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**COHERENT STATE PATH-INTEGRAL
REPRESENTATION
OF SUPERSYMMETRIC LATTICE MODELS**

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International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
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COHERENT STATE PATH-INTEGRAL REPRESENTATION OF SUPERSYMMETRIC LATTICE MODELS

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ABSTRACT

A kind of high temperature superconductivity related lattice model is investigated within the framework of supergroup coherent state path-integral representation. Symmetry properties are analyzed and the Hamiltonians are written in the symmetric form explicitly in terms of generators of the supergroup $U(N/M)$. By a standard approach, general supergroup coherent states are constructed. Holstein-Primakoff realizations of the supergroup $U(N/M)$ on the coset space $\frac{U(N/M)}{U(1) \otimes U(N-1/M)}$ are obtained. Vacuum persistence amplitudes are expressed in terms of parameters on the coset space $\frac{U(2/M)}{U(1) \otimes U(1/M)}$. Symmetry-breaking terms in the Hamiltonian are taken into account separately. The Lagrangians of these models are quadratic in Grassmann variables. Thus fermionic fields can be integrated out. Nonlinear σ model is arrived at as effective continuum field theory describing the low energy excitations of the supersymmetric lattice models.

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1 Introduction

It is well-known that the BCS theory[1] gives viable mechanisms for superconductivity in metals and superfluidity of liquid He₃. This theory has also successfully explained a variety of related phenomena in diverse areas of physics. However, the discovery of copper-oxide superconductivity[2][3] with critical temperature as high as 120 K raised doubts about it. Due to the nature of the phonon-mediated electron-electron interaction in BCS theory, there are upper bounds on the critical temperatures much lower than those achieved with the copper oxides. The lack of a significant isotope effect with substitution of the oxygen sites seems to rule out the possibility that the phonon Debye frequency is the characteristic energy scale entering in the fundamental equations of the high temperature superconductivity. Because the superconducting phase occurs near a metal-insulator transition, an antiferromagnetic as well as a structural instability, the phase diagram of high T_c superconductivity materials is rich. There is a growing suspicion that a different mechanism may be responsible for the high T_c superconductivity. Anderson[4][5] suggested that the novel quantum spin fluctuations in the CuO₂ planes may be responsible for the superconductivity. Interesting magnetic properties revealed by neutron-scattering experiments provide further support for this idea. It was conjectured that such fluctuations might destroy the antiferromagnetic long-range order in the ground state giving rise to a new state of the spin system, a quantum spin-liquid state. The superconductivity in these materials was then conjectured to arise from the behavior of a novel quantum fluid created out of a highly correlated set of electronic degrees of freedom.

Anderson argued that the appropriate model for the high T_c superconductivity is the two-dimensional single-band Hubbard model[6] in the strong on-site Coulomb repulsion limit,

$$H_{\text{Hubb.}} = -t \sum_{\langle jk \rangle} \sum_{\sigma=\pm 1} (c_{j\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{j\sigma}) + U \sum_j c_{j1}^\dagger c_{j1} c_{j,-1}^\dagger c_{j,-1}, \quad (1)$$

where t is the hopping matrix elements and U is the on-site Coulomb interaction. In this model the fermion creation operators create electrons at the outer $d_{x^2-y^2}$ orbital of the Cu atoms, which is hybridized in an antibonding symmetry with the P_x and P_y orbitals of the two oxygen atoms in the CuO₂ cell.

Standard strong-coupling perturbation treatment of the single-band Hubbard Hamil-

tonian produces the effective t - J model[7][8] with the superexchange coupling $J = \frac{4t^2}{U}$,

$$H_{t-J} = -t \sum_{\langle jk \rangle} \sum_{\sigma=\pm 1} (c_{j\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{j\sigma}) + J \sum_{\langle jk \rangle} (S_j^\dagger S_k + S_k^\dagger S_j + 2S_j^3 S_k^3) . \quad (2)$$

The t - J Hamiltonian is an interesting model on its own. This model can be obtained in the strong-coupling limit from a more realistic model that takes into account the more detailed orbital structure of the CuO_2 cell even when the holes created by doping sit primarily on the oxygen sites[9].

At half-filling, apart from a constant, the t - J Hamiltonian is equivalent to the spin- $\frac{1}{2}$ antiferromagnetic Heisenberg model on square lattice,

$$H_{\text{Heisen.}} = J \sum_{\langle jk \rangle} (S_j^\dagger S_k + S_k^\dagger S_j + 2S_j^3 S_k^3) . \quad (3)$$

The spin model has been studied for many years[10][11]. Since 1950's, the importance of field theory to condensed matter physics, and vice versa, has been recognized based on the pioneer works of Landau and Feynman. Conformal field theory describes effectively critical phenomena. Renormalization group, which is developed initially by field theorists, has become as the main tool for the interpretations of experimental data, the conceptual framework and computational algorithm in condensed matter physics. Haldane first used the $SU(2)$ group coherent state path-integral representation[12] to map the spin operators to unit 3-vectors on the large spin limit,

$$\begin{aligned} \vec{S}_j &\longrightarrow s\vec{\varphi}_j , \\ \vec{\varphi}_j \cdot \vec{\varphi}_j &= 1 , \end{aligned} \quad (4)$$

and the system becomes classical. On the other hand, in the long wavelength limit the system approaches a continuum field theory appropriate for the study of low energy excitations. When both approximations are used in conjunction the system is described by a non-linear σ model,

$$L = \frac{1}{2g} \partial_\mu \vec{\varphi} \cdot \partial^\mu \vec{\varphi} , \quad (5)$$

with the constraint, $\vec{\varphi} \cdot \vec{\varphi} = 1$.

Semi-classically we assume spontaneous symmetry breaking[13] and write

$$\begin{aligned} \vec{\varphi} &= \left(\varphi_1, \varphi_2, \sqrt{1 - \sum_{i=1}^2 \varphi_i^2} \right) \\ &\doteq (\varphi_1, \varphi_2, 1) . \end{aligned} \quad (6)$$

Then we have

$$L = \frac{1}{2g} \sum_{i=1}^2 2\partial_\mu \varphi_i \partial^\mu \varphi_i = \frac{1}{2g} \partial_\mu \phi \partial^\mu \phi^\dagger , \quad (7)$$

where we have used the notations $\phi = \varphi_1 + i\varphi_2$, $\phi^\dagger = \varphi_1 - i\varphi_2$.

It is widely believed that the quantum nonlinear σ model provides an accurate description of the long wavelength, low energy properties of the spin system, even in the small spin regime. Affleck and Randjbar-Daemi, Salam and Strathdee (RSS) have generalized the approach to $SU(N)$ and general Lie group cases[14][15].

By defining the so-called spin-hole coherent states

$$|\Omega, \xi\rangle_s = |\Omega\rangle_s |0\rangle + |\Omega\rangle_{s-1/2} \xi f^\dagger |0\rangle , \quad (8)$$

Auerbach and Larson[16] used the coherent state representation to deal with the t - J model. By a Hartree-Fock approximation, a model which is quadratic in the Grassmann variables is arrived. The effective Lagrangian of the t - J model is of the form

$$\begin{aligned} L_{t-J}^{\text{eff.}} &= \sum_j (2s - \rho_j) \vec{A}(\Omega_j) \cdot \dot{\vec{\Omega}}_j - H^J(\Omega) \\ &\quad - \beta^{-1} T \log \left(1 + T_r \exp \left(- \int_0^\beta d\tau (H^J(\Omega_j) - \mu) \right) \right) . \end{aligned} \quad (9)$$

However, no σ -type field theory was obtained explicitly. They argued that although the expansion around the Néel state is formally controlled by the large spin size s , at low doping the success of this approximation for the $s = \frac{1}{2}$ antiferromagnetism can be relied on.

The spin-spin interactions are essential to the high T_c superconductivity related lattice models. As Haldane has shown in the absence of doping the spin-spin interactions can be presented by the 2 + 1-dimensional nonlinear σ model. It is natural to hope that, at least in the low doping case, the spin-spin interactions, in general, should be described by a σ -type field theory.

It is well-known that the novel feature of supersymmetry is that it operates between bosons and fermions. The odd generators of the supergroup corresponding to

transformations between bosons and fermions. Thus supersymmetry provides a unified description of mixed systems of bosons and fermions. Supersymmetry is introduced originally for applications in high energy physics and becomes more and more popular in other fields of physics.

In this paper, we begin with discussing the symmetry properties of these models and rewrite the Hamiltonians in a supersymmetric form. Here the supergroup $U(N/M)$ [17]–[19] plays a fundamental role. The Hamiltonians are written in terms of the generators of $U(N/M)$. For convenience, we introduce Schwinger boson (slave fermion) representation[20][21]. This representation is widely used by condensed matter physicists in dealing with doping problems of high T_c superconductivity. Total particle number of bosons and fermions is used to label representations of $U(N/M)$. The elements of $U(N/M)$ are identified as transformations among states which have the same total particle number. To arrive at a path-integral representation of the supersymmetric models, we construct supergroup coherent states[18][22] from unitary irreducible representations of the supergroup. The supergroup coherent states have a natural topological coset space structure $\frac{U(N/M)}{U(1) \otimes U(N-1/M)}$. Parameterization of the coset space is presented explicitly. A complex projective representation of $\frac{U(N/M)}{U(1) \otimes U(N-1/M)}$ is introduced. We show that two important properties of ordinary group coherent states are maintained by the supergroup coherent states, i.e., they are in general non-orthogonal but are normalized to unity. According to Schurr's lemma, we also give the resolution of identity of the supergroup coherent states. This is crucial for obtaining a path-integral representation. Following the method of RSS[15], we express the vacuum persistence amplitudes of the supersymmetric systems in terms of parameters on the coset space $\frac{U(2/M)}{U(1) \otimes U(1/M)}$ through the Holstein-Primakoff realizations[23]. To agree with the Néel character (at least short range) of the high T_c superconductivity materials, we give two kinds of parameterizations of the coset space on bipartite lattices. After taking account of the symmetry-breaking terms of the Hamiltonians, we get a Lagrangian which describes sensibly the spin fluctuations of the high T_c superconductivity materials. It should be noticed that the Lagrangian is quadratic in Grassmann variables. Thus we can integrate out the fermionic fields and get an effective Lagrangian which only involves bosonic fields. This is just the familiar nonlinear σ model. This result proves the predictions of the weak-coupling mean-field-theory in the Fermion coherent state path-integral representation.

This paper is organized as follows. In Section 2, we show symmetry-properties of the high T_c superconductivity related lattice models. Section 3 is devoted to discussing the Schwinger boson representations of supergroup. Supergroup coherent states are constructed in Section 4. We give the supergroup coherent state path-integral representations of these models in Section 5. Concluding remarks are given in Section 6.

2 High T_c Superconductivity Related Supersymmetric Lattice Models

The existence of antiferromagnetism in the absence of doping is an evidence for strong electron correlations for the high- T_c materials. The extended Hubbard model[24]–[26] is one of the simplest models for describing the electron correlations on a 2-dimensional lattice, which includes nearest-neighbor interactions such as density-density and spin-spin coupling and additional interactions such as a bond charge term and a pair hopping term. The Hamiltonian of the model is given by

$$\begin{aligned} H_{\text{Hubb.}} = & -t \sum_{\langle jk \rangle} \sum_{\sigma=\pm 1} (c_{j\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{j\sigma}) - \frac{t}{2} \sum_{\langle jk \rangle} (n_j - 1)(n_k - 1) \\ & + t \sum_{\langle jk \rangle} (c_{j1}^\dagger c_{j,-1} c_{k,-1}^\dagger c_{k1} + c_{j,-1}^\dagger c_{j1} c_{k1}^\dagger c_{k,-1}) \\ & + t \sum_{\langle jk \rangle} \sum_{\sigma=\pm 1} (c_{j\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{j\sigma})(n_{j,-\sigma} + n_{k,-\sigma}) \\ & - t \sum_{\langle jk \rangle} (c_{j1}^\dagger c_{j,-1}^\dagger c_{k,-1} c_{k1} + c_{k1}^\dagger c_{k,-1}^\dagger c_{j,-1} c_{j1}) - 4t \sum_j \left(n_{j1} - \frac{1}{2}\right) \left(n_{j,-1} - \frac{1}{2}\right) \\ & + U \sum_j c_{j1}^\dagger c_{j1} c_{j,-1}^\dagger c_{j,-1} , \end{aligned} \quad (10)$$

where the operator $c_{j\sigma}$ ($c_{j\sigma}^\dagger$) ($j = 1, 2, \dots, L$ (total number of lattice sites)) describe electrons on lattice and satisfy the anti-commutation relation,

$$\begin{aligned} \{c_{j\sigma}, c_{k\sigma'}\} &= 0 , \\ \{c_{j\sigma}, c_{k\sigma'}^\dagger\} &= \delta_{jk} \delta_{\sigma\sigma'} , \\ \{c_{j\sigma}^\dagger, c_{k\sigma'}^\dagger\} &= 0 . \end{aligned} \quad (11)$$

Here we have used the notations $n_{j\sigma} = c_{j\sigma}^\dagger c_{j\sigma}$ and $n_j = \sum_{\sigma=\pm 1} n_{j\sigma}$, $\sum_{(jk)}$ implies the sum over the nearest neighbors in which the pairs (j, k) and (k, j) have to be counted once each so that the sum can be always considered as symmetric under interchanging $j \leftrightarrow k$.

The Hilbert space of the extended Hubbard model is spanned by the states of the form

$$|a_1\rangle \otimes |a_2\rangle \otimes \cdots \otimes |a_j\rangle \otimes \cdots \otimes |a_L\rangle \quad (12)$$

where $|a_j\rangle \in \{|1_j\rangle = c_{j1}^\dagger |0\rangle_j, |2_j\rangle = c_{j,-1}^\dagger |0\rangle_j, |3_j\rangle = |0\rangle_j, |4_j\rangle = c_{j1}^\dagger c_{j,-1}^\dagger |0\rangle_j\}$. Here $|0\rangle_j$ is defined as $c_{j\sigma} |0\rangle_j = 0$ and the vacuum $|0\rangle = \otimes |0\rangle_j$ satisfies $c_{j\sigma} |0\rangle = 0$.

The system has $SU(2)$ spin symmetry. The spin operator $\vec{S}_j$ is defined by

$$\vec{S}_j = \frac{1}{2} \sum_{\sigma, \sigma'=\pm 1} c_{j\sigma}^\dagger \vec{\tau}_{\sigma\sigma'} c_{j\sigma'} \quad (13)$$

where $\vec{\tau}$ are the three Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (14)$$

Another symmetry of the system is the so-called η -pairing[27] [28] generated by

$$\eta_j = c_{j1} c_{j,-1}, \quad \eta_j^\dagger = c_{j,-1}^\dagger c_{j1}^\dagger, \quad \eta_j^3 = -\frac{1}{2} (n_j - 1). \quad (15)$$

Together with the spin $SU(2)$ symmetry, this gives an $SO(4)$ symmetry.

In terms of the operators of spin and η -spin, the extended Hubbard Hamiltonian is of the form

$$\begin{aligned} H_{\text{Hub.}} = & -t \sum_{(jk)} \sum_{\sigma=\pm 1} (c_{j\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{j\sigma}) - \frac{t}{2} \sum_{(jk)} (n_j - 1)(n_k - 1) \\ & + t \sum_{(jk)} (S_j^1 S_k^1 + S_k^1 S_j^1 + 2S_j^3 S_k^3) \\ & + t \sum_{(jk)} \sum_{\sigma=\pm 1} (c_{j\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{j\sigma}) (n_{j,-\sigma} + n_{k,-\sigma}) \\ & - t \sum_{(jk)} (\eta_j^\dagger \eta_k + \eta_k^\dagger \eta_j + 2\eta_j^3 \eta_k^3) - 4t \sum_j \left(n_{j1} - \frac{1}{2}\right) \left(n_{j,-1} - \frac{1}{2}\right) \\ & + U \sum_j c_{j1}^\dagger c_{j1} c_{j,-1}^\dagger c_{j,-1}. \end{aligned} \quad (16)$$

The above Hamiltonian can be divided formally into two-parts, the supersymmetric part $H_{\text{Hub.}}^0$ and the symmetry-breaking part $H_{\text{Hub.}}^{\text{int}}$,

$$\begin{aligned} H_{\text{Hub.}}^0 = & -t \sum_{(jk)} \sum_{\sigma=\pm 1} (c_{j\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{j\sigma}) - \frac{t}{2} \sum_{(jk)} (n_j - 1)(n_k - 1) \\ & + t \sum_{(jk)} (S_j^1 S_k^1 + S_k^1 S_j^1 + 2S_j^3 S_k^3) \\ & + t \sum_{(jk)} \sum_{\sigma=\pm 1} (c_{j\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{j\sigma}) (n_{j,-\sigma} + n_{k,-\sigma}) \\ & - t \sum_{(jk)} (\eta_j^\dagger \eta_k + \eta_k^\dagger \eta_j + 2\eta_j^3 \eta_k^3) - 4t \sum_j \left(n_{j1} - \frac{1}{2}\right) \left(n_{j,-1} - \frac{1}{2}\right), \\ H_{\text{Hub.}}^{\text{int}} = & U \sum_j c_{j1}^\dagger c_{j1} c_{j,-1}^\dagger c_{j,-1}. \end{aligned} \quad (17)$$

As we know there are four states at each lattice site, two of them are fermionic and the others are bosonic. The terms H_{jk}^0 ($t=1$) act as a minus graded permutations[29] of the electron states at sites j and k . By “graded” we mean that there is an extra minus sign if the two states that are permuted are both single electron states. For example,

$$\begin{aligned} H_{jk}^0 c_{j1}^\dagger |0\rangle &= -c_{k1}^\dagger |0\rangle, \\ H_{jk}^0 c_{j1}^\dagger c_{k,-1}^\dagger |0\rangle &= c_{j,-1}^\dagger c_{k1}^\dagger |0\rangle. \end{aligned} \quad (18)$$

Thus the symmetric part $H_{\text{Hub.}}^0$ can be written as invariant form in terms of generators of the supergroup $U(2/2)$,

$$H_{\text{Hub.}}^0 = -t \sum_{(jk)} \sum_{\alpha, \beta=1}^{16} \text{Str}(X^\alpha X^\beta) X^\alpha X^\beta. \quad (19)$$

The supergroup $U(2/2)$ here is the group of the unitary rotations of the four allowed states, $|a_j\rangle$ ($a=1, \dots, 4$), into one another.

It is easy to show that the off-diagonal matrix element ($j \neq k$) $\frac{\langle \Phi_m | c_{j,-1}^\dagger c_{j1}^\dagger c_{k1} c_{k,-1} | \Phi_m \rangle}{\langle \Phi_m | \Phi_m \rangle}$ is constant for long distances $|j - k|$. Here $|\Phi_m\rangle = (\eta^\dagger)^m |0\rangle$. Indeed we have

$$\frac{\langle \Phi_m | c_{j,-1}^\dagger c_{j1}^\dagger c_{k1} c_{k,-1} | \Phi_m \rangle}{\langle \Phi_m | \Phi_m \rangle} = \frac{m(L - m)}{L(L - 1)}. \quad (20)$$

This fact established the properties of off-diagonal-long-range-order (ODLRO) for the state $|\Phi_m\rangle$. In the thermodynamic limit we have ODLRO as soon as $\frac{m}{L} \rightarrow D$ becomes finite. The ODLRO is characteristic for superconductivity[30].

If the double occupied sites are kinetically forbidden, the restricted Hilbert space of the system now consists of configurations made of empty sites (holes) and up and down spins ($|a_j\rangle$, $a = 1, 2, 3$). The kinetic term will allow for charge motion since empty sites will be able to move. These holes carry electric charge but they have no spin. The effective Hamiltonian now has the form of the t-J model

$$H_{t-J} = -t \sum_{\langle jk \rangle} \sum_{\sigma=\pm 1} (c_{j\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{j\sigma}) + J \sum_{\langle jk \rangle} \left(S_j^\dagger S_k + S_j^\dagger S_j + 2S_j^3 S_k^3 - \frac{n_j n_k}{4} \right). \quad (21)$$

Notice that H_{t-J} acts on the Hilbert space where no double occupancy of sites are allowed.

Also the $t-J$ Hamiltonian can be divided into a symmetric part H_{t-J}^0 and a symmetry-breaking part H_{t-J}^{int} [31]–[33],

$$\begin{aligned} H_{t-J}^0 &= -t \sum_{\langle jk \rangle} \sum_{\sigma=\pm 1} (c_{j\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{j\sigma}) + 2t \sum_{\langle jk \rangle} \left(S_j^\dagger S_k + S_j^\dagger S_j + 2S_j^3 S_k^3 - \frac{n_j n_k}{4} \right), \\ H_{t-J}^{\text{int}} &= (J - 2t) \sum_{\langle jk \rangle} \left(S_j^\dagger S_k + S_j^\dagger S_j + 2S_j^3 S_k^3 - \frac{n_j n_k}{4} \right). \end{aligned} \quad (22)$$

H_{t-J}^0 is of $U(2/1)$ supersymmetric invariance. It generates the unitary rotations among the allowed states of the $t-J$ model, $|a_j\rangle$ ($a = 1, 2, 3$). And the terms H_{t-J}^{int} act as a minus graded permutations of these states.

At the half-filled case of the extended Hubbard model, only the spin up and down states $|a_j\rangle$ ($a = 1, 2$) are allowed and no fermionic states are left. The permutation group in this case becomes as the Lie group $U(2)$. And the symmetric Hamiltonian reduces to the usual spin- $\frac{1}{2}$ quantum Heisenberg Hamiltonian

$$H_{\text{Heisen.}} = J \sum_{\langle jk \rangle} (S_j^\dagger S_k + S_k^\dagger S_j + 2S_j^3 S_k^3). \quad (23)$$

Generally speaking, from n states $|a\rangle$ ($a = 1, 2, \dots, n$) which satisfy

$$\langle a|b\rangle = \delta^{ab}, \quad (24)$$

one can construct $n \times n$ independent real operators X^{ab}

$$X^{ab} = |a\rangle\langle b| \quad a, b = 1, 2, \dots, n. \quad (25)$$

With the local states $|a_j\rangle$ which satisfy the relation

$$\langle a_j|b_k\rangle = \delta_{jk}\delta^{ab} \quad (26)$$

we can define the local operators $X_j^{ab} = |a_j\rangle\langle b_j|$.

With the local Hubbard operators X_j^{ab} the extended Hubbard Hamiltonian can be cast into the following form

$$H_{\text{Hub.}} = -t \sum_{\langle jk \rangle} \sum_{a,c=1}^4 X_j^{ac} X_k^{ca} (-1)^{F(a) \cdot F(c)} + U \sum_j X_j^{44}, \quad (27)$$

where F is the graded number

$$F(x) = \begin{cases} 1 & \text{for } x = 1, 2 \\ 0 & \text{for } x = 3, 4 \end{cases}.$$

The $t-J$ and Heisenberg Hamiltonians in terms of the local Hubbard operators are

$$H_{t-J} = -t \sum_{\langle jk \rangle} \sum_{a,c=1}^4 X_j^{ac} X_k^{ca} (-1)^{F(a) \cdot F(c)} + (J - 2t) \sum_{\langle jk \rangle} \sum_{a,c=1}^2 X_j^{ac} X_k^{ca}; \quad (28)$$

$$H_{\text{Heisen.}} = J \sum_{\langle jk \rangle} \sum_{a,c=1}^2 X_j^{ac} X_k^{ca}. \quad (29)$$

We clearly see that the extended Hubbard model, the t-J model and Heisenberg model can be described in an unified manner. The Hamiltonians of these models can be written into a supersymmetric form explicitly in terms of the generators of the supergroup $U(2/M)$. In the following, we investigate the supergroup coherent state path-integral presentations of the $U(2/M)$ models.

It should be noticed that, so far, the discussion has been entirely about the $s = 1/2$ case. Before going ahead further, we would like to generalize to high s . Let us imagine that the band electrons have an orbital degeneracy labeled by an index $a = 1, 2, \dots, \mathcal{N}$, where $\mathcal{N}$ is the number of degenerate bands. The total band spin at a given site j is now given by $\tilde{S}_j = \frac{1}{2} \sum_{\sigma, \sigma'=\pm 1} \sum_{a=1}^{\mathcal{N}} c_{j\sigma,a}^\dagger \vec{\tau}_{\sigma\sigma'} c_{j\sigma',a}$. This system still has the global $SU(2)$ invariant of spin rotations. At the half-filling case, the local spin gets to be as large as possible. The equivalent Heisenberg model has a total spin quantum number s at each site equal to $s = \mathcal{N}/2$. The limit $\mathcal{N} \rightarrow \infty$ is the same as the semiclassical limit $s \rightarrow \infty$ [34].

3 Schwinger Bosons (Slave Fermions) Representations

Arovas and Auerbach[21] introduced two commuting Schwinger bosons b^i ($b^{i\dagger}$) ($i = 1, 2$) to describe states of spin. The operator b^i ($b^{i\dagger}$) obeys the commutation relation

$$[b^i, b^{j\dagger}] = \delta^{ij}, \quad (30)$$

and satisfies the constrain

$$\sum_{i=1}^2 b^{i\dagger} b^i = 1. \quad (31)$$

The bilinear forms of the Schwinger bosons yield the spin operators

$$\vec{S} = \frac{1}{2}(b^{1\dagger}, b^{2\dagger})\vec{\tau} \begin{pmatrix} b^1 \\ b^2 \end{pmatrix}. \quad (32)$$

Furthermore, a slave fermion f ($f^\dagger$)[16] and the two Schwinger bosons can be used to represent the allowed states of the projected Hilbert space of the $t - J$ model. The slave fermion obeys the anticommutation relation

$$\{f^\dagger, f\} = 1, \quad (33)$$

and satisfies the constrains

$$\sum_{i=1}^2 b^{i\dagger} b^i + f^\dagger f = 1. \quad (34)$$

The $t - J$ model is faithfully represented by

$$H^{t-J} = t \sum_{\langle jk \rangle} f_j^\dagger f_k \mathcal{F}_{jk} - J \sum_{\langle jkl \rangle} (\delta_{jl} - f_l^\dagger f_j) \mathcal{A}_{lk}^\dagger \mathcal{A}_{lk} (1 - f_k^\dagger f_k), \quad (35)$$

where

$$\begin{aligned} \mathcal{A}_{jk} &= b_j^{1\dagger} b_k^{2\dagger} - b_j^{2\dagger} b_k^{1\dagger}, \\ \mathcal{F}_{jk} &= b_k^{1\dagger} b_j^{1\dagger} - b_k^{2\dagger} b_j^{2\dagger}. \end{aligned}$$

Here $\langle jkl \rangle$ are triads of nearest neighbors.

In the general case, to denote the allowed states of the high temperature superconductivity related lattice models, one can introduce a set of Schwinger bosons and slave

fermions at each site, b^i ($b^{i\dagger}$) ($i = 1, 2, \dots, N$) and f^α ($f^{\alpha\dagger}$) ($\alpha = 1, 2, \dots, M$), which satisfy the following commutation (anticommutation) relations

$$\begin{aligned} [b^i, b^{j\dagger}] &= \delta^{ij}, \quad i, j = 1, 2, \dots, N, \\ \{f^\alpha, f^{\beta\dagger}\} &= \delta^{\alpha\beta}, \quad \alpha, \beta = 1, 2, \dots, M, \\ [b^i, f^\alpha] &= 0 = [b^i, f^{\alpha\dagger}], \\ \{f^\alpha, f^\beta\} &= 0 = [b^i, b^j]. \end{aligned} \quad (36)$$

For the sake of compactness, we denote the boson and fermion operators generically as ξ^A ($\xi^{\dagger A}$)

$$\begin{aligned} \xi^A (\xi^{\dagger A}), \quad A &= 1, 2, \dots, N + M, \\ \xi^i &\equiv b^i, \quad i = 1, 2, \dots, N, \\ \xi^{N+\alpha} &\equiv f^\alpha, \quad \alpha = 1, 2, \dots, M, \end{aligned} \quad (37)$$

and write symbolically

$$[\xi^A, \xi^{B\dagger}] = \delta^{AB}, \quad (38)$$

where the product $[\cdot, \cdot]$ is to be understood as an anticommutator between any two fermionic components and as a commutator otherwise.

It is well-known that the bosonic bilinears $b^i b^{j\dagger}$ and the fermionic bilinears $f^\alpha f^{\beta\dagger}$ generate the Lie algebras $u(N)$ and $u(M)$ under commutation, respectively. The bose-fermi bilinears $b^i f^{\alpha\dagger}$ and $f^\alpha b^{i\dagger}$ close into the set $b^i b^{j\dagger}$ and $f^\alpha f^{\beta\dagger}$ under anticommutation

$$\begin{aligned} \{f^{\alpha\dagger} b^i, b^{j\dagger} f^\beta\} &= f^{\alpha\dagger} f^\beta \delta^{ij} + b^{j\dagger} b^i \delta^{\alpha\beta}, \\ \{f^{\alpha\dagger} b^i, f^{\beta\dagger} b^j\} &= 0 = \{b^{i\dagger} f^\alpha, b^{j\dagger} f^\beta\}. \end{aligned} \quad (39)$$

Thus, considering the boson-fermion bilinears as the odd generators and bosonic bilinears and fermionic bilinears as the even generators, one finds that the operators $\xi^{A\dagger} \xi^B$ form the superalgebra $u(N/M)$ [17,18].

Indeed, we have

$$\begin{aligned} X^{AB} &= \xi^{A\dagger} \xi^B, \\ \{X^{AC}, X^{BD}\} &= X^{AD} \delta^{BC} \pm X^{BC} \delta^{AD}. \end{aligned} \quad (40)$$

Thus X^{AB} generates the supergroup $U(N/M)$.

The Fock space of the system is of the form

$$\begin{aligned}
|0\rangle &= \text{vacuum} , \\
b^{\dagger}|0\rangle, f^{\alpha\dagger}|0\rangle &= 1 - \text{particle states} , \\
b^{i_1\dagger}b^{i_2\dagger}|0\rangle, b^{i_1\dagger}f^{\alpha_1\dagger}|0\rangle, f^{\alpha_1\dagger}f^{\alpha_2\dagger}|0\rangle &= 2 - \text{particle states} , \\
&\vdots \\
b^{i_1\dagger}b^{i_2\dagger}\dots b^{i_k\dagger}f^{\alpha_1\dagger}f^{\alpha_2\dagger}\dots f^{\alpha_{\mathcal{L}-k}\dagger}|0\rangle &= \mathcal{L} - \text{particle states} .
\end{aligned} \tag{41}$$

Generally, the $\mathcal{L}$ -particle states may contain k -bosons and $\mathcal{L} - k$ fermions, where $1 \leq k \leq \mathcal{L}$.

It is easy to see that the total particle number operator

$$\begin{aligned}
N_{bf} &= \sum_i b^{\dagger}b^i + \sum_{\alpha} f^{\alpha\dagger}f^{\alpha} \\
&= \sum_A \xi^{A\dagger}\xi^A
\end{aligned} \tag{42}$$

has the fixed eigenvalue $\mathcal{L}$ for the $\mathcal{L}$ -particle states of any value of k , $1 \leq k \leq \mathcal{L}$. It should be used to label the Fock space of the supersymmetric systems.

Supposing that the systems we are interested in are of $\mathcal{N}$ -fold orbital degeneracy, we would focus on the $\mathcal{N}$ -particle states of the Fock space. The $\mathcal{N}$ -particle states may be written in terms of the operator $\xi^{A\dagger}$,

$$\xi^{A_1\dagger}\xi^{A_2\dagger}\dots\xi^{A_{\mathcal{N}}\dagger}|0\rangle . \tag{43}$$

If we analyze the $\mathcal{N}$ -particle state as a tensor in the boson and fermion indices, we find it is symmetric in bosonic indices $(i_1, i_2, \dots, i_k)$ and antisymmetric in the fermionic indices $(\alpha_1, \alpha_2, \dots, \alpha_{\mathcal{N}-k})$. Thus, in terms of Young tableau, we may represent it as

$$\left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \underbrace{\heartsuit\heartsuit\dots\heartsuit\heartsuit}_k, \heartsuit \\ \vdots \\ \heartsuit \\ \heartsuit \end{array} \right) \mathcal{N} - k . \tag{44}$$

It indicates a direct product of a symmetric representation of $U(N)$ and an antisymmetric representation of $U(M)$. Thus, in terms of $U(N) \otimes U(M)$ representations, the $\mathcal{N}$ -particle state are the direct sum

$$\left(\underbrace{\heartsuit\heartsuit\dots\heartsuit\heartsuit}_{\mathcal{N}}, 0 \right) \otimes \left(\underbrace{\heartsuit\heartsuit\dots\heartsuit\heartsuit}_{\mathcal{N}-1}, \heartsuit \right) \otimes \left(\underbrace{\heartsuit\heartsuit\dots\heartsuit\heartsuit}_{\mathcal{N}-2}, \heartsuit, \heartsuit \right) \otimes \dots \otimes \left(0, \begin{array}{c} \heartsuit \\ \heartsuit \\ \vdots \\ \heartsuit \\ \heartsuit \end{array} \right) \mathcal{N} . \tag{45}$$

We will indicate the supertensor by a super Young tableau

$$\left(\underbrace{\spadesuit\spadesuit\dots\spadesuit\spadesuit}_{\mathcal{N}} \right) . \tag{46}$$

Thus, the $U(N/M)$ supertableau, if decomposed into its $U(N) \otimes U(M)$ parts, is seen to be Eq.(45).

Therefore, the $\mathcal{N}$ -particle state may be understood as a collection of irreducible representations of the $U(N) \otimes U(M)$ subgroup. X^{ij} ($X^{\alpha\beta}$) kills a boson (fermion) and creates another boson (fermion) and $X^{\alpha i}$ ($X^{i\alpha}$) kills a boson (fermion) and creates a fermion (boson). Thus,

$$\begin{aligned}
X^{ij} \left| \left(\underbrace{\heartsuit \dots \heartsuit}_k, \dots, \heartsuit \right) \mathcal{N}-k \right\rangle &= \left| \left(\underbrace{\heartsuit \dots \heartsuit}_k, \dots, \heartsuit \right) \mathcal{N}-k \right\rangle, \\
X^{\alpha\beta} \left| \left(\underbrace{\heartsuit \dots \heartsuit}_k, \dots, \heartsuit \right) \mathcal{N}-k \right\rangle &= \left| \left(\underbrace{\heartsuit \dots \heartsuit}_k, \dots, \heartsuit \right) \mathcal{N}-k \right\rangle, \\
X^{\alpha i} \left| \left(\underbrace{\heartsuit \dots \heartsuit}_k, \dots, \heartsuit \right) \mathcal{N}-k \right\rangle &= \left| \left(\underbrace{\heartsuit \dots \heartsuit}_{k-1}, \dots, \heartsuit \right) \mathcal{N}-k+1 \right\rangle, \\
X^{i\alpha} \left| \left(\underbrace{\heartsuit \dots \heartsuit}_k, \dots, \heartsuit \right) \mathcal{N}-k \right\rangle &= \left| \left(\underbrace{\heartsuit \dots \heartsuit}_{k+1}, \dots, \heartsuit \right) \mathcal{N}-k-1 \right\rangle.
\end{aligned} \tag{47}$$

So that the $X^{\alpha i}$ ($X^{i\alpha}$) acts as a ladder operator hopping to the right (left) in Eq.(45). If we apply all the operators X^{AB} taken to any power on the $\mathcal{N}$ -particle state, we remain within the set of $\mathcal{N}$ -particle states. This shows that the supertableau (46) represents an irreducible set of states under all transformations of supergroup $U(N/M)$.

4 Supergroup Coherent States

Consider an arbitrary quantum mechanical lattice model, whose Hamiltonian is expressed in terms of generators of a supergroup G , $X_j^I \in g$,

$$H = H(X_j^I). \tag{48}$$

We are now interested only in the case of the Hamiltonian involving linear and quadratic terms of X_j^I ,

$$H = \sum_{j,k} \sum_{l,m} c_{lm}^{jk} X_j^l X_k^m + \sum_j \sum_l c_j^l X_j^l. \tag{49}$$

The superalgebra g spanned by the operators X_j^I is closed under commutation (anti-commutation)

$$[X_j^k, X_j^l] = \sum_m c_m^{kl} X_j^m, \tag{50}$$

where c_m^{kl} are the structure constants of g .

The group-theoretic approach to coherent states involves the use of unitary group representations[35]–[37]. We know that the standard supergroup elements connected to the identity are obtained by exponentiating superalgebra elements. There is thus one-to-one correspondence between such elements of the supergroup and points in the flat superspace $RM_L^{m,n}$ [38]–[40], which is the Cartesian product of m copies of RB_{L0} with n copies of RB_{L1} . A Grassmann algebra RB_L is an associative algebra generated by identity 1 and by L elements β_a ($a = 1, 2, \dots, L$) obeying the anticommutation relations $\{\beta_a, \beta_b\} = 0$. The algebra is spanned by the identity and all independent nonvanishing products of β_a . There are 2^L linear independent basis elements. The subset of basis elements consisting of identity and all even products of the generators spans the even part RB_{L0} of RB_L ; the remaining basis elements span the odd part RB_{L1} .

The set of left translations on a supergroup forms a supermodule W . This is the direct product of RB_L with a superalgebra. We denote the superalgebra generators[41] by $X^1, X^2, \dots, X^m; X^{m+1}, X^{m+2}, \dots, X^{m+n}$. To obtain a unitary representation $T(g)$, we must work with the super-Hermitian basis of the superalgebra by choosing the generators so that $X^{j\dagger} = X^j$ for $j = 1, 2, \dots, m$ and $X^{j\dagger} = -X^j$ otherwise. Thus the operator

$$\begin{aligned}
T(g) &= \exp \left(\sum_{j=1}^{N^2+M^2} iA^j X^j + \sum_{j=1}^{2NM} i\theta^j X^{m+j} \right), \\
X^{j\dagger} &= X^j, \quad \text{for } j = 1, 2, \dots, N^2 + M^2, \\
X^{j\dagger} &= -X^j, \quad \text{for } j = N^2 + M^2 + 1, N^2 + M^2 + 2, \dots, (N + M)^2, \\
A^j &\in RB_{L0}, \quad \theta^j \in RB_{L1},
\end{aligned} \tag{51}$$

defines an unitary representation of the supergroup $U(N/M)$.

To go to the superHermitian basis we must choose Hermitian combinations of the even generators and antiHermitian combinations of the odd ones. For the even generators we thus choose

$$X^k = \begin{cases} X^{ij} + X^{ji}, \\ i(X^{ij} - X^{ji}), \\ X^{\alpha\beta} + X^{\beta\alpha}, \\ i(X^{\alpha\beta} - X^{\beta\alpha}), \end{cases} \quad k = 1, 2, \dots, N^2 + M^2. \tag{52}$$

And for the odd generators we choose the following antiHermitian operators

$$X^k = \begin{cases} X^{\alpha i} - X^{i\alpha}, \\ i(X^{\alpha i} + X^{i\alpha}), \end{cases} \quad k = N^2 + M^2 + 1, \dots, (N^2 + M^2). \tag{53}$$

For the Hamiltonian (49), the Hilbert space V^Λ is a direct sum of unitary irreducible representations Γ_j of the supergroup G at each sites of lattice. In principle, we can choose an arbitrary state $|\Phi_0\rangle = |\Lambda\rangle$ within each unitary irreducible representation Γ_j^Λ , which can be normalized to unity $\langle\Phi_0|\Phi_0\rangle = 1$. Acting by unitary representation Γ_j^Λ on the reference state Λ , we get the supergroup coherent states.

It should be noticed that there is a subgroup of $U(N/M)$ that consists of all the group elements h that leaves the reference state invariant up to a phase factor, called the maximum-stability subgroup H , i.e.,

$$h|\Lambda\rangle = |\Lambda\rangle e^{i\psi(h)}, \quad h \in H. \tag{54}$$

Every element g of the supergroup $U(N/M)$ can be uniquely decomposed into a product of two elements, one in H and the other in the quotient $U(N/M)/H$,

$$g = \Omega h, \quad g \in U(N/M), \quad h \in H, \quad \Omega \in U(N/M)/H. \tag{55}$$

So that we have

$$\begin{aligned}
g|\Lambda\rangle &= \Omega h|\Lambda\rangle \\
&= \Omega|\Lambda\rangle e^{i\psi(h)}.
\end{aligned} \tag{56}$$

$|SC\rangle \equiv \Omega|\Lambda\rangle$ is the supergroup coherent states. This definition of supergroup coherent states guarantees that there is a one-to-one correspondence with the coset space $U(N/M)/H$. So that the coordinates of the supergroup coherent states, $z^i \in RB_{L0}$ ($\mu = 1, 2, \dots, p$), $z^\alpha \in RB_{L1}$ ($\alpha = 1, 2, \dots, q$; and $p + q = \dim U(N/M)/H$), may be chosen in any convenient parametrize representatives, $T(\Omega)$, from the cosets $U(N/M)/H$.

It is easy to see that

$$\begin{aligned}
X^{\alpha\beta} \frac{1}{\sqrt{\mathcal{N}!}} (b^\dagger)^\mathcal{N} |0\rangle &= 0, \\
X^{ij} \frac{1}{\sqrt{\mathcal{N}!}} (b^\dagger)^\mathcal{N} |0\rangle &= 0, \quad \text{for } j \neq 1, \\
X^{i\alpha} \frac{1}{\sqrt{\mathcal{N}!}} (b^\dagger)^\mathcal{N} |0\rangle &= 0, \\
X^{\alpha i} \frac{1}{\sqrt{\mathcal{N}!}} (b^\dagger)^\mathcal{N} |0\rangle &= 0, \quad \text{for } i \neq 1.
\end{aligned} \tag{57}$$

These operators span a subalgebra $u(1) \otimes u(N-1/M)$ of $u(N/M)$. The corresponding supergroup $U(1) \otimes U(N-1/M)$ is just the stability subgroup of $U(N/M)$ which leaves the state $\frac{1}{\sqrt{\mathcal{N}!}} (b^\dagger)^\mathcal{N} |0\rangle$ invariant.

Choosing $\frac{1}{\sqrt{\mathcal{N}!}} (b^\dagger)^\mathcal{N} |0\rangle$ as the reference state $|\Lambda\rangle$ in the representation denoted by the super Young tableau (46), we obtain the supergroup coherent states by acting

$$\Omega = \exp \left(\sum_{j=1}^N \eta^j X^{1j} + \sum_{\alpha=1}^M \eta^\alpha X^{1\alpha} - \sum_{j=1}^N \eta^{j\dagger} X^{j1} - \sum_{\alpha=1}^M \eta^{\alpha\dagger} X^{1\alpha} \right) \tag{58}$$

on the chosen reference state $\frac{1}{\sqrt{\mathcal{N}!}} (b^\dagger)^\mathcal{N} |0\rangle$,

$$\begin{aligned}
|SC\rangle &= \Omega|\Lambda\rangle \\
&= \exp \left(\sum_{j=1}^N \eta^j X^{1j} + \sum_{\alpha=1}^M \eta^\alpha X^{1\alpha} - \sum_{j=1}^N \eta^{j\dagger} X^{j1} - \sum_{\alpha=1}^M \eta^{\alpha\dagger} X^{1\alpha} \right) \frac{1}{\sqrt{\mathcal{N}!}} (b^\dagger)^\mathcal{N} |0\rangle,
\end{aligned} \tag{59}$$

where $\eta^j \in CB_{L0}$, $\eta^\alpha \in CB_{L1}$ and CB_{L0} , CB_{L1} denotes the even and odd part of a complex Grassmann algebra with L elements, respectively.

These supergroup coherent states have a natural topological coset space $\frac{U(N/M)}{U(1) \otimes U(N-1/M)}$.

The fundamental representation of the superalgebra $u(N/M)$ is generated by the $(N+M)$ -dimensional matrices,

$$(X^{AB})_{mn} = \delta_m^A \delta_n^B. \quad (60)$$

Thus, we have

$$\begin{aligned} \Omega &= \exp \left(\sum_{j=1}^N \eta^j X^{1j} + \sum_{\alpha=1}^M \eta^\alpha X^{1\alpha} - \sum_{j=1}^N \eta^{j\dagger} X^{j1} - \sum_{\alpha=1}^M \eta^{\alpha\dagger} X^{1\alpha} \right) \\ &= \exp \begin{pmatrix} \bigcirc & \eta \\ -\eta^\dagger & \bigcirc \end{pmatrix}, \end{aligned} \quad (61)$$

where η is an $(N+M-1)$ -component rank vector with the first $(N-1)$ elements ordinary complex numbers while the last M elements complex Grassmann numbers. Therefore, in the fundamental representation, the finite transformation Ω is of the form

$$(\Omega)_C^D(z) = \begin{pmatrix} \sqrt{1-zz^\dagger} & z \\ -z^\dagger & \sqrt{1-z^\dagger z} \end{pmatrix}, \quad (62)$$

where $z = \eta \frac{\sin \sqrt{\eta^\dagger \eta}}{\sqrt{\eta^\dagger \eta}}$, is also an $(N+M-1)$ -component rank vector with the same grad with η . This is a supersymmetric generalization of the bosonic case discussed in [15].

If we explicitly introduce a complex projective representation τ of $\frac{U(N/M)}{U(1) \otimes U(N-1/M)}$,

$$\tau = \frac{z}{\sqrt{1-z^\dagger z}}, \quad (63)$$

any group transformation g acting on the coset space $\frac{U(N/M)}{U(1) \otimes U(N-1/M)}$ must be a holomorphic transformation,

$$\tau' = g\tau = \frac{A\tau + B}{C\tau + D}, \quad (64)$$

where

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(N+M),$$

A (D) is a 1×1 ($(N+M-1) \times (N+M-1)$) complex matrix, B (C) is a $1 \times (N+M-1)$ ($(N+M-1) \times 1$) matrix.

Just as the ordinary group coherent states, two important algebraic properties are maintained by the supergroup coherent states. From the definition of the supergroup coherent states, we have

$$\begin{aligned} \langle SC(z) | SC(z') \rangle &= \langle \Lambda | \Omega^{-1}(z) \Omega(z') | \Lambda \rangle \\ &= \langle \Lambda | \Omega(z'') | \Lambda \rangle e^{i\psi} \neq 0, \end{aligned} \quad (65)$$

and

$$\begin{aligned} \langle SC(z) | SC(z) \rangle &= \langle \Lambda | \Omega^{-1}(z) \Omega(z) | \Lambda \rangle \\ &= \langle \Lambda | \Lambda \rangle = 1. \end{aligned} \quad (66)$$

Thus the supergroup coherent states are general non-orthogonal but normalized to unity.

Define the operator O as

$$O = \int |SC(z)\rangle d\mu(\Omega(z)) \langle SC(z)|, \quad (67)$$

we obtain that

$$\begin{aligned} gO &= \int g |SC(z)\rangle d\mu(\Omega(z)) \langle SC(z)| g^{-1} g \\ &= \int |SC(z')\rangle e^{i\phi'(h)} d\mu(\Omega(z')) e^{-i\phi'(h)} \langle SC(z')| g \\ &= Og, \quad g \in U(N/M), \end{aligned} \quad (68)$$

where $d\mu(\Omega)$ is the supergroup-invariant measure of $U(N/M)$ possessing the property $d\mu(\Omega(z)) = d\mu(\Omega(z'))$.

Therefore, according to Schurr's lemma, the operator O must be proportional to the identity operator. Hence, with an appropriately normalized measure $d\mu$, we can get

$$\int |SC(z)\rangle d\mu(\Omega(z)) \langle SC(z)| = 1. \quad (69)$$

The supergroup coherent states can be expanded in a complete orthonormal basis $|\lambda\rangle$,

$$|SC(z)\rangle = \sum_{\lambda} |\lambda\rangle \langle \lambda | SC(z) \rangle, \quad (70)$$

and it is possible to project the orthonormal basis vectors from the supergroup coherent states by making use of Eq.(69),

$$|\lambda\rangle = \int d\mu(\Omega(z)) |SC(z)\rangle \langle SC(z)|\lambda\rangle. \quad (71)$$

The supergroup coherent states therefore constitute an over-complete basis.

5 Path-integral Representations

Typically we are interested in studying both zero-temperature and finite-temperature properties of a system. At finite temperature, the equilibrium properties are determined by the partition function

$$Z = \text{Tr} e^{-H/k_B T}. \quad (72)$$

At zero temperature we are interested in the vacuum persistence amplitudes

$$Z = \text{Tr} \hat{T} \exp \left(-\frac{i}{\hbar} \int_{-\infty}^{\infty} H d\tau \right), \quad (73)$$

where $\hat{T}$ is the time order operator.

In this section, we generalize the method of RSS[15] to supersymmetric systems and give the supergroup coherent state path-integral representation of the high T_c superconductivity related lattice models. For a supersymmetric system with the supergroup theoretic Hamiltonian H , the standard path integral is derived as follows. We first split up the time interval τ into N_τ segments of infinitesimal length $\Delta\tau$ such that $N_\tau \cdot \Delta\tau = \tau$. For infinitesimal intervals $\Delta\tau \rightarrow 0$ we can write

$$\begin{aligned} Z &= \text{Tr} \hat{T} \exp \left(-\frac{i}{\hbar} \sum_{j=1}^{N_\tau} H(\tau_j) \Delta\tau \right) \\ &= \text{Tr} \hat{T} \prod_{j=1}^{N_\tau} \exp \left(-\frac{i}{\hbar} H(\tau_j) \Delta\tau \right). \end{aligned} \quad (74)$$

Then we insert the resolution of identity (69) at each intermediate time τ_j . We get

$$\begin{aligned} Z &= \lim_{\Delta\tau \rightarrow 0} \int \langle SC(z_f) | e^{-iH\Delta\tau/\hbar} | SC(z_f - \frac{dz}{d\tau} \Delta\tau) \rangle d\mu(\Omega(z_f - \frac{dz}{d\tau} \Delta\tau)) \\ &\quad \times \langle SC(z_f - \frac{dz}{d\tau} \Delta\tau) | e^{-iH\Delta\tau/\hbar} | SC(z_f - \frac{dz}{d\tau} 2\Delta\tau) \rangle d\mu(\Omega(z_f - \frac{dz}{d\tau} 2\Delta\tau)) \\ &\quad \times \langle SC(z_f - \frac{dz}{d\tau} 2\Delta\tau) | \dots \dots \dots | SC(z_i + \frac{dz}{d\tau} \Delta\tau) \rangle d\mu(\Omega(z_i + \frac{dz}{d\tau} \Delta\tau)) \\ &\quad \times \langle SC(z_i + \frac{dz}{d\tau} \Delta\tau) | e^{-iH\Delta\tau/\hbar} | SC(z_i) \rangle \\ &= \int \mathcal{D}[d\mu(\Omega(\tau))] e^{\frac{i}{\hbar} \int_{-\infty}^{\infty} L d\tau}, \end{aligned} \quad (75)$$

where $\mathcal{D}[d\mu(\Omega(\tau))]$ is the functional measure of the path integral and L is the Lagrangian of the system.

The overlap between neighboring coherent states is

$$\begin{aligned} \langle SC(z + dz) | SC(z) \rangle &= \langle \Lambda | \Omega^{-1}(z + dz) \Omega(z) | \Lambda \rangle \\ &= 1 - \frac{1}{2} \mathcal{N}(z dz^\dagger - dz z^\dagger). \end{aligned} \quad (76)$$

Hence the Lagrangian of the system takes the form

$$L = -\frac{\hbar}{2i} \mathcal{N}(z \partial_\tau z^\dagger - \partial_\tau z z^\dagger) - H(z, z^\dagger). \quad (77)$$

The canonical momenta of the system are

$$\begin{aligned} \pi_i &= \frac{\partial L}{\partial \dot{z}^i} \\ &= \frac{\hbar}{i} \mathcal{N} z^{i\dagger}. \end{aligned} \quad (78)$$

Canonical quantization gives the commutation rules

$$\begin{aligned} [z^i, z^j] &= 0, \\ [z^i, z^{j\dagger}] &= \frac{1}{\mathcal{N}} \delta_{ij}, \\ [z^{i\dagger}, z^{j\dagger}] &= 0. \end{aligned} \quad (79)$$

To construct the generators out of these operators it is helpful to consider firstly their coherent state expectation values,

$$\begin{aligned} \langle SC(z) | X^{AB} | SC(z) \rangle &= \langle \Lambda | \Omega^{-1}(z) X^{AB} \Omega(z) | \Lambda \rangle \\ &= (\Omega^{-1}(z))^{AC} \langle \Lambda | X^{CD} | \Lambda \rangle (\Omega(z))^{DB} \\ &= \mathcal{N}(\Omega^{-1}(z) Y \Omega(z))^{AB}, \end{aligned} \quad (80)$$

where

$$Y = \begin{pmatrix} \mathcal{N} & 0 \\ 0 & 0 \end{pmatrix}.$$

With the matrices $\Omega(z)$ given by Eq.(62) one obtains,

$$\langle SC(z) | X^{AB} | SC(z) \rangle = \begin{pmatrix} \mathcal{N}(1 - zz^\dagger) & \mathcal{N}\sqrt{1 - zz^\dagger}z \\ \mathcal{N}z^\dagger\sqrt{1 - zz^\dagger} & \mathcal{N}z^\dagger z \end{pmatrix}. \quad (81)$$

The classical expressions on the r.h.s. of the above equation suggest the following operator realization

$$\begin{aligned} X^{11}(z) &= \mathcal{N} \left(1 - \sum_{k=1}^{N+M-1} z^{k\dagger} z^k \right), \\ X^{1(j+1)}(z) &= \mathcal{N} \sqrt{1 - \sum_{k=1}^{N+M-1} z^{k\dagger} z^k} z^j, \quad j = 1, 2, \dots, N+M-1, \\ X^{(i+1)1}(z) &= \mathcal{N} z^{i\dagger} \sqrt{1 - \sum_{k=1}^{N+M-1} z^{k\dagger} z^k}, \quad i = 1, 2, \dots, N+M-1, \\ X^{(i+1)(j+1)}(z) &= \mathcal{N} z^{i\dagger} z^j. \end{aligned} \quad (82)$$

It is easy to verify, using the commutation rules (79) that these operators satisfy the superalgebra $u(N/M)$.

From now on we discuss the high temperature superconductivity related lattice models and so that we restrict ourselves at the case of $N = 2$. We expect that the high temperature superconductivity materials, at least the short-range order, should have Néel character, it is natural to consider the staggered and uniform components of the z -field, i.e., on sublattice A we use the above z -parameterization and on sublattice B we use another set of parameterization. Making use of a parameter transformation of the supergroup coherent states,

$$\begin{aligned} z^1 &\rightarrow -z^1 \sqrt{1 - \sum_{i=1}^{1+M} z^{i\dagger} z^i}, \\ z^{1\dagger} &\rightarrow -\sqrt{1 - \sum_{i=1}^{1+M} z^{i\dagger} z^i} z^{1\dagger}, \\ z^i &\rightarrow \frac{z^{1\dagger}}{\sqrt{z^{1\dagger} z^1}} z^i, \quad i = 2, 3, \dots, 1+M, \\ z^{i\dagger} &\rightarrow z^{i\dagger} \frac{z^{1\dagger}}{\sqrt{z^{1\dagger} z^1}}, \quad i = 2, 3, \dots, 1+M. \end{aligned} \quad (83)$$

we get the other set of expectations of X_j^{ac} ,

$$\langle SC(z') | X^{ac} | SC(z') \rangle = \begin{pmatrix} \mathcal{N} z^{1\dagger} z^1 & -\mathcal{N} \sqrt{1 - \sum_{i=1}^{1+M} z^{i\dagger} z^i} z^1 & \mathcal{N} z^{1\dagger} z' \\ -\mathcal{N} z^{1\dagger} \sqrt{1 - \sum_{i=1}^{1+M} z^{i\dagger} z^i} & \mathcal{N} (1 - \sum_{i=1}^{1+M} z^{i\dagger} z^i) & -\mathcal{N} \sqrt{1 - \sum_{i=1}^{1+M} z^{i\dagger} z^i} z' \\ \mathcal{N} z^{i\dagger} z^1 & -\mathcal{N} z^{i\dagger} \sqrt{1 - \sum_{i=1}^{1+M} z^{i\dagger} z^i} & \mathcal{N} z^{i\dagger} z' \end{pmatrix}, \quad (84)$$

where we have used the notation $z' = (z^2, z^3, \dots, z^{1+M})$.

This set of expectations arise another Holstein-Primakoff realization of the superalgebra $u(2/M)$

$$\begin{aligned} X^{11}(z) &= \mathcal{N} z^{1\dagger} z^1, \\ X^{12}(z) &= -\mathcal{N} \sqrt{1 - \sum_{k=1}^{1+M} z^{k\dagger} z^k} z^1, \\ X^{1(j+1)} &= \mathcal{N} z^{1\dagger} z^j, \quad j = 2, 3, \dots, 1+M, \\ X^{21}(z) &= -\mathcal{N} z^{1\dagger} \sqrt{1 - \sum_{k=1}^{1+M} z^{k\dagger} z^k}, \\ X^{22} &= \mathcal{N} \left(1 - \sum_{k=1}^{1+M} z^{k\dagger} z^k \right), \\ X^{2(j+1)} &= -\mathcal{N} \sqrt{1 - \sum_{k=1}^{1+M} z^{k\dagger} z^k} z^j, \\ X^{(i+1)1} &= \mathcal{N} z^{i\dagger} z^1, \quad i = 2, 3, \dots, 1+M, \\ X^{(i+1)2} &= -\mathcal{N} z^{i\dagger} \sqrt{1 - \sum_{k=1}^{1+M} z^{k\dagger} z^k}, \\ X^{(i+1)(j+1)}(z) &= \mathcal{N} z^{i\dagger} z^j. \end{aligned} \quad (85)$$

Let H^0 be the Hamiltonian for the system with $U(2/M)$ symmetry on an two-dimensional lattice

$$H^0 = -t \sum_{(jk)} \sum_{a,c=1}^{2+M} X_j^{ac} X_k^{ca} (-1)^{F(a) \cdot F(c)}, \quad (86)$$

$$\text{where } F(x) = \begin{cases} 1 & x = 1, 2 \\ 0 & x = 3, 4, \dots, 2+M \end{cases}.$$

Then we have

$$\begin{aligned}
H^0(z) &= \langle SC(z)|H|SC(z) \rangle \\
&= -t \sum_{(jk)} \sum_{a,c=1}^{2+M} \langle X_j^{ac} \rangle \langle X_k^{ca} \rangle (-1)^{F(a) \cdot F(c)} \\
&= \frac{t}{2} \sum_{(jk)} \sum_{a,c=1}^2 \left(\langle X_j^{ac} \rangle + \langle X_k^{ac} \rangle \right) \left(\langle X_j^{ca} \rangle + \langle X_k^{ca} \rangle \right) \\
&\quad - t \sum_j \sum_{a,c=1}^2 \langle X_j^{ac} \rangle \langle X_j^{ca} \rangle \\
&\quad + \frac{t}{2} \sum_{(jk)} \sum_{a=1}^2 \sum_{c=3}^{2+M} \left(\langle X_j^{ac} \rangle + \langle X_k^{ac} \rangle \right) \left(\langle X_j^{ca} \rangle - \langle X_k^{ca} \rangle \right) \\
&\quad + \frac{t}{2} \sum_{(jk)} \sum_{a=3}^{2+M} \sum_{c=1}^2 \left(\langle X_j^{ac} \rangle + \langle X_k^{ac} \rangle \right) \left(\langle X_j^{ca} \rangle - \langle X_k^{ca} \rangle \right) \\
&\quad + \frac{t}{2} \sum_{(jk)} \sum_{a,c=3}^{2+M} \left(\langle X_j^{ac} \rangle - \langle X_k^{ac} \rangle \right) \left(\langle X_j^{ca} \rangle - \langle X_k^{ca} \rangle \right) \\
&\quad - t \sum_j \sum_{a,c=3}^{2+M} \langle X_j^{ac} \rangle \langle X_j^{ca} \rangle .
\end{aligned} \tag{87}$$

We will not consider here frustrated systems. Thus, and for the sake of simplicity, we will consider the case of $U(2/M)$ models on bipartite lattices. We split the staggered z_j^1 -field into a slowly varying piece $\phi(j)$, the order parameter field, and a small rapidly varying part, $\xi(j)$,

$$\begin{aligned}
z_j^1 &= \phi(j) + a_0 \xi(j) , & \text{for sublattice } A , \\
z_j^1 &= \phi(j) - a_0 \xi(j) , & \text{for sublattice } B ,
\end{aligned} \tag{88}$$

where a_0 denotes the lattice spacing.

We work with the constrain which requires that

$$\sum_{a,c=1}^{2+M} \langle X^{ac}(z) \rangle \langle X^{ca}(z) \rangle (-1)^{F(a) \cdot F(c)} = \sum_{a,c=1}^{2+M} \langle X^{ac}(\phi) \rangle \langle X^{ca}(\phi) \rangle (-1)^{F(a) \cdot F(c)} . \tag{89}$$

By making use of the large $\mathcal{N}$ approximation, we obtain

$$H^0(\phi, \xi; z') = \frac{t\mathcal{N}^2}{2} \int dxdy \left(\partial_\mu \phi \partial^\mu \phi^\dagger + 4\xi \xi^\dagger \right) , \tag{90}$$

where $\partial_\mu \phi \partial^\mu \phi^\dagger \equiv \partial_x \phi \partial_x \phi^\dagger + \partial_y \phi \partial_y \phi^\dagger$.

We know that the Lagrangian of the $U(2/M)$ model is

$$L^0(z) = -\frac{i\hbar}{2} \mathcal{N} \sum_j \left\{ (-1)^{S(j)} \left(z_j^1 \partial_\tau z_j^{1\dagger} - \partial_\tau z_j^1 z_j^{1\dagger} \right) + \sum_{i=2}^{1+M} \left(z_j^i \partial_\tau z_j^{i\dagger} - \partial_\tau z_j^i z_j^{i\dagger} \right) \right\} - H^0(z) , \tag{91}$$

where $S(j) = \begin{cases} 0 , & \text{for sublattice } A , \\ 1 , & \text{for sublattice } B . \end{cases}$

To complete the discussion of the continuum limit, it is necessary to consider the kinetic term,

$$-\frac{i\hbar}{2} \mathcal{N} \sum_j \left\{ (-1)^{S(j)} \left(z_j^1 \partial_\tau z_j^{1\dagger} - \partial_\tau z_j^1 z_j^{1\dagger} \right) + \sum_{i=2}^{1+M} \left(z_j^i \partial_\tau z_j^{i\dagger} - \partial_\tau z_j^i z_j^{i\dagger} \right) \right\} .$$

Substituting Eq.(88), we obtain after discarding a total derivative

$$-\frac{i\hbar}{2} \mathcal{N} \sum_j (-1)^{S(j)} \left(z_j^1 \partial_\tau z_j^{1\dagger} - \partial_\tau z_j^1 z_j^{1\dagger} \right) \doteq \frac{-i\hbar \mathcal{N}}{a_0} \int dxdy \left(\xi \partial_\tau \phi^\dagger - \partial_\tau \phi \xi^\dagger \right) . \tag{92}$$

Thus, the Lagrangian becomes as

$$\begin{aligned}
L^0(\phi, \xi; z_i) &= -\frac{i\hbar \mathcal{N}}{a_0} \int dxdy \left(\xi \partial_\tau \phi^\dagger - \partial_\tau \phi \xi^\dagger \right) \\
&\quad - \frac{i\hbar \mathcal{N}}{2a_0^2} \int dxdy \sum_{i=2}^{1+M} \left(\partial_\tau z^i z^{i\dagger} + \partial_\tau z^{i\dagger} z^i \right) \\
&\quad - \frac{t\mathcal{N}^2}{2} \int dxdy \left(\partial_\mu \phi \partial^\mu \phi^\dagger + 4\xi \xi^\dagger \right) .
\end{aligned} \tag{93}$$

The auxiliary fields ξ can be eliminated by solving the Euler-Lagrange equation $\frac{\delta L}{\delta \xi} = 0$,

$$\begin{aligned}
2t\mathcal{N}^2 \xi^\dagger &= -\frac{i\hbar \mathcal{N}}{a_0} \partial_\tau \phi^\dagger , \\
2t\mathcal{N}^2 \xi &= \frac{i\hbar \mathcal{N}^{a_0}}{a_0} \partial_\tau \phi .
\end{aligned} \tag{94}$$

Therefore, we obtain

$$\begin{aligned}
L^0(\phi, z_i) &\doteq \int dxdy \left(\frac{\hbar^2}{2ta_0^2} \partial_\tau \phi \partial_\tau \phi^\dagger - \frac{t\mathcal{N}^2}{2} \partial_\mu \phi \partial^\mu \phi^\dagger \right) \\
&\quad - \frac{i\hbar \mathcal{N}}{2a_0^2} \int dxdy \sum_{i=2}^{1+M} \left(\partial_\tau z^i z^{i\dagger} + \partial_\tau z^{i\dagger} z^i \right) .
\end{aligned} \tag{95}$$

To deal with the realistic models related to high temperature superconductivity, we must consider the interactive Hamiltonian. The whole Lagrangian of the $2D$ extended Hubbard model is

$$\begin{aligned}
L_{\text{Hub.}}(\phi, z_i) &\doteq \int dxdy \left(\frac{\hbar^2}{2ta_0^2} \partial_\tau \phi \partial_\tau \phi^\dagger - \frac{t\mathcal{N}^2}{2} \partial_\mu \phi \partial^\mu \phi^\dagger \right) \\
&\quad - \frac{i\hbar\mathcal{N}}{2a_0^2} \int dxdy \sum_{i=2}^3 \left(\partial_\tau z^i z^{i\dagger} + \partial_\tau z^{i\dagger} z^i \right) \\
&\quad - \frac{U}{a_0^2} \int dxdy z^{3\dagger} z^3 .
\end{aligned} \tag{96}$$

The Lagrangian of the $t - J$ model on two-dimensional lattice is

$$\begin{aligned}
L_{t-J}(\phi, z_i) &\doteq \int dxdy \left(\frac{\hbar^2}{2Ja_0^2} \partial_\tau \phi \partial_\tau \phi^\dagger - \frac{J\mathcal{N}^2}{2} \partial_\mu \phi \partial^\mu \phi^\dagger \right) \\
&\quad - \frac{i\hbar\mathcal{N}}{2a_0^2} \int dxdy \sum_{i=2}^3 \left(\partial_\tau z^i z^{i\dagger} + \partial_\tau z^{i\dagger} z^i \right) .
\end{aligned} \tag{97}$$

And the Lagrangian of the 2D Heisenberg model is simply

$$L_{\text{Heisen.}}(\phi, z_i) \doteq \int dxdy \left(\frac{\hbar^2}{2Ja_0^2} \partial_\tau \phi \partial_\tau \phi^\dagger - \frac{t\mathcal{N}^2}{2} \partial_\mu \phi \partial^\mu \phi^\dagger \right) . \tag{98}$$

Making use of the property of Grassmann integrals,

$$\int \mathcal{D}z^{i\dagger} \mathcal{D}z^i e^{-\int z^{i\dagger} M z^i} = \text{Det} M , \quad z^i, z^{i\dagger} \in CB_{L1} , \tag{99}$$

we can integrate out the fermionic fields and obtain an effective Lagrangian which involves only bosonic part,

$$\begin{aligned}
L_{\text{Hub.}}^{\text{eff.}}(\phi) &= \int dxdy \left(\frac{\hbar^2}{2ta_0^2} \partial_\tau \phi \partial_\tau \phi^\dagger - \frac{t\mathcal{N}^2}{2} \partial_\mu \phi \partial^\mu \phi^\dagger \right) \\
&\quad - i \ln \left(\text{Det} \left(\frac{i\hbar\mathcal{N}}{a_0^2} \partial_\tau \right) \text{Det} \left(\frac{i\hbar\mathcal{N}}{a_0^2} \partial_\tau + \frac{U}{a_0^2} \right) \right) ;
\end{aligned} \tag{100}$$

$$\begin{aligned}
L_{t-J}^{\text{eff.}}(\phi) &= \int dxdy \left(\frac{\hbar^2}{2Ja_0^2} \partial_\tau \phi \partial_\tau \phi^\dagger - \frac{J\mathcal{N}^2}{2} \partial_\mu \phi \partial^\mu \phi^\dagger \right) \\
&\quad - i \ln \text{Det} \left(\frac{i\hbar\mathcal{N}}{a_0^2} \partial_\tau \right) .
\end{aligned} \tag{101}$$

Therefore, up to a constant, the low energy excitations of the extended Hubbard model, $t - J$ model and Heisenberg model is described, in a unified manner, by the nonlinear σ model. In other words, the nonlinear σ model gives a sensible description of the spin fluctuations of the high T_c superconductivity materials. This agrees with the predictions of the weak-coupling mean-field-theory.

6 Concluding Remarks

In this paper, we have shown that the supergroup $U(N/M)$ plays an important role in a kind of high temperature superconductivity related lattice model. To arrive at a continuum field theory description of these models, we constructed supergroup coherent states by a standard approach. A nonlinear σ model was obtained as an effective field theory of low energy excitations of spin-spin interactions of high T_c superconductivity materials in the supergroup coherent state path-integral representation. This result is within our expectations, the spin-spin interactions can be described by the nonlinear σ model in the absence of doping and doping do not violate the Néel ground state at least in the low doping case. This also proves the prediction of weak-coupling mean-field-theory in fermion coherent state path-integral representation. In the Fermion coherent state path-integral representation[34][42,43], the Lagrangian of the Hubbard model is of the form

$$L = \frac{i}{\hbar} \sum_{(jk)} \sum_{\sigma=\pm 1} \Phi_{j\sigma}^\dagger \partial_\tau \Phi_{k\sigma} + 2t \sum_{(jk)} \sum_{\sigma=\pm 1} \Phi_{j\sigma}^\dagger \Phi_{k\sigma} + \frac{U}{6} \sum_j \left(\Phi_{j\sigma}^\dagger \vec{\tau}_{\sigma\sigma'} \Phi_{j\sigma'} \right)^2 . \tag{102}$$

The associated path-integral contains quartic terms, the interaction, and hence we do not know how to complete the partition function. Introducing a vector real boson field $\vec{\phi}$ and using the Hubbard-Stratonovich transformation[44,45]

$$\int d\vec{\phi} \exp \left(-\frac{i}{\hbar} \left(\frac{1}{2} \vec{\phi}^2 + \lambda \vec{\phi} \Phi^\dagger \vec{\tau} \Phi \right) \right) = \text{const.} \times \exp \left(\frac{i}{2\hbar} \lambda^2 \left(\Phi^\dagger \vec{\tau} \Phi \right)^2 \right) , \tag{103}$$

one can get an equivalent Lagrangian of the Hubbard model,

$$\begin{aligned}
L' &= \frac{i}{\hbar} \sum_{(jk)} \sum_{\sigma=\pm 1} \Phi_{j\sigma}^\dagger \partial_\tau \Phi_{k\sigma} + 2t \sum_{(jk)} \sum_{\sigma=\pm 1} \Phi_{j\sigma}^\dagger \Phi_{k\sigma} \\
&\quad - \sqrt{\frac{U}{3}} \sum_j \vec{\phi}_j \Phi_{j\sigma}^\dagger \vec{\tau}_{\sigma\sigma'} \Phi_{j\sigma'} - \frac{1}{2} \sum_j \vec{\phi}_j^2 .
\end{aligned} \tag{104}$$

The $\vec{\phi}$ fields represent the collective modes associated with spin fluctuations. The effective Lagrangian is given by

$$L^{\text{eff}}(\vec{\phi}) = -\frac{1}{2} \sum_j \vec{\phi}^2 - i \ln \text{Det} \left(\frac{i}{\hbar} \partial_\tau - M(\vec{\phi}) \right) , \tag{105}$$

where $M(\vec{\phi})$ is a functional matrix of the $\vec{\phi}$ fields. On the mean-field-theory approximation, the effective Lagrangian shows the relevant nonlinear effects. The properties of the effective Lagrangian suggests that nonlinear σ model

$$L^{\text{eff}} = \frac{\rho}{2} \left(\partial_t \vec{\phi} \cdot \partial_t \vec{\phi} - v_s^2 \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} \right) + \dots, \quad (106)$$

describes the slow spin fluctuations. In this paper we have proved the assumption but not used any approximation except large spin and long wavelength.

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