MIXED COLD-HOT DARK MATTER MODEL WITH FALLING AND QUASI-FLAT INITIAL PERTURBATION SPECTRA

Dmitri Yu. Pogosyan*
CITA, Univ. of Toronto, Toronto ONT M5S 1A7, Canada
pogosyan@cita.utoronto.ca

Alexei A. Starobinsky
Yukawa Institute for Theoretical Physics,
Kyoto University, Uji 611, Japan
and
Landau Institute for Theoretical Physics,
Kosygina St. 2, Moscow 117334, Russia
alstar@yisun1.yukawa.kyoto-u.ac.jp

September 28, 1994

*On leave of absence from Tartu Astrophysical Observatory, Tõravere, EE2444, Estonia
Abstract

The mixed cold-hot dark matter cosmological model (CHDM) with $\Omega_{\text{tot}} = 1$ and a falling power-law initial spectrum of Gaussian adiabatic perturbations ($n > 1$) is tested using recent observational data. It is shown that its fit to the data becomes worse with the growth of $n - 1$, and may be considered as unreasonable for $n > 1.1$ for all possible values of the Hubble constant. Thus, the CHDM model with a falling initial spectrum is worse than the same model with the approximately flat ($|n - 1| < 0.1$) spectrum. On the other hand, the CHDM model provides a rather good fit to the data if $n$ lies in the range $(0.9 - 1.0)$, the Hubble constant $H_0 < 60 \text{ km/s/Mpc}$ ($H_0 < 55$ for $n = 1$) and the neutrino energy density $\Omega_\nu < 0.25$. So, the CHDM model provides the best possibility for the realization of the simplest variants of the inflationary scenario having the effective slope $n \approx (0.95 - 0.97)$ between galaxy and horizon scales, including a modest contribution of primordial gravitational wave background to large-angle $\Delta T/T$ fluctuations of the cosmic microwave background (resulting in the increase of their total rms amplitude by $(5 - 10)\%$) expected in some variants. A classification of cosmological models according to the number of fundamental parameters used to fit observational data is presented, too.

1 Introduction

It remains an ambitious goal of the inflationary scenario, as well as of any other fundamental cosmological theory of the early Universe, to explain all observed structure of the present-day Universe using a minimal number of additional microphysical “fundamental” constants, apart from those already known from the particle physics. Of course, we don’t know how many parameters is really needed to describe the whole Universe, so one can’t say apriori that a theory having more parameters is worse than a theory with a less number of them. However, following the the Occam’s razor principle, classification of different cosmological models according to the number of additional phenomenological parameters used in them gives us a natural logical sequence of their consideration and comparison with observational data. We call these parameters fundamental, if they appear in basic equations (as in the inflationary scenario), not in initial conditions or other assumptions.

As is well known, a power spectrum of perturbations producing the observed structure of the Universe is a product of an initial (primordial) spectrum formed in the early Universe and a transfer function $C^2(k)$ which depends on the type of dark matter at present (e.g., on masses and concentrations of neutrinos). From the inflationary scenario point of view, the initial spectrum is completely determined by a phenomenological Lagrangian of an effective scalar field (or fields) $- \text{inflaton(s)} -$ at the de Sitter (inflationary) stage in the very early...
Universe. Then parameters determining both the initial spectrum and the present dark matter content are fundamental and should be considered and counted on equal footing. If such a classification is applied to inflationary models (see, e.g., Starobinsky 1993), then a model of the first level having only one fundamental parameter – an amplitude of perturbations – is the CDM model with the approximately flat (Harrison-Zeldovich, or $n \approx 1$) spectrum of initial adiabatic perturbations. Because of theoretical considerations and observational uncertainties, it is better to include “weakly-tilted” models with $|n - 1| \leq 0.1$ into this class, too. Hereafter this model will be referred as the Standard Cold Dark Model (SCDM). There exists another model belonging to this level: the CDM model with the approximately flat spectrum of isocurvature fluctuations ($n \approx -3$). But that model has been known to be excluded by observations long ago, because it produces excessive large-scale $\Delta T/T$ fluctuations. Strictly speaking, SCDM has one more parameter which defines an amplitude of the approximately flat spectrum of primordial gravitational waves and which is directly connected to the Hubble parameter $\dot{a} \equiv a$ at the de Sitter stage (Starobinsky 1979), $a(t)$ being the scale factor of the Friedmann-Robertson-Walker cosmological model. However, this is a rather small effect which may be seen in a slight increase of large-scale $\Delta T/T$ fluctuations only (apart, of course, from a remote possibility of direct detection of this relic gravitational background), see the discussion section below. So, it is better to consider parameters connected with the gravitational wave background separately.

At present, it is clear already that SCDM predictions, though being not far from observational data (that is remarkable for such a simple model with only one free parameter), still definitely do not agree with all of them. Namely, if the free parameter is chosen to fit the data on scales $(100 - 1000) h^{-1}_{50} \text{Mpc}$, discrepancy of about twice in perturbation amplitude arises on scales $(1 - 10) h^{-1}_{50} \text{Mpc}$, and vice versa ($h_{50} = H_0/50$, where $H_0$ is the Hubble constant in km/s/Mpc). Thus, models of the next (second) level having one more additional constant have to be considered. Among these models, the best is certainly the mixed cold+hot dark matter model (CHDM), Shafi & Stecker 1984, for recent analysis see Pogosyan & Starobinsky 1993 (hereafter PS), Liddle & Lyth 1993, Klypin et al. 1993 and references therein. In this model, the hot component is assumed to be the most massive of 3 neutrino species (presumably, $\tau$-neutrino) with the standard concentration following from the textbook Big Bang theory. Then the only new fundamental parameter is the neutrino mass $m_\nu$ (masses of the other two types of neutrinos are supposed to be much less and, therefore, unimportant for cosmology). If, on the contrary, masses of two neutrino types are assumed to be comparable or even equal, the resulting model will belong to the third level, until the mass ratio will be either confirmed in laboratory experiments, or theoretically derived from some underlying theory (we shall return to the discussion of this case at the end of the paper). $m_\nu$ is related to the energy density of the hot component (in terms of the
critical one) \( \Omega_\nu \) by

\[
m_\nu = 23.3 \Omega_\nu h_{50}^2 \text{ eV}
\]

for \( T_\gamma = 2.735 \text{K} \). The CDM model with the cosmological constant seems to be on the second place by a number of difficulties (and still marginally admissible), and the two tilted CDM models with a power-law initial spectrum of adiabatic perturbations (with \( n < 1 \)) and isocurvature ones (with \( n > -3 \)) to avoid excessive large-scale \( \Delta T/T \) fluctuations) are marginally or completely excluded.

Still the CHDM model with \( n \approx 1 \) is not without difficulties. The main of them is connected with later galaxy and quasar formation in this model as compared to the SCDM model. As a result, only a small region in the \( H_0 - \Omega_\nu \) plain remains permitted (PS, see also a more pessimistic view in Cen & Ostriker 1994). Recently, this difficulty exacerbated due to the problem of producing sufficient number of damped \( Ly - \alpha \) systems (Subramanian & Padmanabhan 1994, Mo & Miralda-Escude 1994, Kauffmann & Charlot 1994). However, latest analysis based on N-body simulations suggests that the latter problem may be solved if \( \Omega_\nu \) is taken smaller than it was supposed before: \( \Omega_\nu \leq 0.25 \) (Klypin et al. 1994) or even \( \Omega_\nu \leq 0.2 \) (Ma & Bertschinger 1994) for \( h_{50} = 1 \), in complete agreement with the restrictions on the model following from quasar and galaxy formation which were earlier obtained in PS using linear theory.

Therefore, it is desirable to have more power on small scales in the CHDM model. This was one motivation for us to consider a CHDM model belonging to the next (third) level, i.e., having one more parameter. We assume the standard neutrino concentration and use one of three adjustable parameters as the neutrino mass as earlier. Then we are left with two parameters to characterize an initial perturbation spectrum. The most natural possibility is to assume a power-law spectrum of adiabatic perturbations with \( n \neq 1 \), then one of the parameters gives a rms amplitude of perturbations at some scale (say, at the present horizon scale), while the other defines the slope. For the reason stated above, we consider the case \( n > 1 \) in this paper (with some results relevant to the approximately flat case \( |n - 1| < 0.1 \), too). The case \( n < 1 \) for the CHDM model has been already considered in detail in Liddle & Lyth 1993 and briefly mentioned in PS, with the conclusion that the “really” tilted case with \( n < 0.9 \) is excluded, but typical chaotic inflationary spectra with \( n \approx 0.95 - 0.97 \) (which we count as approximately flat ones) are possible.

The other motivation to consider such a model is that the best fit to the COBE data is given by \( n \) slightly larger than 1: \( n \approx 1.2 \) (Bennett et al. 1994, Górski et al. 1994), although \( n = 1 \) lies inside 1\( \sigma \) error bars (and, as advocated by Górski et al. 1994, becomes even the best fit if the quadrupole is completely excluded). \( n > 1 \) is also needed to account for the results of the Tenerife experiment (Hancock et al. 1994). Of course, \( n \) significantly larger
than 1 produces too much power at small scales and, thus, may be rejected for different reasons. E.g., if the perturbation spectrum is assumed to maintain its power-law form up to very small scales where metric perturbations (gravitational potential) become comparable to unity, then the upper limit \( n < 1.4 \) follows from the consideration of production of primordial black holes (Carr, Gilbert & Lidsey 1994). Even if a falling power-law spectrum is assumed over a much smaller scale range of a few orders of magnitude, another upper limit \( n < 1.54 \) follows from the absence of spectral distortions of the cosmic microwave background (Hu, Scott & Silk 1994). For these reasons, we investigate the range \( n \lesssim 1.3 \) only.

On the other hand, in spite of the advent of the Hubble telescope, there is still no general agreement about the value of the Hubble constant \( H_0 \). One methods produce the value of \( H_0 \) around 50 km/s/Mpc (Branch & Miller 1993, Sandage et al. 1994), others lead to a significantly larger value (70 – 80) km/s/Mpc (Schmidt et al. 1994), while the accuracy of new methods based, e.g., on the Sunyaev-Zeldovich effect in rich clusters of galaxies, is still not enough to discriminate between these two cases (Birkinshaw & Hughes 1994). That is why we have to investigate a dependence of the model predictions on \( H_0 \) additionally. We assume \( H_0 \) in the range (40 – 80) km/s/Mpc. Of course, \( H_0 \) is neither a new, nor a fundamental parameter, so it should not be counted in our classification scheme.

Finally, to justify the use of the notion of fundamental parameters, we have to present at least one inflationary model producing a falling power-law spectrum over some range of scales. The simplest way is to take the well-known case of a test massive scalar field in the de Sitter background, that is equivalent to assuming the inflaton potential to be of the form \( V(\phi) = V_0 + \frac{m^2\phi^2}{2} \) with \( m^2 \sim H_1^2 \equiv \frac{8\pi G V_0}{3} \ll M_P^2 \) and the field \( \phi \) being in the regime \( |\phi| \ll M_P \) (\( M_P \equiv G^{-1/2} \) is the Planck mass, and \( \hbar = c = 1 \) is used in this paragraph). The inflationary stage in this model ends when \( \phi \) drops to a very small value \( \phi_f \ll M_P \) either as in the new inflationary model, or due to a second-order phase transition destroying \( V_0 \) (Linde 1994). The exact expression for the slope is \( n - 1 = 3 - \sqrt{9 - 4 \frac{m^2}{M_P^2}} \). The slope \( n = 1.1 \) corresponds to \( m = 0.38H_1 \) (of course, then it is possible to use an approximate form of this formula: \( n \approx 1 + \frac{2m^2}{3M_P^2} \)). Therefore, we may take \( m_\nu, \frac{H_1}{M_P} \) and \( \frac{m}{M_P} \) as the three new fundamental parameters of the cosmological scenario considered.
2 CHDM Linear Perturbation Spectrum

Following our previous notations for the case of the flat initial spectrum, we write the linear power spectrum of perturbations of the gravitational potential $\Phi$ in the form

$$P_\Phi(k,t) = \frac{9A^2}{200\pi^2k^3} \left( \frac{2ck}{H_0} \right)^{n-1} \cdot C_{\text{CHDM}}^2(k,t),$$

specifying an initial spectrum produced at an early (inflationary) stage of the evolution of the Universe to be the power-law one: $P_{\Phi,\text{in}} = \frac{9A^2}{200\pi^2k^3} \left( \frac{2ck}{H_0} \right)^{n-1}$. The transfer function $C_{\text{CHDM}}$ was determined in our previous paper (PS) numerically by solving a system of the Einstein-Vlasov equations for the evolution of adiabatic perturbations with a neutrino component treated kinetically and a CDM component as dust.

The resulting $C_{\text{CHDM}}$ depends on present values of the neutrino fractional density $\Omega_\nu$, the Hubble constant $H_0 = 50h_{50}$ km/s/Mpc and the CMB temperature $T_\gamma = 2.735t_\gamma$ K as parameters, as well as on time $t$ (redshift $z$). If we define $C_{\text{CHDM}}(k, \Omega_\nu, z) = C_{\text{CDM}}(k) \cdot D(k, \Omega_\nu, z)$, our numerical calculations can be described by the following fitting formula

$$D(k, \Omega_\nu, z) = \left( \frac{1 + (Ak)^2 + (1 - \Omega_\nu)^{\frac{3}{4}}(a_\nu/a_0)(1 + z)(Bk)^4}{1 + (Bk)^2 - (Bk)^3 + (Bk)^4} \right)^\beta,$$

with $\beta = \frac{3 - \sqrt{23 - 44\Omega_\nu}}{4}$. This formula satisfies asymptotic regimes for the transfer function discussed in PS. Here $a_0/a_\nu = (1.681 \Omega_\nu)^{-1}$ is the expansion factor of the Universe from the matter-radiation equality moment $t_{eq}$ until the present time $t_0$. The scale $B = R_{eq} \cdot (1 + \Omega_\nu)/\Omega_\nu + \Omega_{\text{crit}}$, where $R_{eq} = 10.80 h_{50}^{-1/2}$ Mpc, is the neutrino non-relativization scale for $\Omega_\nu = 1$. For the critical value $\Omega_\nu = \Omega_{\text{crit}} = 0.1435$, neutrinos become nonrelativistic at $t = t_{eq}$. The best fit of the only scale remained gives $A = R_\ast (1 + 10.912\Omega_\nu)\sqrt{\Omega_\nu(1 - 0.9465\Omega_\nu)/(1 + (9.259\Omega_\nu)^2)}$ with $R_\ast = 69.52 h_{50}^{-1/2}$ Mpc. The deviation of this fit from the calculated $D(k)$ is better than 2% in the region $k < 5h_{50}^{-1/2}$ Mpc$^{-1}$ for the present moment $z = 0$ and $\Omega_\nu < 0.7$, while staying within 5% for $z < 30$. We used the Bardeen et al. (1986) expression for $C_{\text{CDM}}(k)$ in actual fitting.

If the present CMB temperature is fixed (we use $T_\gamma = 2.735$ K), the model involves four parameters, namely, $A$, $n$, $\Omega_\nu$ and $H_0$. If the adiabatic mode is solely responsible for the observed large-angle CMB fluctuations, we have

$$\langle \left( \frac{\Delta T}{T} \right)^2_{\text{in}} \rangle = \frac{A^2 \Gamma(3-n)}{100 2^{3-n} \Gamma^2(2-n/2) \Gamma(l + n/2)} \frac{\Gamma(l + n/2)}{\Gamma(l + 2 - n/2)},$$

(4)
Note also a simple form of the corresponding large-angle correlation function (with the dipole components excluded) which we did not notice in literature:

\[
\xi_T(\vartheta) \equiv \left\langle \frac{\Delta T}{T}(0), \frac{\Delta T}{T}(\vartheta) \right\rangle \\
= \frac{A^2}{200\pi^2} \frac{\Gamma(n) \cos \frac{\pi(n-1)}{2}}{2^{n-1}(n-1)(2-n)} \left[ \left( \frac{\sin \frac{\vartheta}{2}}{2} \right)^{1-n} - \frac{2}{3-n} \left( 1 + \frac{3(n-1)}{(5-n) \cos \vartheta} \right) \right] \\
= a_{2}^2 \frac{(3-n)(5-n)(7-n)}{10(n^2 - 1)} \left[ \left( \frac{\sin \frac{\vartheta}{2}}{2} \right)^{1-n} - \frac{2}{3-n} \left( 1 + \frac{3(n-1)}{(5-n) \cos \vartheta} \right) \right] \\
\]  

where \( a_{2}^2 \equiv \frac{5}{4\pi} \langle \left( \frac{\Delta T}{T} \right)^2 \rangle_{2m}^2 \). This expression is valid for both \( n > 1 \) and \(-1 < n < 1\). The known correlation function for the \( n = 1 \) case (Starobinsky 1983) follows from here by limiting transition. If the tensor mode (gravitational waves) is responsible for a part of observed \( \Delta T/T \) fluctuations, the value \( A \) derived from (4) for given \( \langle \left( \frac{\Delta T}{T} \right)^2 \rangle_{2m} \) serves as an upper limit on the amplitude of the adiabatic mode. However, we don’t expect a noticeable tensor contribution for \( n > 1 \), unless a new parameter is introduced into the model that would shift it into the next, fourth level.

### 3 Confrontation with Observational Tests

Our way of comparison of the model with observational tests closely follows our previous paper (PS). However, here we confine our consideration to the following tests shown to be the most restrictive for the CHDM model in PS.

1. The COBE measurement of large-angle \( \Delta T/T \) fluctuations.
2. Value of the total rms mass fluctuation \( \sigma_8 \) at \( R = 16h^{-1}_{50}\text{Mpc} \) (we use the index “8”, not “16”, to be in accordance with standard notation caused by the habit of measuring \( H_0 \) in units of 100 km/s/Mpc).
5. Large-scale peculiar velocities following from the POTENT reconstruction (Bertschinger et al. 1990, Dekel 1993).
Other tests included in the extended list of PS do not lead to additional limitations on the model.

We adopted the following numerical values for the observational data considered.

The COBE result for the total rms value of the $l = 4$ multipole ($\Delta T/T)_4 = (12.8 \pm 2.3)$ $\mu$K/$T_c$ (Wright et al. 1993) is used to put limits on the amplitude $A$, because it seems to be the most spectrum independent. For $n = 1$, this corresponds to the amplitude $A = (4.38 \pm 0.79) \times 10^{-4}$ and the total quadrupole value $Q_{\text{rms}-PS} = (17.4 \pm 3.1)$ $\mu$K which are somewhat higher than those used in PS.

The total rms mass fluctuation $\sigma_8 \equiv \langle \left( \frac{M}{M_\odot} (16 h^{-1}_\text{Mpc}) \right)_{\text{rms}} \rangle$ is calculated for the lower limit of the COBE amplitude ($\Delta T/T)_4 = 10.5$ $\mu$K/$T_c$ (note that the inverse quantity $\sigma_8^{-1}$ is usually understood as the biasing parameter $b$ for optical galaxies). Based on cluster abundance data (White, Efstathiou & Frenk 1993), we consider $\sigma_8 < 0.67$ as a conservative upper limit on $\sigma_8$.

The Stromplo-APM counts-in-cells represent nine data points for the mass variance in cubic cells $\sigma^2(l)$ over the range $l = (20 - 150) h^{-1}_\text{Mpc}$. Fixing normalization by the condition $\sigma^2(l) = 1$ for $l = 25 h^{-1}_\text{Mpc}$ which agrees with the data, we eliminate a constant redshift correction (Kaiser 1987) and become able to compare directly the shape of the power spectrum with theoretical predictions for these scales. To find the best fit to the data, we formally applied the $\chi^2$ test with $N = 9 - 2 = 7$ degrees of freedom for a fixed $n$.

The standard model of quasar formation assumes that they arise as a result of formation of massive black holes in nuclei of galaxies with total masses $(10^{11} - 10^{12}) M_\odot$. Recent estimates of the fraction of mass in bound objects which can serve as quasar hosts at $z = 4$ give $f(> 10^{11} M_\odot) \geq 10^{-4}$ (Haehnelt 1993). In this paper, Haehnelt also presented the estimate for a fraction of mass in large galaxies: $f(> 10^{12} M_\odot) \geq 10^{-5}$ at the same redshift $z = 4$. We use the simple Press-Schechter formalism to connect the mass fraction $f(\geq M)$ to a linear mass fluctuation $\sigma(M, z)$ on a scale $M$ at a redshift $z$:

$$f(\geq M) = 1 - e r f c \left( \frac{\delta_c}{\sqrt{2} \sigma(M, z)} \right).$$

Then we get the limitation

$$[(1 + z) \sigma(M, z)]_{z=4} \geq \alpha \delta_c$$

where $\alpha = 1.285$ (1.132) for $M = 10^{11}$ (10$^{12}$) $M_\odot$. The left-hand side of Eq. (7) depends on $z$ only due to the $z$-dependence of the transfer function (3). There is no consensus on the threshold value $\delta_c$ to be used. The standard one for top-hat fluctuations is $\delta_c = 1.69$, while Klypin et al. (1994) advocate for $\delta_c = 1.4$ as the best fit to a mass distribution in the CHDM model at high $z$. In the latter case, however, the Gaussian filtering was used to
calculate $\sigma(M)$ for a given mass scale. Let us note that the choice $M = 10^{12} \, M_\odot$ provides a tighter limitation than $M = 10^{11} \, M_\odot$. Therefore, we adopt (7) with $\delta_c = 1.4$ (but with the Gaussian filtering) as a conservative restriction, while the limit on $10^{12} \, M_\odot$ objects with the more standard $\delta_c = 1.69$ for top-hat fluctuations as, may be, more realistic one. Also, we found useful to use an amplitude-independent combination of the $\sigma_8$ limitation $\sigma_8 < \sigma_*$ and the number of quasars test in the form

$$[(1+z)\, \sigma(M, z)]_{z=4} \sigma_8^{-1} \geq \alpha \delta_c \sigma_*^{-1}.$$  

(8)

From the POTENT data, we use two values of large-scale bulk velocities $v(80h^{-1}_{50} \, \text{Mpc}) = (405 \pm 60) \, \text{km/s}$ and $v(120h^{-1}_{50} \, \text{Mpc}) = (340 \pm 50) \, \text{km/s}$ (Dekel 1993). These values are in agreement with results obtained by other groups, see, e.g., Courteau et al. 1993 where the values $v(80h^{-1}_{50} \, \text{Mpc}) = (385 \pm 38) \, \text{km/s}$ and $v(120h^{-1}_{50} \, \text{Mpc}) = (360 \pm 40) \, \text{km/s}$ for our bulk motion are presented. Note, however, that all these values refer to bulk flow velocities in our vicinity and, thus, may differ from their rms values calculated for the whole Universe ("cosmic variance").

In Fig. 1, we display restrictions in the $H_0 - \Omega_\nu$ plane for several values of the slope of initial spectrum $n = 0.85, 0.95, 1.1, 1.2$ based on a combination of tests, namely the counts-in-cells $\sigma^2(l)$ values and the combined quasar density $\sigma_8$ condition (8). The case $n = 1.0$ was extensively discussed in PS (see Fig. 6 therein). The main conclusion made is that CHDM model parameters are restricted to the narrow range of a low Hubble constant $H_0 < 60 \, \text{km/s/Mpc}$ and the neutrino fraction $\Omega_\nu = 0.17 - 0.28$ for $H_0 = 50$. The first of these limits reflects a problem with unavoidable high mass fluctuations at the $16h^{-1}_{50} \, \text{Mpc}$ scale, as well as a wrong shape of the perturbation spectrum over the $l = (20 - 150)h^{-1}_{50} \, \text{Mpc}$ interval if the Hubble constant is high. The upper bound on $\Omega_\nu$ comes from the combined condition (8) that implies that the slope of the spectrum in the scale range $(0.7h^{-2/3}_{50} - 16h^{-1}_{50}) \, \text{Mpc}$ cannot be too steep. The lower bound on $\Omega_\nu$ arises both by matching of the spectrum slope to the counts-in-cells $\sigma^2(l)$ values (that shows more relative power on large scales than in the SCDM model), as well as from the result that $\sigma_8$ is too high for low-$\Omega_\nu$ models with the COBE normalization.

Fig. 1 shows how these results are affected by allowing an initial power-law spectrum to be non-flat: $n \neq 1$. Two main conclusions can be drawn. The amplitude independent tests remain in essentially the same mutual relation favouring somewhat higher values of $\Omega_\nu$ for larger $n$ as expected. For illustration, one may follow the point of intersection of the $\chi^2 = 2$ contour with the limiting line for $10^{11} \, M_\odot$ objects. This point moves from $H_0 = 62, \, \Omega_\nu = 0.3$ for $n = 0.85$ to $H_0 = 52, \, \Omega_\nu = 0.38$ for $n = 1.2$. On the other hand, restrictions from the $\sigma_8$ test become much tighter as the slope of initial spectrum increases. For $n = 1.2$, all
models with \( H_0 \geq 50 \) are antibiased: \( b < 1 \). Thus, models with large \( n \) have too large mass fluctuations at the 16\( b_{50}^1 \) Mpc scale that cannot be compensated by increasing the neutrino fraction \( \Omega_\nu \).

We present another look at these results by plotting the \( n - \Omega_\nu \) cross-section of the parameter space for the same three tests in Fig. 2. One can see that, for low values of the Hubble constant \( H_0 = 40, 50 \), there is a significant degree of degeneracy between \( n \) and \( \Omega_\nu \), as far as the tests on the spectrum shape are considered, which do not disallow any value of \( n \). Note, however, that if we adopt the more stringent quasar abundance condition \( f(\geq 10^{12} M_\odot) \geq 10^{-3} \) at \( z = 4 \), then we get an upper limit on \( \Omega_\nu \) for \( H_0 = 50 \) roughly coinciding with the lower boundary of the best \( \chi^2 = 2 \) region for the counts-in-cells fit. So both tests are only in a marginal agreement for parameters close to the line \( 3\Omega_\nu = n - 1/4 \).

As we increase \( H_0 \), the fit to the counts-in-cells values begins to fail for large spectrum indexes, so that the best \( \chi^2 = 2 \) contour sets the limit \( n < 1.1 \) for \( H_0 = 60 \), while for \( H_0 = 70 \), this limit follows even from the more conservative condition \( \chi^2 \leq 7 \). Moreover, the best-fit contours tend to select higher values of the neutrino fraction than is allowed by the quasar test. In this way, for \( H_0 \geq 55 \), no model can satisfy the condition (7) for \( M = 10^{12} M_\odot \) and have \( \chi^2 < 2 \) (for \( H_0 > 70 \), even \( \chi^2 < 7 \)) simultaneously. If the less stringent quasar test for \( M = 10^{11} M_\odot \) is used, it becomes possible to achieve the best fit for the counts-in-cells data for \( H_0 = 60 \) if \( n < 0.9 \), but not for the Hubble constant as high as \( H_0 = 70 \). On this basis, we conclude once again that the CHDM model is incompatible with high values of the Hubble constant \( H_0 \geq 60 \) even for \( n \neq 1 \) initial spectra.

Although tests on the spectrum shape do set a tight limitation on the initial slope for high \( H_0 \), too, the strongest argument against high \( n \) comes directly from the \( \sigma_8 \) condition. To achieve \( \sigma_8 < 0.67 \) for \( H_0 = 50 \), one must restrict the model to \( n \leq 1 \), while for \( H_0 = 60 \), even \( \sigma_8 < 1 \) leads to \( n < 1.05 \).

All the tests on the CHDM model become more in agreement with each other for \( H_0 = 40 \). However, we are not sure that such low values of the Hubble constant are possible. Therefore, on the basis of Figs. 1, 2, we consider a small region of the parameter space in the vicinity of \( H_0 = 50, \Omega_\nu = 0.23, n = 0.95 \) as the best parameter set for the model. It corresponds to the neutrino mass \( m_\nu \approx 5 \text{ eV} \).

The Stromplo-APM counts-in-cells \( \sigma^2(l) \) are given in redshift space. Although we excluded any constant redshift correction using these data as a test on the slope of the initial spectrum, it may be asked how the results would change if the redshift correction depended on scale significantly. Here we note that the counts-in-cells test in the form we used it serves primarily to indicate the necessity for relatively high \( \Omega_\nu \). Probably, the redshift correction can only increase with scale. Then underlying real space mass fluctuations depend less steeply on scale and are better fitted by lower \( \Omega_\nu \) models than the straightforward use of
$\sigma^2(l)$ predicts.

To support the conclusions derived from the counts-in-cells test, we present $\chi^2$ contours of the direct fit of theoretical power spectra to a power spectrum reconstructed from the galaxy angular correlation function $w(\theta)$ (Baugh & Efstathiou 1993, 1994) in Fig. 3. These data give the power spectrum directly in real space. We used their values for $P(k)$ in the range $k = (0.07 - 0.025)h_{50}$ Mpc$^{-1}$ where errors are not as large as on larger scales and nonlinear corrections are not yet as important as on smaller scales. In fact, we have only 4 data points in the considered range which are fitted with the 3-parametric model (for fixed n or $H_0$). In Fig. 3, we compare $\chi^2$ contours of this one and the counts-in-cells tests in the $H_0 - \Omega_\Lambda$ plane for $n = 0.95$ and in the $n - \Omega_\Lambda$ plane for $H_0 = 50$. First, we should note that no CHDM model fits the Bough & Efstathiou data too well (except for $H_0 = 40$ or low $n$). In particular, $\chi^2 < 2$ is achieved only for $H_0 \leq 50$ if the spectral index is $n = 0.95$. Second, $\chi^2$ contours for these two tests have a rather close structural resemblance. Since the number of $P(k)$ points used is small and the errors given by Bough & Efstathiou (1993) may be questioned, we don’t use this test to set specific restrictions on the model. However, using a rather relaxed limitation $\chi^2 < 4$ for the angular correlation function test, we can confirm if not strengthen the lower limit on $\Omega_\Lambda$ previously obtained from the counts-in-cells data for parameters in the most interesting region around $H_0 = 50, \Omega_\Lambda = 0.25$. Other conclusions made as the deterioration of the fit with increasing of $n$ or $H_0$ are also confirmed by this test.

In the previous consideration, we have not discussed the size of the allowed region in the amplitude $A$ dimension of the parameter space. In Fig. 4, we present the $\Omega_\Lambda - A$ cross-section of the parameter space with fixed $H_0 = 50$. The shaded strip selects the adiabatic amplitude $A$ following from the COBE result $(\Delta T/T)_4 = (12.8 \pm 2.3)\mu K/T_\gamma$. Now it depends on $n$ (according to Eq. (4)) but not, practically, on $\Omega_\Lambda$. This is also the case for the estimation of bulk velocities $v(R)$ in spheres of radii $R = 80, 120h_{50}^{-1}$ Mpc. The condition $\sigma_8 < 0.67$ and the quasar number test (7) for $M = 10^{11} M_\odot$ leave the triangular area left and below the point of intersection of solid curves as the allowed range of parameters. Exactly this point of intersection produces the upper dashed line in Figs. 1, 2.

In Fig. 4, we see new aspects of the failure of the CHDM model to be successful for large $n$. Not only the $\sigma_8$ condition can’t be fulfilled, but also large-scale bulk velocities become simultaneously too high to be compatible with $\sigma_8 < 0.67$ for $n \geq 1.1$ and too low in comparison with the COBE amplitude for $n > 1.2$.

We understand that there are two effects which might make these limits less strong. First, some part of the observed $\Delta T/T$ can be due to gravitational waves. Then the adiabatic amplitude $A$ derived from the COBE data will be lower than that in Fig. 4. Second, bulk velocities in our vicinity can differ from average ones in the Universe by
cosmic variance. Therefore, assuming both that the velocities $v(R)$ given by POTENT are at least 40% larger than their rms values and that the primordial gravitational wave background is responsible for the increase of the observed large-angle $\Delta T/T$ by at least 1/3, one can, in principle, reconcile the amplitude $A$ from these tests with $\sigma_8 < 0.67$ for $n = 1.2$. However, the former assumption is equivalent to introducing one more fundamental dimensionless parameter that shifts the model to the fourth level according to our classification. Really, since adiabatic and tensor fluctuations are statistically independent, we have $\langle (\Delta T/T)_{\text{tot}}^2 \rangle_{\text{tot}} = \langle (\Delta T/T)^2 \rangle_{\text{ad}} + \langle (\Delta T/T)^2 \rangle_{\text{gw}}$. Therefore, to increase the rms value of $\langle (\Delta T/T)_{\text{tot}}^2 \rangle_{\text{tot}}$ by 40% or more, $\langle (\Delta T/T)^2 \rangle_{\text{gw}}$ should be no less than $\langle (\Delta T/T)^2 \rangle_{\text{ad}}$ - a kind of additional “fine tuning” in the $n > 1$ case (in contrast to the $n < 1$ case where such a condition arises naturally). According to the main idea of classification of cosmological models presented in the Introduction, this does not mean that such a model is bad (because dimensionless fundamental parameters may be fine-tuned to some number), it simply shows that more new significant fundamental parameters are necessary for explanation of observational data than it was assumed initially. Thus, this way is not admissible if we want to stay among models of the third level.

This can be illustrated using the simple inflationary model presented in the end of Introduction. Under the condition $|\phi| \ll M_P$ assumed earlier, the gravitational wave contribution is small: assuming $n \approx 1$ and using well-known expressions for the slow-roll motion, it is straightforward to show that

$$\frac{T}{S} \equiv \frac{\langle (\Delta T/l_m)^2 \rangle_{\text{gw}}}{\langle (\Delta T/l_m)^2 \rangle_{\text{ad}}} = \left(4.16 \frac{m^2 \phi}{H^2 M_P} \right)^2 \left(1 + \frac{m^2 \phi^2}{2V_\phi} \right)^{-2}$$

for $2 \ll l \leq 30$ (the numerical coefficient in round brackets is 6% more for $l = 2$, but 6% less for $l = 3$ and 9% less for $l = 5, 6$). Here the value of $\phi$ is, as usually, taken at the moment of the first horizon crossing during the inflationary stage ($\approx 60$ e-folds before its end). Now let us try to get a larger value of $T/S$ by assuming that $|\phi| \sim M_P$ at this moment. To achieve this, we have to assume some specific relation between $\phi_f$ and $M_P$. Earlier, a value of the dimensionless constant $\phi_f/M_P$ was not important for comparison with observational data, it was enough to assume that it was sufficiently small. But now this value becomes the new significant fundamental parameter of the model. Of course, then the power spectrum cannot be considered as an exactly power-law one. Note that, incidentally, this specific model of the fourth level does not achieve the aim of having $T > S$ for $n > 1.1$, too, if we assume the abovementioned upper limit on the spectral index at very small scales following from the absence of excessive PBH formation: $n \leq 1.4$. This requires $m \leq 0.8H_1$, so, for scales where the local index $n > 1.1$, the temperature fluctuation amplitude increases by $\sqrt{1 + \frac{T}{S}} < 1.2$.  


1/2
For values of the spectral index $n < 1$, the situation with bulk velocities becomes inverse. For $n \leq 0.85$, they are higher than those following from the normalization to the large-scale $\Delta T/T$. Now the possible effect of gravitational waves only worsens the discrepancy (for $n = 0.85$, in the framework of inflationary models with the exponential inflaton potential, the adiabatic amplitude $A$ is 40% smaller than the one given in Fig. 4 due to this effect). Therefore, following Liddle & Lyth (1993), we exclude models with $n < 0.9$.

4 Conclusions and discussion

We compared the CHDM model with a falling power-law primordial spectrum of adiabatic perturbations ($n > 1$) with observational data. This model has one more adjustable dimensionless parameter ($n - 1$, which can be expressed in terms of an additional parameter of an inflaton potential) than the CHDM model with the approximately flat ($n \approx 1$) spectrum. It might be thought naively that the model with more parameters fits data better. Remarkably, we found just the opposite: fit to the data becomes worse with the growth of $n - 1$, and may be considered as unreasonable already for $n > 1.1$.

Combining this with the previously known result that the CHDM model is hardly compatible with observations for $n < 0.9$ (Liddle & Lyth 1993, PS), we arrive to the conclusion that the CHDM model requires the approximately flat ($|n - 1| < 0.1$) primordial spectrum among all possible power-law spectra of adiabatic perturbations. This shows the robustness of the CHDM model with the simplest inflationary initial conditions. Our conclusion agrees qualitatively with that in the recent paper by Lyth & Liddle (1994). Moreover, the best fit to the data is achieved for $n$ slightly less than 1, and around values of the effective slope expected in the simplest inflationary models: either with a scalar field with a polynomial potential $V = \frac{m^2 \phi^2}{2}$, $n \approx 0.97$ or $V = \frac{\lambda \phi^4}{4}$, $n \approx 0.95$ (chaotic inflation, Linde 1983), or in the higher-derivative gravity $R + R^2$ model (Starobinsky 1980) where $n \approx 0.97$, too, or in the new inflationary model with the Coleman-Weinberg potential (Linde 1982, Albrecht & Steinhardt 1982) with $n \approx 0.95$.

Vice versa, the CHDM model presents the best possibility for the realization of these inflationary models, because the other alternative, the CDM+$\Lambda$ model, has more serious problems. The most promising purely CDM models seem now to be based on a non-scale-invariant, step-like initial spectrum of adiabatic perturbations produced in more complicated inflationary models (Gootlöber, Mück & Starobinsky 1993, Peter, Polarski & Starobinsky 1994), they belong to the third level in our classification. Thus, unexpectedly, the fate of the simplest inflationary models appears to be closely tied to the fate of the CHDM model.

In addition, there is a place in the CHDM model with $0.9 < n < 1$ for a modest but
noticeable gravitational wave contribution to large-angle $\frac{\Delta T}{T}$ fluctuations expected for chaotic inflationary models with polynomial inflaton potentials (but not for the new inflationary model and the $R + R^2$ model). It leads to the increase in the total rms amplitude of these fluctuations by, e.g., \( \sqrt{1 + \frac{T}{S}} - 1 \approx 10\% \) for the $V = \frac{\lambda \phi^4}{4}$ inflationary model (Starobinsky 1985). Note that all the simplest inflationary models listed above do not require additional fundamental parameters to specify their predictions for (weakly scale dependent) values of the spectral index $n - 1$ and the ratio $T/S$. So, they belong to the second level if combined with the CHDM model for dark matter. The increase in $\frac{\Delta T}{T}$ may be even found desirable in view of recent papers by Górski et al. (1994), Banday et al. (1994) where further rise of the COBE results up to $Q_{rms, P S} = (19 - 20) \mu K$ is suggested. On the other hand, it is clear that a significantly larger gravitational wave contribution is incompatible with the CHDM model.

Of course, the most crucial confirmation of the CHDM model would be a direct or indirect (through neutrino oscillations parameters) discovery that the $\tau$-neutrino mass is really around 5 eV as predicted by the best fit to the model. But of no less importance is the precise determination of the Hubble constant because the model can’t work for $H_0 > 60$ km/s/Mpc, and it is better to have its value around 50 km/s/Mpc. The third critical test for the model is the abundance of galaxies and quasars at large redshifts.

Finally, let us return to the case of two (or even three) comparable neutrino masses mentioned in the Introduction. If neutrino concentrations are the standard ones and $\Omega_\nu$ denotes the total neutrino energy density in terms of the critical one, then what stands in the left-hand side of Eq. (1) is actually the sum of masses of all neutrino species. In this case, the relative transfer function $D(k, \Omega_\nu, z)$ (3) will have the same step-like form discussed in PS, but the transition region from one plateau to another will be shifted to larger scales due to increase of $R_{nr}$. So, in the first approximation, we may expect that the best fit is given by the condition that the sum of all neutrino masses should be about 5 eV. However, due to the abovementioned shift of characteristic scales in the transfer function, more careful analysis of this higher level model is needed that will be carried elsewhere.

Acknowledgments

A.S. is grateful to Profs. Y. Nagaoka and J. Yokoyama for their hospitality at the Yukawa Institute for Theoretical Physics, Kyoto University. A.S. was supported in part by the Russian Foundation for Basic Research, Project Code 93-02-3631, and by Russian Research Project “Cosmomicrophysics”. D.P. is grateful to Profs. Alan Omont and Francois Bouchet for their hospitality at the Institute d’Astrophysique de Paris where this project was started as well as to Prof. Simon White and Dr. Thomas Buchert from Max-Planck-Institut für Astrophysik, München where the final version of the paper was prepared.
References


[38] Starobinsky, A. A. 1979, JETP Lett., 30, 682.


Figure captions

Fig. 1 Restrictions in the $H_0 - \Omega_v$ parameter plane for $n = 0.85$, 0.95 (upper row) and $n = 1.1, 1.2$ (lower row) following from: a) Stromlo-APM counts-in-cells test. Solid lines correspond to $\chi^2 = 2, 7$ contours. b) The $\sigma_8$ condition. The values $\sigma_8 < 1, 0.67$ are achieved left to the dashed (correspondingly right and left) lines. c) The combination of the $\sigma_8 < 0.67$ condition with quasar and galaxy formation conditions (dotted lines). The allowed region lies below the upper dotted line if the limitation set for objects of the mass $M = 10^{11} \, M_\odot$ is used and below the lower one if $M = 10^{12} \, M_\odot$.

Fig. 2 Same tests as in Fig. 1 are given in the $n - \Omega$ parameter plane for $H_0 = 40, 50$ (upper row) and $H_0 = 60, 70$ (lower row).

Fig. 3 Comparison of the $\chi^2$ contours for fits to the cloud-in-cells values (solid contours $\chi^2 = 2, 7$) and the power spectrum $P(k)$ from the galaxy angular correlation function (dashed contours $\chi^2 = 1, 2, 4$). The left panel shows the $H_0 - \Omega_v$ plane for $n = 0.95$, the right one shows the $n - \Omega_v$ plane for $H_0 = 50$.

Fig. 4 Amplitude of perturbations $A$ in the CHDM model as follows from POTENT data for bulk peculiar velocity $v(R)$ [regions between dotted and dashed lines correspond to $\pm 1\sigma$ error bars for $v(80h^{-1}_{\text{Mpc}})$ and $v(120h^{-1}_{\text{Mpc}})$ respectively] compared to the COBE limits [shaded region]. The region above the solid line (i) is allowed by the quasar number condition $f(\geq 10^{11} \, M_\odot, z = 4) \geq 10^{-4}$. The region below the solid line (ii) is allowed by the condition $\sigma_8 < 0.67$. 