_c^{E_R, \epsilon_L}$$

in the classical region. This situation such that the ultra-eigenfunctions representing the classical ensembles defined by the equivalence relation $[[A]]$ provide the many QM physical spaces as the background seems to be a quite beautiful description for representing the situation that we can find differences between any equivalent macroscopic objects by performing the observations from the QM level, that is, by performing the observations of the internal motions.

5 Decoherence among ultra-eigenfunctions and many QM physical spaces

As was shown in the last section, the most general expression of the ultra-eigenfunctions given in (68) has the $[C_L([l_n])]$ -dependence which arises from the effect of the almost infinitely large number of different QM(quantum mechanical) physical spaces. Through the observations of classical quantities written only by the CM(center of mass) variables, we can not fix the coefficients $[C_L([l_n])]$ at all. In other words the CM energy is determined only within the error $\Delta E([C_L([l_n])])$, for which only the constraint $st_{macro}(\Delta E([C_L([l_n])])) = 0$ is required. Therefore, we may introduce integration procedures with respect to the coefficients $[C_L([l_n])]$ in order to take off the apparent dependence on those unobservable parameters in the density matrices. Let us study this situation in the density matrix for the following superposed state of two ultra-eigenfunctions with different energies, $st_{macro}(E - E') \neq 0$,

$$\Psi^{E,E';\Delta E} = c_E \Psi_c^{E,\Delta E} + c_{E'} \Psi_c^{E',\Delta E}. \quad (80)$$

where $|c_E|^2 + |c_{E'}|^2 = 1$. The density matrix is given by

$$\begin{aligned} \rho_c^{E,E';\Delta E} &= |c_E|^2 \rho_c^{E,E;\Delta E} + |c_{E'}|^2 \rho_c^{E',E';\Delta E} \\ &+ (c_E c_{E'}^* \Psi_c^{E,\Delta E} \Psi_c^{E',\Delta E\dagger} + h.c.). \end{aligned} \quad (81)$$

In order to obtain the density matrix for the CM motions which is independent of the coefficients, we introduce the integrations with respect to the undetermined complex coefficients $[C_L([l_n])]$. Here we would like to introduce more convenient expressions for the coefficients instead of $[C_L([l_n])]$ s. The number of the coefficients is counted as

$$\hat{W} \equiv \sum_{L=0}^{L_I} W(N, L), \quad (82)$$

where $W(N, L)$ is the number of the different combinations for $[l_n]$ given in (18). The multiplicity $\hat{W}$ is same as that of the equivalent internal wave-functions $\phi_L([l_n])$. Then

we can rewrite the internal state $\phi([C_L([l_n])])$ given in (60) as

$$\phi(I; [\rho_n]) = \sum_{I=1}^{\hat{W}} C_I \phi_I([\rho_n]), \quad (83)$$

where $[C_I] \equiv [C_1, C_2, \dots, C_{\hat{W}}]$ are the new coefficients and $\phi_I([\rho])$ stands for the internal wave-function corresponding to the number I . Of course, they satisfy the relation

$$\langle \phi_I, \phi_{I'} \rangle = \delta_{I,I'}.$$

The energy expectation value given in (61) is also rewritten by

$$\Delta E([C_I]) = \sum_{I=1}^{\hat{W}} |C_I|^2 \epsilon_I. \quad (84)$$

Using these coefficients, we can write the integrations with respect to $[C_I]$ as follows;

$$\hat{\rho}_c^{E,E'} \equiv \prod_{I=1}^{\hat{W}} \int d^2 C_I \mathcal{G}(C_I) \rho_c^{E,E';\Delta E}, \quad (85)$$

where $\int d^2 C_I$ stands for the integrals with respect to the real and imaginary parts of C_I and $\mathcal{G}(C_I)$ is the metric function for C_I satisfying the condition

$$st_{macro}(\prod_{I=1}^{\hat{W}} \int d^2 \mathcal{G}(C_I) \sum_{I'=1}^{\hat{W}} |C_{I'}|^2) = 1 \quad (86)$$

so as to derive the normalization condition

$$st_{macro}(\text{Tr}(\hat{\rho}_c^{E,E'})) = 1.$$

Note that the constraint $\sum_I |C_I|^2 = 1$ is not required here. We may impose the relation for the metric function

$$\mathcal{G}(C_I) \geq 0 \quad (87)$$

at least in the region for finite values of $|C_I|$, because the coefficient should be in general taken as an arbitrary value except the normalization condition (86). For the simplicity of discussions and also from the constraint (87) we shall take as

$$\mathcal{G}(C_I) = \frac{1}{2\pi} \mathcal{G}(|C_I|) > 0, \quad (88)$$

that is, no dependence on the phase of C_I .

In the density matrix the integrations are written down as follows;

$$\prod_I \int d^2 C_I \mathcal{G}(C_I) \sum_{I'} \sum_{I''} C_{I'} C_{I''}^* \frac{\hat{N}_E \hat{N}_{E'}^*}{\sqrt{u_{cl}^{E, \Delta E}(R) u_{cl}^{E', \Delta E'}(R)}} \times e^{i \frac{1}{\hbar} (W_{cl}^{E, \Delta E}(R) - W_{cl}^{E', \Delta E'}(R))} \phi_{I'}([\rho_n]) \phi_{I''}([\rho_n])^*, \quad (89)$$

where $\Delta E = \sum_{I'} |C_{I'}|^2 \epsilon_{I'}$ and $\Delta E' = \sum_{I''} |C_{I''}|^2 \epsilon_{I''}$. Let us start from the study of the diagonal term with $E = E'$, which is given by

$$\prod_I \int d^2 C_I \mathcal{G}(C_I) \sum_{I'} \sum_{I''} C_{I'} C_{I''}^* \phi_{I'}([\rho_n]) \phi_{I''}([\rho_n])^* \frac{|\hat{N}_E|^2}{\sqrt{u_{cl}^{E, \Delta E}(R) u_{cl}^{E, \Delta E}(R)}}. \quad (90)$$

It is rewritten as

$$\prod_I \frac{1}{2\pi} \int d^2 C_I \mathcal{G}(|C_I|) \left[\sum_{I'} |C_{I'}|^2 |\phi_{I'}|^2 + \sum_{I' \neq I''} C_{I'} C_{I''}^* \phi_{I'} \phi_{I''}^* \right] \frac{|\hat{N}_E|^2}{\sqrt{u_{cl}^{E, \Delta E}(R) u_{cl}^{E, \Delta E}(R)}}. \quad (91)$$

The second term becomes zero because of the integrations with respect to the phases of $C_I = |C_I| e^{i\theta_I}$ from zero to 2π . We may consider that this result, that is, the disappearance of the off-diagonal terms with respect to the internal states, looks similar to that derived by the internal trace operation with respect to the internal states[6] in quantum measurement processes. However, we notice that the introduction of the internal trace operation is not needed to derive the disappearance in the present model. Then we can evaluate the diagonal term as

$$\hat{\rho}_c^{E, E}(R, [\rho_n]) = \prod_I \int d|C_I| \mathcal{G}(|C_I|) \left[\sum_{I'} |C_{I'}|^2 |\phi_{I'}|^2 \right] \frac{N_E^2}{u_c^{E, \Delta E}(R)}. \quad (92)$$

Trivially the relation

$$st_{macro} Tr(\hat{\rho}_c^{E, E}(R, [\rho_n])) = 1$$

is certified by the constraint (86).

Now let us estimate the interference terms. Taking into account that differences with the order $O(\hbar)$ having no contribution in the st_{macro} -operation are allowed in the expression of $\Phi_c^E(R)$ and also the order of the error ΔE is $O(\hbar)$, we can use the following

equivalent expression for $W_{cl}^{E, \Delta E}(R)$ in the classical region $0 < st_{macro}(E - V(R)) \in \mathcal{R}_+$

$$W_{cl}^{E, \Delta E}(R) \approx_{macro} \int^R \sqrt{2M(E - V(R'))} dR' - \frac{1}{2} \Delta E (|C_I|) \int^R \sqrt{\frac{2M}{E - V(R')}} dR'. \quad (93)$$

Then we can write the off-diagonal term as

$$\hat{\rho}_c^{E, E'}(R, [\rho_n]) \approx_{macro} \prod_I \int d^2 C_I \mathcal{G}(C_I) \sum_{I'} \sum_{I''} C_{I'} C_{I''}^* \frac{\hat{N}_E \hat{N}_{E'}^*}{\sqrt{u_{cl}^E(R) u_{cl}^{E'}(R)}} \times e^{-\frac{i}{2} \sum_{I'} I' |C_{I'}|^2 \omega(f(E; R) - f(E'; R))} \times e^{i \frac{1}{\hbar} (W_{cl}^E(R) - W_{cl}^{E'}(R))} \phi_{I'}([\rho_n]) \phi_{I''}([\rho_n])^*, \quad (94)$$

where

$$\begin{aligned} u_{cl}^E(R) &= \sqrt{2M(E - V(R))}, \\ W_{cl}^E(R) &= \int^R \sqrt{2M(E - V(R'))} dR', \\ f(E; R) &= \int^R \sqrt{\frac{2M}{E - V(R')}} dR' \end{aligned} \quad (95)$$

and $u_{cl}^{E, \Delta E}(R) \approx_{macro} u_{cl}^E(R)$ is used. Note that the function $f(E; R)$ is a real function in the classical region. As same as the second term in (91), the terms with $I' \neq I''$ disappear by the integrations over the phases of C_I s. The remaining terms with $I' = I''$ include the following integrals with respect to $|C_I|$ s

$$\prod_{I=1}^{\tilde{N}} \int d|C_I| \mathcal{G}(|C_I|) e^{i a_I |C_I|^2} \sum_{I'} |C_{I'}|^2, \quad (96)$$

where

$$a_I = -\frac{1}{2} (f(E; R) - f(E'; R)) \omega \neq 0 \quad (97)$$

in the classical region. The normalization (86) can be rewritten as

$$\begin{aligned} \sum_{I'=1}^{\tilde{N}} \langle |C_{I'}|^2 \rangle &= \prod_{I \neq I'}^{\tilde{N}} \langle 1 \rangle = \\ \langle 1 \rangle^{\tilde{N}-1} \langle \sum_{I'=1}^{\tilde{N}} |C_{I'}|^2 \rangle &= 1, \end{aligned} \quad (98)$$

where

$$\langle A_I \rangle \equiv \int d|C_I| \mathcal{G}(|C_I|) A_I.$$

Taking into account that this equation must be satisfied for arbitrary number of $\hat{W} \in \mathcal{N}$, it is reasonable to impose the following relations

$$\sum_{I'=1}^{\hat{W}} \langle |C_{I'}|^2 \rangle = 1 \quad (99)$$

and

$$\langle 1 \rangle = 1. \quad (100)$$

We obtain the relations

$$q_I \equiv \langle e^{iI a_I |C_I|^2} \rangle < \langle 1 \rangle = 1 \quad (101)$$

and

$$\left| \sum_{I'=1}^{\hat{W}} \langle |C_{I'}|^2 e^{iI' a_{I'} |C_{I'}|^2} \rangle \right| < \sum_{I'=1}^{\hat{W}} \langle |C_{I'}|^2 \rangle = 1 \quad (102)$$

because of $\mathcal{G}(|C_I|) > 0$ and $\forall a_I \neq 0$ and $\neq 0$ for $\forall I \in \mathcal{N}$ in the classical region. We may estimate the integrations as

$$\left| \sum_{I'=1}^{\hat{W}} \langle |C_{I'}|^2 e^{iI' a_{I'} |C_{I'}|^2} \rangle \prod_{I \neq I'} \langle e^{iI a_I |C_I|^2} \rangle \right| < (q_{max})^{\hat{W}-1} \approx_{macro} 0, \quad (103)$$

where q_{max} denotes the maximum number among q_I s and the last equality is derived from the fact that $q_{max} < 1$ and $\hat{W}$ goes to infinity in the macroscopic limit where the number of constituents is taken to be infinity. From the above result we know that the magnitudes of the off-diagonal terms in $\hat{\rho}_c^{E,E'}$ are infinitesimal. It means that the contributions from the off-diagonal terms are always infinitesimal in the evaluation of expectation values for all operators ($\mathcal{O}$) which are written only in terms of the CM variables and also have finite expectation values with respect to $\hat{\rho}_c^{E,E}$ in the macroscopic limit, i.e.

$$sl_{macro}(Tr(\mathcal{O} \hat{\rho}_c^{E,E})) \in \mathcal{R}. \quad (104)$$

Note that the above constraint (104) is not a special constraint but it must be required for the observability of $\mathcal{O}$ in the macroscopic limit. Now we may say that the interference terms in $\hat{\rho}_c$ disappear and then the so-called wave-function collapse represented by the transition from pure states to mixed states has been obtained in the macroscopic limit. It should be also noted that this decoherence property among the ultra-eigenfunctions in the macroscopic limit is a necessary property in order that the ultra-eigenfunctions represent the classical motions of the macroscopic objects.

6 Comment on coherent states for eigenfunctions

In §3 we obtained the eigenfunctions $\Psi_{E_R,L}$, satisfying the exact equality on $\mathcal{H}$, such that

$$(H_R - E_R) \Psi_{E_R,L} = 0$$

with $E_R = (n_R + 1/2)\hbar\omega$. In this section we use the notation $\omega = \sqrt{\hat{K}/M}$ instead of ω_R used up to now because of the simplification of the notation in the following equations. Here we comment on the relation between the eigenfunctions and classical trajectories.[3,4] The eigenfunctions are separable into the CM(center of mass) variable and the internal variables as

$$\Psi_{n_R,L} = \Phi_{n_R}(R) \phi_L([\rho_n]),$$

where

$$\sum_n \hat{H}_n \phi_L = \epsilon_L \phi_L.$$

The coherent states for the CM motions can be obtained as follows;

$$\Psi_{ch}(R, [\rho_n]; t) = \sum_{n_R=0}^{\infty} b_{n_R} \Psi_{n_R,L} e^{-i(E_R + \epsilon_L)t/\hbar}, \quad (105)$$

where

$$b_{n_R} = \sqrt{\frac{\alpha}{\sqrt{\pi}}} \frac{(A/\sqrt{2})^{n_R}}{n_R!} e^{-|A|^2/2} \quad (106)$$

with $A = |A|e^{i\phi}$. We have the coherent state

$$\begin{aligned}\Psi_{ch}^E(R, [\rho_n]; t) &= \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} \exp\left[-\frac{\alpha^2}{2}\left(R - \frac{\sqrt{2}|A|}{\alpha}\cos(\omega t - \phi)\right)^2\right] \\ &\times \exp\left[-i\left(\frac{\omega}{2}t + \sqrt{2}|A|\alpha\sin(\omega t - \phi)\left(R - \frac{\sqrt{2}|A|}{\alpha}\cos(\omega t - \phi)\right)\right)\right] \\ &\times \phi_L([\rho_n])e^{-ie_L t/\hbar},\end{aligned}\quad (107)$$

where

$$\langle E_R \rangle = (|A|^2 + \frac{1}{2})\hbar\omega \approx_{macro} |A|^2\hbar\omega \equiv E_0$$

and

$$\langle n_R \rangle = |A|^2.$$

As was shown in Ref.s[3,4], the density matrix of the coherent state

$$\begin{aligned}\rho_{ch}^{E_0} &\equiv |\Psi_{ch}^{E_0}|^2 \\ &= \sqrt{\frac{M\omega}{\pi\hbar}} \exp\left[-\frac{\omega}{\hbar}\left(R - \frac{\sqrt{2}|A|}{\alpha}\cos(\omega t - \phi)\right)^2\right] \phi_L \phi_L^*,\end{aligned}\quad (108)$$

We see that in the macroscopic limit the part representing the CM motions in the density matrix is reduced to the δ -function as

$$\lim_{macro} \sqrt{\frac{M\omega}{\pi\hbar}} \exp\left[-\frac{M\omega}{\hbar}\left(R - \frac{\sqrt{2}|A|}{\alpha}\cos(\omega t - \phi)\right)^2\right] = \delta\left(R - \sqrt{\frac{2E_0}{M\omega^2}}\cos(\omega t - \phi)\right) \quad (109)$$

in the real number space $\mathcal{R}$. From (109) it is apparent that the coherent state reproduces the classical trajectory in the macroscopic limit.[3,4]

Let us study decoherence property among the coherent states. The density matrix of the superposed state

$$\Psi_{ch}^{E_0, E'_0} = c_{E_0} \Psi_{ch}^{E_0} + c_{E'_0} \Psi_{ch}^{E'_0}, \quad (110)$$

with $st_{macro}(E_0 - E'_0) \neq 0$ is given by

$$\begin{aligned}\rho_{ch}^{E_0, E'_0} &= |c_{E_0}|^2 \rho_{ch}^{E_0} + |c_{E'_0}|^2 \rho_{ch}^{E'_0} \\ &+ (c_{E_0} c_{E'_0}^* \Psi_{ch}^{E_0} \Psi_{ch}^{E'_0*} + h.c.).\end{aligned}\quad (111)$$

In the macroscopic limit $\rho_{ch}^{E_0}$ and $\rho_{ch}^{E'_0}$ behave as the δ -functions for the CM variables as was shown in (109). On the other hand the interference terms are obtained as follows;

$$\begin{aligned}\rho_{ch}^{E_0, E'_0, off} &= \sqrt{\frac{M\omega}{\pi\hbar}} \exp\left[-\frac{M\omega}{\hbar}\left(R - \frac{1}{\sqrt{2M\omega^2}}(\sqrt{E_0}e^{i(\omega t - \phi)} + \sqrt{E'_0}e^{-i(\omega t - \phi')})^2\right)\right] \\ &\times \exp\left[-\frac{1}{2\hbar\omega}(\sqrt{E_0} - \sqrt{E'_0})^2 - \frac{1}{2\hbar\omega}\sqrt{E_0 E'_0}(1 - e^{-i(\phi - \phi')})\right] \\ &\times \phi_L \phi_L^* e^{-i(\epsilon_L - \epsilon_{L'})t/\hbar}.\end{aligned}\quad (112)$$

For observables ($\mathcal{O}$) which do not contain any divergence in the st -operation for the trace of the diagonal terms of (111), $st_{macro}(Tr(\mathcal{O}\rho_{ch}^{E_0}))$ for $\forall E_0$ with $st_{macro}(E_0) \in \mathcal{R}_+$, we have

$$st_{macro}(Tr(\mathcal{O}\rho_{ch}^{E_0, E'_0, off})) = 0 \quad (113)$$

because the Gaussian dumping factor, $\exp[-\frac{1}{2\hbar\omega}(\sqrt{E_0} - \sqrt{E'_0})^2]$, leads it to zero in the macroscopic limit. We may say that all the interferences among coherent states disappear in the macroscopic limit. The decoherence property among the coherent states has been proved.

7 Discussions and remarks on decoherence mechanisms

We have explicitly obtained the wave-functions for macroscopic objects, which contain undetectable errors for the CM(center of mass) energy induced from unobserved internal motions of the macroscopic objects in the macroscopic limit, in terms of the ultra-power representation of nonstandard analysis. The equivalence relation given by $[[A]]$ for the ultra-powers may be considered to be the right equivalence relation to define the classical ensemble on ${}^*\mathcal{H}$ (the nonstandard extension of Hilbert space $\mathcal{H}$). Actually we can show that the density matrices of the ultra-eigenfunctions reproduce the right distributions for the corresponding classical motions and, furthermore, all the interference among the ultra-eigenfunctions with different (not infinitesimal difference) energies disappears in the

macroscopic limit, whereas the superposition of the ultra-eigenfunctions are allowed in the same way of the quantum mechanics on the usual Hilbert space $\mathcal{H}$. That is to say, there is no difference between pure states and mixed states on $\mathcal{S}_{macro}(\mathcal{H})$ (the physical space for the CM motions of macroscopic objects).

Another important property of the equivalence relation appears in the fact that many different QM (quantum mechanical) physical spaces exist in the background of the equivalence relation. This situation is quite similar to many Hilbert space theory proposed in the quantum theory of measurements by Machida and Namiki.[7] The integrations over different QM spaces have intuitively introduced like a super-selection rule in their model, while they are naturally introduced as the integrations with respect to the freedom in the representation of the ultra-eigenfunctions in the present model. We would like to comment on the difference between two ideas for the introduction of many QM spaces. In their model the origin of many Hilbert spaces has been interpreted as the repetition of measurements in QM experiments. That is to say, one experiment consists of many repetitions of measurements for an ensemble of objects. In each measurement the Hamiltonian of the detector which is always constructed from macroscopic objects is not the same as those in other measurements. This is the reason why many Hilbert spaces are introduced in QM measurement processes in their idea. In our model, however, the introduction of many QM spaces are due to the fact that the representation of each wave-function (ultra-eigenfunction) for macroscopic objects has the freedom corresponding to undetectable ambiguities which arise from the unobserved internal motions contained in the macroscopic motions of macroscopic objects. The repetition of the measurements is not the base of the many QM spaces in our model. In order to understand whether both ideas are equivalent or not and also whether they are each other compatible in one process if they are different, we have to investigate much more about the relation between the two ideas.

Here we would like to remark the difference between the decoherence mechanisms

presented in §§5 and 6. The mechanism for the ultra-eigenfunctions in §5 stems from many QM spaces induced from the unobserved internal motions, whereas that for the coherent states in §6 does not depend on such ambiguities at all. In classical mechanics the former represents the statistical distribution of the CM variable for the classical ensemble of macroscopic objects, whereas the latter does the classical trajectory of the CM variable of one macroscopic object. It is very interesting that the decoherence mechanisms are quite different between the classical solutions representing the statistical nature (ultra-eigenfunctions) of the wave-functions and those representing the particle nature (coherent states) of the wave-functions. It seems that this difference of the decoherence mechanisms separates classical mechanics into wave mechanics and particle mechanics.

Finally we have to argue about the difference between the decoherence mechanism for the ultra-eigenfunctions proposed in Refs.[3,4] and that presented in §5. In the former the decoherence is derived from the diverging phases between arbitrary two ultra-eigenfunctions with different energies, $st_{macro}(E - E') \neq 0$ and $E, E' \in \mathcal{R}$, described as

$$st_{macro}((W_{cl}^E - W_{cl}^{E'})/\hbar) = \infty. \quad (114)$$

Although those phases are included in the density matrix (94), the decoherence is not due to such phases in the latter. The difference may be interpreted as follows: In the former we cannot directly show that the interference terms become infinitesimal and we can only show that the contributions from the interference terms to any expectation values for classical observables are infinitesimal. In the latter, however, we can directly show that the interference terms become infinitesimal without taking any expectation values. At present we cannot say which mechanism is important. We have to investigate other cases more before giving the conclusion. It seems that there are some different mechanisms leading the decoherence and one or some of those mechanisms simultaneously appear in one physical process. We must make efforts for finding all possible decoherence mechanisms. And then each physical process should be characterized in terms of the decoherence

mechanisms acting on the process, as was noted in the last discussion for wave mechanics and particle mechanics in classical mechanics. We will be able to classify all possible mechanics in terms of the difference of the decoherence mechanisms.

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