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The Energy-Energy Correlation Function Revisited

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Abstract

The $\mathcal{O}(\alpha_s^2)$ coefficient of the energy-energy correlation function (EEC) has been calculated by four groups with differing results. This discrepancy has lead to some confusion over how to measure the strong coupling constant using the EEC and the asymmetry of the energy-energy correlation function (AEEC) in electron-positron annihilation at the Z resonance. For example, SLD average the four values of α_s extracted from each of the different calculations. To resolve this situation, we present a new calculation of this coefficient using three separate numerical techniques to cancel the infrared poles. All three methods agree with each other and confirm the results of Kunszt and Nason that form the benchmark for other $\mathcal{O}(\alpha_s^2)$ quantities. As a consequence, the central values and theoretical errors of the strong coupling constant derived by SLD from the EEC and AEEC are altered. Using the SLD data, we find, $\alpha_s^{EEC}(M_Z^2) = 0.125^{+0.002}_{-0.003}$ (exp.) $\pm$ 0.012 (theory) and $\alpha_s^{AEEC}(M_Z^2) = 0.114 \pm 0.005$ (exp.) $\pm$ 0.004 (theory).

The energy-energy correlation function Σ_{EEC} has recently been used to measure the strong coupling constant, α_s, in e^+e^- annihilation on the Z resonance[1] [1, 2, 3, 4]. It is defined in terms of the angle χ_{ij} between two particles i and j such that,

$$\frac{1}{\sigma}\frac{d\Sigma_{EEC}(\chi)}{d\cos\chi} = \frac{1}{\Delta\cos\chi\ N_{events}} \sum_{N_{events}} \sum_{ij} \frac{E_i E_j}{E^2}, \tag{1}$$

where E_i and E_j are the energies of the particles and E the total energy in the event, $E = \sqrt{s}$. The sum runs over all pairs ij lying within a bin in $\cos\chi$ of width $\Delta\cos\chi$ so that $\cos\chi - \Delta\cos\chi/2 < \cos\chi_{ij} < \cos\chi + \Delta\cos\chi/2$. For $i \neq j$, each pair enters twice in the sum so the integral of the distribution is normalised to one when integrated over the whole range of χ.

The energy-energy correlation function can be described by a perturbative series,

$$\frac{1}{\sigma}\frac{d\Sigma_{EEC}(\chi)}{d\cos\chi} = \left(\frac{\alpha_s(\mu)}{2\pi}\right) A(\chi) + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \left(2\pi b_0 \log\left(\frac{\mu^2}{s}\right) A(\chi) + B(\chi)\right) + \mathcal{O}(\alpha_s^3), \tag{2}$$

where $b_0 = (11N - 2n_f)/12\pi$ and σ is the first order hadronic cross section,

$$\sigma = \sigma_0 \left(1 + \frac{\alpha_s(\mu)}{\pi}\right),$$

with σ_0 being the Born cross section for $e^+e^- \to$ hadrons. More often, the energy-energy correlation function is described relative to the Born cross section,

$$\frac{1}{\sigma_0}\frac{d\Sigma_{EEC}(\chi)}{d\cos\chi} = \left(\frac{\alpha_s(\mu)}{2\pi}\right) \bar{A}(\chi) + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \left(2\pi b_0 \log\left(\frac{\mu^2}{s}\right) \bar{A}(\chi) + \bar{B}(\chi)\right) + \mathcal{O}(\alpha_s^3), \tag{3}$$

where $\bar{A} = A$, $\bar{B} = B + 2A$. The lowest order contribution to the energy-energy correlation function for $0° < \chi < 180°$ is obtained from the $Z,\ \gamma^* \to q\bar{q}g$ process, where χ can be the angle between any of the three partons. An analytic form for the leading order coefficient A has been obtained by Basham et al [7],

$$\bar{A}(\chi) = A(\chi) = (1+\omega)^3 \frac{1+3\omega}{3\omega}\left[(2-6\omega^2)\log\left(1+\frac{1}{\omega}\right) + 6\omega - 3\right], \tag{4}$$

where $\omega = \cot^2(\chi/2)$. At next-to-leading order, the relevant processes are $Z,\ \gamma^* \to q\bar{q}g$ at one-loop and $Z,\ \gamma^* \to q\bar{q}q\bar{q},\ q\bar{q}gg$ at tree level. The matrix elements for these processes were first computed by Ellis, Ross and Terrano (ERT) [6] in the $\overline{MS}$ scheme and have formed the basis for all subsequent estimates of the second order coefficient $\bar{B}$. However, $\bar{B}$ has been computed several times with differing results, first by Ali and Barreiro (AB) [8, 9] and by Richards, Stirling and Ellis (RSE) [10, 11] and more recently by Falck and Kramer (FK) [12]

[1] A significant amount of data has also been collected at lower energies, see for example [5] and references therein.

and Kunszt and Nason (KN) [13]. Despite this disagreement, the radiative corrections are known to be relatively large while hadronisation corrections are also significant. Therefore, to extract the strong coupling constant, it is more usual to define the asymmetry,

$$\frac{1}{\sigma}\frac{d\Sigma_{AEEC}(\chi)}{d\cos\chi} = \frac{1}{\sigma}\frac{d\Sigma_{EEC}(180^\circ-\chi)}{d\cos\chi} - \frac{1}{\sigma}\frac{d\Sigma_{EEC}(\chi)}{d\cos\chi}, \tag{5}$$

where the corrections are smaller and the intra-jet region at $\chi \sim 0^\circ$ and the back-to-back two jet region at $\chi \sim 180^\circ$ are suppressed.

Nevertheless, the disagreement of the theoretical calculations has caused confusion in the extraction of α_s using the $\mathcal{O}(\alpha_s^2)$ coefficient from both the EEC and AEEC. For example, OPAL use the Kunszt-Nason calculation to extract a central value for α_s from the AEEC with a theoretical error on $\Lambda_{\overline{MS}}$ of +55 Mev/-10 MeV to encompass the range of predictions [2]. On the other hand, SLD average the values of α_s obtained from the four theoretical calculations and increase the theoretical error accordingly [4]. They find,

$$\begin{aligned}
\alpha_s^{EEC}(M_Z^2) &= 0.127^{+0.002}_{-0.003}\ (\text{exp.}) \pm 0.013\ (\text{theory}), \\
\alpha_s^{AEEC}(M_Z^2) &= 0.116 \pm 0.005\ (\text{exp.}) \pm 0.006\ (\text{theory}),
\end{aligned}$$

where the theoretical error includes the uncertainty from hadronisation (± 0.002 for EEC and $^{+0.003}_{-0.002}$ for AEEC), renormalisation scale (± 0.011 for EEC and ± 0.003 for AEEC) and an error of ± 0.006 for EEC and ± 0.004 for the AEEC purely from the uncertainty over which $\mathcal{O}(\alpha_s^2)$ coefficient is correct [4].

In this letter, we attempt to resolve the theoretical disagreement. We first review the discrepancies amongst the existing calculations of $\bar{B}$. We then present a new calculation of the energy-energy correlation function at $\mathcal{O}(\alpha_s^2)$ which is in complete agreement with the results of Kunszt and Nason.

In order to compute next-to-leading order quantities in perturbation theory, it is necessary to combine the contribution from n-parton one-loop Feynman diagrams with the $n+1$-parton bremstrahlung process. The virtual matrix elements are divergent and contain both infrared and ultraviolet singularities. The ultraviolet poles are removed by renormalisation, however the soft and collinear infrared poles are only cancelled when the virtual graphs are combined with the bremstrahlung process. Although the cancellation of infrared poles can be done analytically for simple processes, for complicated processes like this, it is necessary to resort to numerical techniques. Since the theoretical calculations of $\bar{B}$ are all based on the same ERT matrix elements, the discrepancies amongst the theoretical calculations appear to be rooted in the numerical implementation.

The numerical problem has been nicely formulated by Kunszt and Soper [14] by means of a simple example integral,

$$\mathcal{I} = \lim_{\epsilon\to 0}\left\{\int_0^1 \frac{dx}{x}x^\epsilon F(x) - \frac{1}{\epsilon}F(0)\right\}, \tag{6}$$

where $F(x)$ is a known but complicated function representing the $n+1$-parton bremstrahlung matrix elements. Here x represents the angle between two partons or the energy of a gluon and the integral over x represents the additional phase space of the extra parton. As $x \to 0$, the integrand is regularised by the x^ϵ factor as in dimensional regularisation, however, the first term is still divergent as $\epsilon \to 0$. This divergence is cancelled by the second term - the n-parton one-loop contribution - so that the integral is finite. A variety of methods to compute $\mathcal{I}$ have been developed.

The original method used by ERT is also known as the subtraction method. Here, a divergent term is subtracted from the first term and added to the second,

$$\begin{aligned} \mathcal{I} &= \lim_{\epsilon\to 0}\left\{\int_0^1 \frac{dx}{x}x^\epsilon(F(x)-F(0)) + F(0)\int_0^1 \frac{dx}{x}x^\epsilon - \frac{1}{\epsilon}F(0)\right\} \\ &= \int_0^1 \frac{dx}{x}\left(F(x)-F(0)\right), \end{aligned} \tag{7}$$

so that the integral is manifestly finite. This method has the advantages of requiring no extra theoretical cutoffs and making no approximations. A disadvantage is that it does require some additional analytic effort to explicitly extract the poles from the analogue of $\int_0^1 \frac{dx}{x}x^\epsilon$. For $\mathcal{O}(\alpha_s^2)$ quantities in electron-positron collisions, this was performed by Ellis, Ross and Terrano [6]. However, this analytic integration has to be carried out from scratch for each process under investigation and may even require a knowledge of the experimental jet algorithm [14].

An alternate approach known as the phase space slicing method has been widely used - see ref. [15] and refences theirin. The integration region is divided into two parts, $0 < x < \delta$ and $\delta < x < 1$. In the first region, the function $F(x)$ can be approximated by $F(0)$ provided the arbitrary cutoff $\delta \ll 1$,

$$\begin{aligned} \mathcal{I} &\sim \lim_{\epsilon\to 0}\left\{\int_\delta^1 \frac{dx}{x}x^\epsilon F(x) + F(0)\int_0^\delta \frac{dx}{x}x^\epsilon - \frac{1}{\epsilon}F(0)\right\} \\ &\sim \int_\delta^1 \frac{dx}{x}F(x) + F(0)\ln(\delta). \end{aligned} \tag{8}$$

This method is extremely portable [15] since the soft and collinear approximations of the matrix elements and phase space are universal. This makes it easy to apply to a wide variety of physically interesting processes. However, the disadvantage is the presence of the arbitrary cutoff δ. The integral should not depend on δ, and the δ dependence of the two terms in Eq. 8 should cancel. Since the approximations are reliable only when δ is small, this can give rise to numerical problems.

Finally, a third method is a hybrid of the two previous techniques. To preserve the portability of the phase space slicing method, we add and subtract only the universal soft/collinear approximations for $x < \delta$,

$$\mathcal{I} \sim \lim_{\epsilon\to 0}\left\{\int_0^1 \frac{dx}{x}x^\epsilon F(x) - F(0)\int_0^\delta \frac{dx}{x}x^\epsilon + F(0)\int_0^\delta \frac{dx}{x}x^\epsilon - \frac{1}{\epsilon}F(0)\right\}$$

	$\bar{A}$	$\bar{B}$
AB$^{(1)}$	$2F$	$4G$
RSE$^{(2)}$	$2g^{(1)}$	$4g^{(2)}$
FK$^{(3)}$	$2C$	$4D$
KN$^{(4)}$	$A_{EEC}/\sin^2\chi$	$B_{EEC}/\sin^2\chi$

Table 1: The definitions of the LO and NLO coefficients of the different calculations; (1) Eq. 1.6 of ref. [9], (2) eq. 1.4 of ref. [11], (3) eq. 1.2 of ref. [12], (4) eq. 4.24 of ref. [13].

$$\sim \int_0^1 \frac{dx}{x}\left(F(x) - F(0)\theta(\delta - x)\right) + F(0)\ln(\delta). \tag{9}$$

A cancellation between the terms still occurs, however only the phase space is approximated, so that this method is valid at larger values of δ. The difference between the latter two approaches is,

$$\int_0^\delta \frac{dx}{x}\left(F(x) - F(0)\right), \tag{10}$$

which clearly tends to zero as $\delta \to 0$. Therefore, provided δ is chosen small enough, all three methods should give equivalent results.

We will present results using all three methods, however, we first turn briefly to a discussion of the previous calculations, each of which uses a different notation for the first and second order coefficients $\bar{A}$ and $\bar{B}$ as shown in Table 1. We follow the event shape description and focus on the perfect resolution limit ($\delta \to 0$). In other words, no jet definition is applied to the partons before computing the energy-energy correlation function.

Both AB and RSE use the subtraction method to compute $\bar{B}$. AB perform the five dimensional integral over the four parton phase space numerically, while RSE relate the angle between partons χ to the invariants and are left with a four dimensional integral. Their results for the energy-energy correlation at $\chi \sim 90°$ are shown in Table 2. Because one of the integrations has been removed, RSE have significantly smaller errors. Nevertheless, within the errors, both agree. This is in contrast to the claim in [9] where B/A of RSE is compared with $\bar{B}/\bar{A}$ of AB[2].

Falck and Kramer [12] present two results for the perfect resolution limit. The first is based on a phase space slicing approach with a theoretical cutoff such that $y_{ij} = (p_i+p_j)^2/s > y_{\min} = 10^{-4}$. For such small values of $y_{\min}$, it is assumed that the limit $y_{\min} \to 0$ can be considered to have been reached - an assumption supported by earlier studies [8]. This result

[2]Table 2 of [9] claims to show the ratio $R^{corr} \sim B/A$ however inspection of the raw numbers in Table 1 indicates that actually $R^{corr} \sim \bar{B}/\bar{A}$ is quoted. As a result the comparison of AB and RSE in Fig. 4 of [9] is flawed. Unfortunately this claim has propagated through the literature [12, 13].

	$\bar{A}(\chi \sim 90^\circ)$	$\bar{B}(\chi \sim 90^\circ)$	$\bar{B}/\bar{A}$
AB[1]	2.434	42.68 ± 1.94	17.53 ± 0.79
RSE[2]	2.426	41.52 ± 0.16	17.1 ± 0.06
FKa[3]	$--$	$--$	$25.1\pm?$
FKb[4]	$--$	$--$	$17.5\pm?$
KN[5]	2.43	51.25 ± 2.67	21.1 ± 1.1
GSa[6]	2.43	52.39 ± 0.83	21.6 ± 0.34
GSb[7]	2.43	51.15 ± 0.68	21.05 ± 0.28
GSc[8]	2.43	52.29 ± 2.08	21.52 ± 0.86

Table 2: The NLO to LO coefficients and the ratio $\bar{B}/\bar{A}$ for the different calculations; (1) Table 1 of ref. [9], (2) Table 3 of ref. [11], (3) Fig. 3 of ref. [12], (4) Fig. 6 of ref. [12], (5) Tables 2 and 3 of ref. [13], (6) The subtraction method, (7) The phase space slicing scheme with $y_{\rm min} = 10^{-5}$, (8) The hybrid subtraction scheme with $y_{\rm min} = 10^{-5}$.

is somewhat larger than AB and RSE, but the numerical errors also appear to be larger. As a check, FK quote a second much smaller result based on the subtraction method which is in rough agreement with RSE (and hence AB). However, in conclusion, FK ascribe these differences to the presence of the cutoff $y_{\rm min}$ and the phase space slicing method used to compute $\bar{B}/\bar{A}$.

Finally, the benchmark calculation of $\mathcal{O}(\alpha_s^2)$ quantities at LEP energies is that of Kunszt and Nason. By using a sophisticated phase space mapping, they have reorganised the ERT matrix elements to give numerically stable results for all of the event shape and three jet quantities measured at LEP. Many of their predictions for other quantities have been checked [16] and it would be rather surprising for a single distribution to be in error. However, despite claiming to agree with FK, the KN result at $\chi \sim 90^\circ$ lies between the other calculations.

We have recalculated the $\mathcal{O}(\alpha_s^2)$ coefficient using all three numerical techniques described earlier. First, we have recoded the ERT matrix elements precisely as described in [6]. However, rather than weight each event by the value of the C parameter [6], we have weighted by the energy-energy correlation function,

$$\Sigma^{(3)} = \Sigma_{ij,k\neq i,j} \frac{2E_iE_j}{s}\ \delta\left(\cos\chi - \left(\frac{y_{ik}y_{jk} - y_{ij}}{y_{ik}y_{jk} + y_{ij}}\right)\right),$$

and,

$$\Sigma^{(4)} = \Sigma_{ij,k\neq i,j,l\neq i,j,k} \frac{2E_iE_j}{s}\ \delta\left(\cos\chi - \left(\frac{y_{ikl}y_{jkl} - y_{kl} - y_{ij}}{y_{ikl}y_{jkl} - y_{kl} + y_{ij}}\right)\right),$$

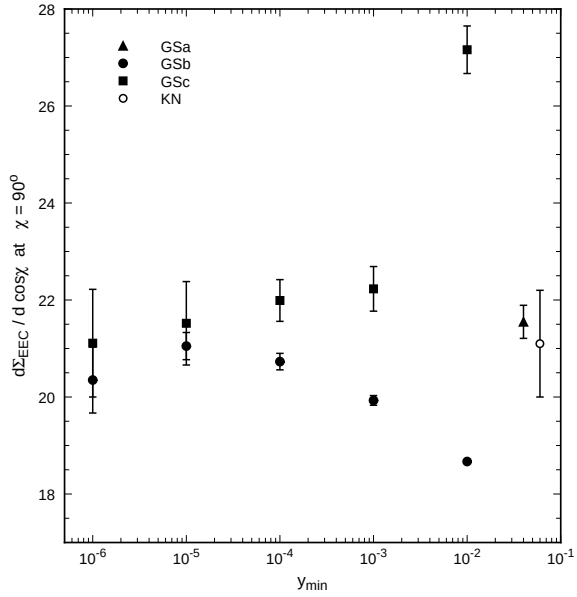

Figure 1: The ratio $\bar{B}/\bar{A}$ at $\chi \sim 90°$ as a function of $y_{\min}$ for the phase space slicing (GSb) and hybrid subtraction (GSc) schemes. The $y_{\min}$ independent values obtained from the subtraction method (GSa) and from ref. [13] are also shown.

for three and four parton final states respectively.

Second, we have constructed a completely independent program along the lines of [15] using squared matrix elements rather than helicity amplitudes. Either phase space slicing or the hybrid subtraction scheme may be selected. As described earlier, these methods rely on an unphysical cut to isolate the divergences. Fig. 1 shows the ratio $\bar{B}/\bar{A}$ at $\chi \sim 90°$ as a function of this cut, $y_{\min}$. At large $y_{\min}$, the predictions using the phase space slicing method varies rapidly with $y_{\min}$. This is because the approximations used to perform the analytic integrations are inaccurate. However, despite the increasing numerical errors, we see that for $y_{\min} < 10^{-4}$, the variation with $y_{\min}$ is small. A reasonable approximation to the $y_{\min} \to 0$ limit is therefore $y_{\min} = 10^{-5}$. In the hybrid subtraction scheme, $\bar{B}/\bar{A}$ also varies rapidly when $y_{\min}$ is large, but the zero resolution limit is also approximated by $y_{\min} = 10^{-5}$.

Table 2 shows the value of $\bar{B}/\bar{A}$ at $\chi \sim 90°$ for these three methods. Within errors, all three agree with each other and with the result of Kunszt and Nason. In principle, the calculation of Falck and Kramer (FKa) should coincide with those of the phase space slicing method for $y_{\min} = 10^{-4}$, and the results of AB and RSE should agree with the subtraction calculation, GSa, however, we see that this is not the case.

So far we have concentrated on a single value of χ. Fig. 2 shows the ratio of the Kunszt-Nason calculation and our phase space slicing results with $y_{\min} = 10^{-5}$ over the whole range of $\cos\chi$. We see that the two calculations are in good agreement with errors of less than $\mathcal{O}(5\%)$. Fitting a constant to this data yields a ratio of 1.0068 with a χ^2/d.o.f of 0.98. The

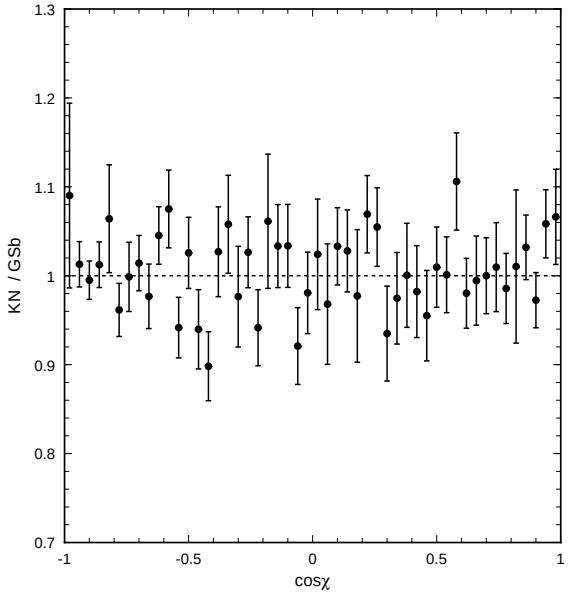

Figure 2: The ratio of KN [13] and our results using the phase space slicing method with $y_{\min} = 10^{-5}$ (GSb) over the whole range of $\cos\chi$.

other numerical methods give similar results.

We therefore conclude that the KN calculation is indeed correct and that the other predictions (AB, RSE and FK) seem to be deficient in some way. This has a direct impact on the measurements of the strong coupling constant made at LEP and SLC. For example, by eliminating this source of theoretical error, the values of α_s extracted from the EEC and AEEC quoted earlier become[3],

$$\begin{aligned} \alpha_s^{EEC}(M_Z^2) &= 0.125^{+0.002}_{-0.003}\ (\text{exp.}) \pm 0.012\ (\text{theory}), \\ \alpha_s^{AEEC}(M_Z^2) &= 0.114 \pm 0.005\ (\text{exp.}) \pm 0.004\ (\text{theory}), \end{aligned}$$

where, in addition to a reduction in the quoted theoretical error, the central values have also changed.

Finally, the $\mathcal{O}(\alpha_s^2)$ coefficients have also provided an input into QCD calculations where logarithms of the form $\log(1/y)$ where $y = \frac{1+\cos\chi}{2}$ have been resummed [17]. It is worth noting that the coefficients of the logarithmic terms computed using resummation techniques have been shown to agree with the coefficients extracted from the numerical results of Kunszt and Nason and not with those of AB, RSE and FK [18]. This provides additional confirmation of the results presented here and of the validity of the resummation method.

SLD have obtained a value of α_s using such resummed calculations for the EEC, again

[3]See Table 1 of [4]

averaging over the four calculations of $\bar{B}$,

$$\alpha_s^{EEC}(M_Z^2) = 0.130^{+0.003}_{-0.004}\ (\text{exp.}) \pm 0.007\ (\text{theory}) \qquad (\text{resummed}).$$

Using the KN calculation alone, this becomes [4],

$$\alpha_s^{EEC}(M_Z^2) = 0.129^{+0.003}_{-0.004}\ (\text{exp.}) \pm 0.005\ (\text{theory}) \qquad (\text{resummed}),$$

which, within errors, is consistent with the measurements from the EEC and AEEC using the $\mathcal{O}(\alpha_s^2)$ calculations.

In conclusion, motivated by an apparent disagreement amongst theoretical calculations of the $\mathcal{O}(\alpha_s^2)$ coefficient of the energy-energy correlation function, we have recomputed it using three different numerical techniques. With all three methods, we reproduce the results of Kunszt and Nason which have formed the benchmark for extracting a value for the strong coupling constant from LEP and SLC data.

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