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Light-cone quantization approach to quantum electrodynamics in $(2+1)$ dimensions

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Abstract

Light-cone quantization techniques are applied to quantum electrodynamics in $(2+1)$ dimensions with four-component spinors. The discretized light-cone Hamiltonian for the model is derived. Hence a Tamm-Dancoff integral equation for the “positronium” bound states is obtained. It is shown that at weak couplings the integral equation reduces to the correct non-relativistic Schrödinger equation, and accurate numerical solutions are obtained for the energy eigenvalues. At stronger couplings we find that the infrared divergences no longer cancel, which we attribute to a lack of gauge invariance in the approximation scheme. Possible remedies are discussed.

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I. INTRODUCTION

The light-cone quantization approach to quantum field theory has emerged in recent years [1] as a possible new method for tackling problems in the strong coupling regime, including eventually (it is hoped) the mass spectrum in quantum chromodynamics (QCD). The apparent advantages of the method include: firstly, that the ground state of the theory is very simple in this formulation; secondly, one may easily calculate the structure functions measured in deep-inelastic scattering experiments, and other features of the relativistic wave-functions; and thirdly, one may address theories containing dynamical fermions in a direct and simple fashion, without many of the difficulties that occur in the standard lattice gauge theory approach. On the other hand, the light-cone formalism carries new difficulties of its own, such as the problem of “zero modes” and spontaneous symmetry breaking, difficulties with “spurious light-cone singularities” and renormalization, and so on [1]. It is still not clear, at this stage, whether light-cone methods will prove useful for QCD or not.

An early success of the light-cone approach was the application of the “discretized light-cone quantization” (DLCQ) method [2] to the case of quantum electrodynamics (QED) in (1+1) dimensions by Eller, Pauli and Brodsky [3]. They restricted the light-cone momenta to discrete values by introducing a finite box size in the light-cone co-ordinates, and derived a matrix form for the light-cone “Hamiltonian”, which could then be solved numerically. They obtained impressive results for the mass spectrum of the model, with a very modest computational effort. Similar techniques have since been applied to other models in (1+1) dimensions, as reviewed by Brodsky *et al.* [1].

Attempts were then made to extend the DLCQ technique to more realistic models in (3+1) dimensions. The DLCQ Hamiltonian for QED in (3+1) dimensions was formulated by Tang, Brodsky and Pauli [4], and further computations were made by Kaluza and Pauli [5]. They were able to reproduce the binding energy and charge radius of positronium to about 10% accuracy for a fine structure constant $\alpha = 0.3$. Hollenberg *et al.* [6] even attacked the problem of QCD itself. The limitations of the technique were already becoming

apparent, however: the number of basis states in the matrix representation proliferates with alarming rapidity in higher dimensions, and the presence of physical massless gauge bosons complicates matters even further.

Meanwhile, a somewhat different light-cone technique involving “Tamm-Dancoff integral equations”, was being developed [7–9]. Krautgärtner, Pauli and Wölz [9], for example, applied it to QED in (3+1) dimensions. A “Tamm-Dancoff cutoff” [10] was applied, restricting the basis states to the (e^+e^-) and $(e^+e^-\gamma)$ sectors only, and then the effect of the $(e^+e^-\gamma)$ sector was approximated using an “effective Hamiltonian” approach, which leads to an integral equation whose eigenvalues are the masses (squared) of the positronium bound states. At weak couplings the results are in excellent agreement with known physical results, and can again be obtained with a very modest computational effort. Further calculations of the mass renormalization [11] and hyperfine splitting effects [12] in the model have also been reported. The crucial question remains, how well can these techniques be extended to stronger coupling values?

Our aim in the present work is to begin testing these light-cone techniques on QED in (2+1) dimensions, rather than (3+1) dimensions. The number of basis states or degrees of freedom is much smaller in the lower dimensional case, of course, which should allow more comprehensive tests of the techniques. On the other hand, the model does possess independent photon degrees of freedom, unlike the (1+1)-dimensional model, and thus exhibits many of the complications of the real world. It also exhibits confinement, like QCD. It should thus provide an excellent test-bed for light-cone methods.

We start by deriving the “discretized” light-cone Hamiltonian for the (2+1)-dimensional case, following Eller *et al.* [3] and Tang *et al.* [4]. A Tamm-Dancoff integral equation is then derived for the “positronium” bound states, following Krautgärtner *et al.* [9]. It is explicitly shown how the “instantaneous” exchange diagrams play a necessary part in cancelling out the “spurious light-cone singularities”. Taking the weak-coupling limit, one obtains a non-relativistic integral equation for the system, which is simply the momentum-space counterpart of the normal Schrödinger equation for this system [13,14]. The infrared divergences

cancel neatly between the self-mass and one-photon exchange diagrams, as in the normal equal-time formalism [13,15]. Numerical solutions are obtained for the non-relativistic binding energies which are substantially more accurate than previous results [14].

A problem arises when attempting to apply the Tamm-Dancoff integral equation at stronger couplings: the infrared divergent terms no longer cancel. We presume this to be associated with the fact that the Tamm-Dancoff cutoff violates gauge invariance [1,4,7,8,16], and extra diagrams, plus extra sectors of states, need to be brought in as one moves to larger couplings. It highlights the major difficulty that light-cone methods will encounter at strong couplings: one must confront head-on the problem of dealing with states containing many particles. A brief discussion of this problem is given in the conclusions.

We hope to apply the DLCQ technique itself to QED in (2+1) dimensions in a future publication.

II. THE DLCQ HAMILTONIAN

To construct the discretized light-cone “Hamiltonian” for QED in (2+1) dimensions, we shall follow much the same procedures as Eller *et al.* [3], Tang [17], and Tang, Brodsky and Pauli [4]. Definitions of the standard light-cone variables and kinematic relations are shown in Table I, following Tang *et al.* [4].

We start from the (2+1)-dimensional QED Lagrangian density

$$\begin{aligned} \mathcal{L} = & \frac{i}{2} \left[\bar{\psi} \gamma^\mu \partial_\mu \psi - \left(\partial_\mu \bar{\psi} \right) \gamma^\mu \psi \right] - m \bar{\psi} \psi \\ & - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - g \bar{\psi} \gamma^\mu \psi A_\mu , \end{aligned} \quad (2.1a)$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu . \quad (2.1b)$$

We choose to employ a four-component spinor representation [18], in which the Dirac matrices, projection operators and spinors obey the relations given in Table II. In the two-component formalism a parity-violating Chern-Simons photon mass term is dynamically generated by the fermions [19]: this is an interesting phenomenon, but also a complication, which we prefer to do without for the present purposes. The four-component model avoids this complication [18,13,20].

The canonical Euler-Lagrange equations of motion are Maxwell's equations

$$\partial_\mu F^{\mu\nu} = g\bar{\psi}\gamma^\nu\psi \equiv gj^\nu \quad (2.2)$$

and Dirac's equation

$$[(i\partial_\mu - gA_\mu)\gamma^\mu - m]\psi = 0. \quad (2.3)$$

We choose the light-cone gauge $A^+ = 0$ while noting the caveat that the “zero mode” of A^+ (with light-cone momentum $k^+ = 0$) cannot simply be gauged away in this fashion [1,21]. (We shall largely ignore the question of zero modes in this paper, on the grounds that they are expected to decouple from the physics of charge-zero bound states [1,21], which is the subject of interest here.) Then one component of the photon field, A^- , is determined (up to a zero mode) by the Maxwell equation

$$(\partial_-)^2 A^- = -\frac{1}{2}gj^+ - \partial_- \partial_\perp A^\perp \quad (2.4)$$

which acts as a constraint; and one of the projections of the Dirac field, ψ_- (defined in Table II), is determined similarly from Dirac's equation

$$4i\partial_- \psi_- = m\gamma^+ \psi_+ + (i\partial_\perp - gA_\perp)\gamma^\perp \gamma^+ \psi_+. \quad (2.5)$$

The remaining independent dynamical fields are therefore ψ_+ and $A^\perp$. They are required to obey position-space (anti-) commutation relations

$$\begin{aligned} & \left\{ \psi_{+\alpha}(\mathbf{x}), \psi_{+\beta}^\dagger(\mathbf{y}) \right\}_{x^+=y^+} \\ &= \Lambda_{\alpha\beta}^+ \delta(x^- - y^-) \delta(x^\perp - y^\perp), \end{aligned} \quad (2.6)$$

$$\begin{aligned}
& \left[A^\perp(\mathbf{x}), A^\perp(\mathbf{y}) \right]_{x^+=y^+} \\
&= -\frac{i}{4} \epsilon \left(x^- - y^- \right) \delta \left(x^\perp - y^\perp \right)
\end{aligned} \tag{2.7}$$

at equal light-cone time, as shown for example by Chang, Root and Yan [22].

The Lagrangian density (2.1) gives rise to conserved “charges”

$$Q = \frac{1}{2} \int dx^- dx^\perp j^+ \tag{2.8}$$

and

$$P^\nu = \frac{1}{2} \int dx^- dx^\perp T^{+\nu} . \tag{2.9}$$

where $T^{\mu\nu}$ is the energy-momentum tensor. The charge Q , the light-cone momenta P^+ and $P^\perp$, and the energy P^- form a set of mutually commuting operators. They are given in terms of the fields by

$$Q = \int dx^- dx^\perp \psi_+^\dagger \psi_+ . \tag{2.10}$$

$$P^+ = \int dx^- dx^\perp \left[i \psi_+^\dagger \partial^+ \psi_+ + \frac{1}{2} \left(\partial^+ A^\perp \right)^2 \right] , \tag{2.11}$$

$$P^\perp = \int dx^- dx^\perp \left[i \psi_+^\dagger \partial^\perp \psi_+ + \frac{1}{2} \left(\partial^+ A^\perp \partial^\perp A^\perp \right) \right] . \tag{2.12}$$

and

$$\begin{aligned}
P^- = \frac{1}{4} \int dx^- dx^\perp & \left\{ i \left[\psi_+^\dagger \gamma^- \gamma^\perp \partial^\perp \psi_- - \left(\partial^\perp \psi_-^\dagger \right) \gamma^+ \gamma^\perp \psi_+ \right] + m \left(\psi_-^\dagger \gamma^+ \psi_+ + \psi_+^\dagger \gamma^- \psi_- \right) \right. \\
& \left. - g A^\perp \left(\psi_-^\dagger \gamma^+ \gamma^\perp \psi_+ + \psi_+^\dagger \gamma^- \gamma^\perp \psi_- \right) + \frac{1}{2} \left(\partial^+ A^- \right)^2 \right\} .
\end{aligned} \tag{2.13}$$

Use has been made of Eqs. (2.4) and (2.5) to eliminate some terms from (2.13), but we choose not to eliminate the dependent fields ψ_- and A^- entirely at this stage, because that would require the introduction of some ill-defined Green’s function or inverse derivative operators [3,23], which we wish to avoid. The explicit construction of the operators Q and P^ν in Fock space is the aim of this section.

The model is now “discretized” by defining it in a box in light-cone space, $-L \leq x^- \leq L$, $-M \leq x^\perp \leq M$. For the fermion field, we choose anti-periodic boundary conditions for ψ_+ in the x^- direction to avoid any possible problems with fermion zero modes, and periodic boundary conditions in the transverse direction, so that the allowed momenta become discrete:

$$\text{fermions: } \begin{cases} k^+ = (2n+1)\pi/L, & n = 0, 1, 2, \dots, \\ k^\perp = l\pi/M, & l = 0, \pm 1, \pm 2, \dots \end{cases} \quad (2.14)$$

(Note the feature of light-cone kinematics, that only positive values of k^+ are allowed for physical, positive-energy particles [1,2].) For the photon field, to achieve consistency we are forced to assume periodic boundary conditions in both directions:

$$\text{photons: } \begin{cases} k^+ = 2n\pi/L, & n = 0, 1, 2, \dots, \\ k^\perp = l\pi/M, & l = 0, \pm 1, \pm 2, \dots \end{cases} \quad (2.15)$$

Then the dynamical fields at a fixed light-cone time, say $x^+ = 0$, can be expanded in momentum space:

$$\begin{aligned} \psi_+(x^-, x^\perp; x^+ = 0) &= \frac{1}{2\sqrt{LM}} \\ &\times \sum_{\sigma=1}^2 u_+^\sigma \sum_{n,l} (b_{nl}^\sigma + d_{nl}^{\sigma\dagger}) e^{i(\xi\xi - \tilde{l}\eta)} \end{aligned} \quad (2.16)$$

$$\begin{aligned} A^\perp(x^-, x^\perp; x^+ = 0) &= \frac{1}{\sqrt{8\pi M}} \\ &\times \sum_{n,l} \frac{1}{\sqrt{n}} (a_{nl} + a_{-l}^\dagger) e^{i(\xi\xi - \tilde{l}\eta)} \end{aligned} \quad (2.17)$$

where $\xi = x^- \pi/L$, $\eta = x^\perp \pi/M$, and the Dirac spinors u_+^σ are defined by their properties in Table II, and we have adopted a “condensed” notation in which

$$\tilde{l} = \begin{cases} +l & \text{when associated with } b^\dagger, d^\dagger, \text{ or } a^\dagger; \\ -l & \text{when associated with } b, d, \text{ or } a. \end{cases} \quad (2.18)$$

$$\hat{n} = \begin{cases} +\left(n + \frac{1}{2}\right) & \text{when associated with } b^\dagger \text{ or } d^\dagger; \\ -\left(n + \frac{1}{2}\right) & \text{when associated with } b \text{ or } d; \\ +n & \text{when associated with } a^\dagger; \\ -n & \text{when associated with } a. \end{cases} \quad (2.19)$$

The photon zero modes $n = 0$ must be excluded from the expansions (2.17).

If we postulate the standard (anti-) commutation relations for the creation and destruction operators:

$$\{b_{nl}^\sigma, b_{n'l'}^{\sigma'\dagger}\} = \{d_{nl}^\sigma, d_{n'l'}^{\sigma'\dagger}\} = \delta_{\sigma\sigma'} \delta_{nn'} \delta_{ll'} \quad (2.20)$$

and

$$[a_{nl}, a_{n'l'}^\dagger] = \delta_{nn'} \delta_{ll'} \quad (2.21)$$

then in the continuum limit $L, M \rightarrow \infty$ one can verify that the position-space (anti-) commutation relations (2.6) and (2.7) are regained from the expansions (2.16) and (2.17).

Given these expansions one can derive corresponding expressions for the dependent fields A^- and ψ_- from Eqs. (2.4) and (2.5) [in a condensed notation $\underline{1} = (n_1, l_1)$, etc.]:

$$\begin{aligned} A^-(x^-, x^\perp; x^+ = 0) &= \frac{g}{4\pi^2} \frac{L}{M} \sum_{\sigma, \underline{1}, \underline{2}} (b_{\underline{1}}^{\sigma\dagger} + d_{\underline{1}}^\sigma) \\ &\quad \times (b_{\underline{2}}^\sigma + d_{\underline{2}}^{\sigma\dagger}) \frac{e^{i[(\tilde{n}_1 + \tilde{n}_2)\xi - (\tilde{l}_1 + \tilde{l}_2)\eta]}}{(\tilde{n}_1 + \tilde{n}_2)^2} \\ &\quad + \frac{L}{\sqrt{8\pi} M^3} \sum_{\underline{1}} (a_{\underline{1}} + a_{\underline{1}}^\dagger) e^{i(\tilde{n}_1 \xi - \tilde{l}_1 \eta)} \frac{\tilde{l}_1}{\tilde{n}_1 \sqrt{\tilde{n}_1}}. \end{aligned} \quad (2.22)$$

and

$$\begin{aligned} \psi_-(x^-, x^\perp; x^+ = 0) &= \frac{1}{2\pi} \sqrt{\frac{L}{M}} \left\{ \sum_{\sigma, \underline{1}} u_-^\sigma (b_{\underline{1}}^\sigma + d_{\underline{1}}^{\sigma\dagger}) \right. \\ &\quad \times \frac{e^{i(\tilde{n}_1 \xi - \tilde{l}_1 \eta)}}{2\tilde{n}_1} \left[(-1)^\sigma m + i\tilde{l}_1 \frac{\pi}{M} \right] \\ &\quad + \frac{ig}{\sqrt{8\pi} M} \sum_{\sigma, \underline{1}, \underline{2}} u_-^\sigma (a_{\underline{1}} + a_{\underline{1}}^\dagger) (b_{\underline{2}}^\sigma + d_{\underline{2}}^{\sigma\dagger}) \\ &\quad \times \frac{e^{i[(\tilde{n}_1 + \tilde{n}_2)\xi - (\tilde{l}_1 + \tilde{l}_2)\eta]}}{(\tilde{n}_1 + \tilde{n}_2) \sqrt{\tilde{n}_1}} \left. \right\} \end{aligned} \quad (2.23)$$

Once again, the zero modes are to be excluded from the expansion (2.22): they are not constrained by Eq. (2.4) in any case.

Inserting expansions (2.16) and (2.17) into Eqs. (2.10) – (2.12), normal ordering and setting vacuum expectation values to zero, one obtains the desired Fock space expressions for the conserved charge and light-cone momentum operators. They take the expected form:

$$Q = \sum_{\sigma, \perp} \left(b_{\perp}^{\sigma\dagger} b_{\perp}^{\sigma} - d_{\perp}^{\sigma\dagger} d_{\perp}^{\sigma} \right) . \quad (2.24)$$

$$P^+ = \frac{2\pi}{L} \sum_{\perp} \left[\left(n_{\perp} + \frac{1}{2} \right) \sum_{\sigma} \left(b_{\perp}^{\sigma\dagger} b_{\perp}^{\sigma} + d_{\perp}^{\sigma\dagger} d_{\perp}^{\sigma} \right) + n_{\perp} a_{\perp}^{\dagger} a_{\perp} \right] , \quad (2.25)$$

$$P^{\perp} = \frac{\pi}{M} \sum_{\perp} l_{\perp} \left[\sum_{\sigma} \left(b_{\perp}^{\sigma\dagger} b_{\perp}^{\sigma} + d_{\perp}^{\sigma\dagger} d_{\perp}^{\sigma} \right) + a_{\perp}^{\dagger} a_{\perp} \right] , \quad (2.26)$$

which are all diagonal in Fock space, and conserved. It is convenient to define a new parameter K , the “harmonic resolution” [2,3], by

$$P^+ = \frac{2\pi K}{L} \quad (2.27)$$

and then the “light-cone Hamiltonian” H_{LC} is defined by

$$P^- = \frac{H_{LC}}{P^+} = \frac{L}{2\pi K} H_{LC} . \quad (2.28)$$

The eigenvalues of H_{LC} , in the case $P^{\perp} = 0$, give the invariant-mass-squared spectrum $\tilde{M}^2$ for the system. The Fock space expression for H_{LC} , obtained from (2.13) using (2.16), (2.17), (2.23), and (2.22), contains several terms:

$$H_{LC} = H_0 + V_1 + V_2 + V_3 + V_4 + V_5 , \quad (2.29)$$

where

$$\begin{aligned} H_0 = K \left\{ \sum_{\sigma, \perp} \frac{1}{\left(n_{\perp} + \frac{1}{2} \right)} \left[m^2 + \left(\frac{\pi l_{\perp}}{M} \right)^2 \right] \right. \\ \times \left(b_{\perp}^{\sigma\dagger} b_{\perp}^{\sigma} + d_{\perp}^{\sigma\dagger} d_{\perp}^{\sigma} \right) \\ \left. + \sum_{\perp} \frac{1}{n_{\perp}} \left(\frac{\pi l_{\perp}}{M} \right)^2 a_{\perp}^{\dagger} a_{\perp} \right\} \end{aligned} \quad (2.30)$$

is the diagonal, free-particle Hamiltonian;

$$\begin{aligned}
V_1 = & \frac{Kg}{\sqrt{8\pi M}} \sum_{\sigma, \underline{1}, \underline{2}, \underline{3}} \delta \left(\sum_{i=1}^3 \hat{n}_i \right) \delta \left(\sum_{i=1}^3 \hat{l}_i \right) \frac{1}{\sqrt{n_3}} \\
& \times \left[\frac{\pi}{M} \left(\frac{2l_3}{n_3} - \frac{l_1}{n_1} - \frac{l_2}{n_2} \right) + im(-1)^\sigma \left(\frac{1}{\hat{n}_1} + \frac{1}{\hat{n}_2} \right) \right] \\
& \times \left[\left(b_1^{\sigma\dagger} b_2^\sigma - d_2^{\sigma\dagger} d_1^\sigma \right) a_3 + a_3^\dagger \left(b_1^{\sigma\dagger} b_2^\sigma - d_2^{\sigma\dagger} d_1^\sigma \right) \right. \\
& \quad \left. + b_1^{\sigma\dagger} d_2^{\sigma\dagger} a_3 + a_3^\dagger d_1^\sigma b_2^\sigma \right]
\end{aligned} \tag{2.31}$$

is the usual 3-particle fermion-photon vertex term;

$$\begin{aligned}
V_2 = & \frac{Kg^2}{4\pi M} \sum_{\sigma, \tau, \underline{1}, \underline{2}, \underline{3}, \underline{4}} \delta \left(\sum_{i=1}^4 \hat{n}_i \right) \delta \left(\sum_{i=1}^4 \hat{l}_i \right) \frac{1}{(\hat{n}_1 + \hat{n}_2)^2} \\
& \times \left(b_1^{\sigma\dagger} d_2^{\sigma\dagger} b_3^\tau b_4^\tau - b_1^{\sigma\dagger} d_2^{\sigma\dagger} d_4^{\tau\dagger} d_3^\tau + b_1^{\sigma\dagger} d_2^{\sigma\dagger} d_3^{\tau\dagger} b_4^\tau \right. \\
& \quad \left. + b_3^{\tau\dagger} b_4^\tau d_1^\sigma b_2^\sigma - d_1^{\tau\dagger} d_3^\tau d_1^\sigma b_2^\sigma + b_3^{\tau\dagger} d_4^{\tau\dagger} d_1^\sigma b_2^\sigma \right)
\end{aligned} \tag{2.32}$$

is a 4-fermion “instantaneous photon” interaction term;

$$\begin{aligned}
V_3 = & \frac{Kg^2}{8\pi M} \sum_{\sigma, \underline{1}, \underline{2}, \underline{3}, \underline{4}} \delta \left(\sum_{i=1}^4 \hat{n}_i \right) \delta \left(\sum_{i=1}^4 \hat{l}_i \right) \\
& \times \frac{1}{\sqrt{n_3 n_4} (\hat{n}_1 + \hat{n}_3)} \\
& \times \left(b_1^{\sigma\dagger} b_2^\sigma + b_1^{\sigma\dagger} d_2^{\sigma\dagger} + d_1^\sigma b_2^\sigma - d_2^{\sigma\dagger} d_1^\sigma \right) \\
& \times \left(a_3^\dagger a_4^\dagger + a_3^\dagger a_4 + a_4^\dagger a_3 + a_3 a_4 \right)
\end{aligned} \tag{2.33}$$

is a 2-photon 2-fermion “instantaneous fermion” interaction term;

$$V_4 = \frac{Kg^2}{8\pi M} \sum_{\sigma, \underline{1}} \left(I_{n_1}^{(1)} + I_{n_1}^{(2)} \right) \left(b_1^{\sigma\dagger} b_1^\sigma + d_1^{\sigma\dagger} d_1^\sigma \right) \tag{2.34}$$

is the “self-induced inertia” term [2,3] for the fermions, and

$$V_5 = \frac{Kg^2}{8\pi M} \sum_{\underline{1}} I_{n_1}^{(3)} a_{\underline{1}}^\dagger a_{\underline{1}} \tag{2.35}$$

is the “self-induced inertia” for the photons. The self-induced inertias arise as contraction terms in the normal-ordering process, and involve the expressions:

$$I_n^{(1)} = 2 \sum_{n', l'} \left[\frac{1}{(n - n')^2} - \frac{1}{(n + n' + 1)^2} \right], \tag{2.36}$$

$$I_n^{(2)} = \frac{1}{2} \sum_{n', l'} \frac{1}{n'} \left(\frac{1}{n + n' + \frac{1}{2}} + \frac{1}{n - n' + \frac{1}{2}} \right), \quad (2.37)$$

and

$$I_n^{(3)} = -\frac{1}{n} \sum_{n', l'} \left(\frac{1}{n' + n + \frac{1}{2}} + \frac{1}{n' - n + \frac{1}{2}} \right). \quad (2.38)$$

The various interaction terms in the light-cone Hamiltonian can be represented by diagrams as in Fig. 1. Note that the light-cone momenta are explicitly conserved at each vertex, as they must be. The “instantaneous” interaction terms are peculiar to the light-cone formalism; they arise via the dependent fields ψ_- and A^- in (2.13). The “self-induced inertia” terms are also unfamiliar: the role played by these terms in the theory will be clarified as we go on. Note also that the discrete momentum sums run over the ranges (2.14) and (2.15). The sums (2.36) – (2.38) are divergent unless regulated by a cutoff at large momentum. Any ill-defined terms corresponding to photon “zero modes” ($n = 0$) are excluded from the sums throughout: justification for this rule is further discussed by Tang *et al.* [4]. We note that the DLCQ Hamiltonian for QED in (2+1) dimensions was obtained previously, in fact, by Yung [24].

With these Fock-space expressions in hand, a DLCQ calculation of the positronium mass spectrum could be set up, following Eller *et al.* [3], and Kaluza and Pauli [5]. We hope to return to this question in future work.

III. THE TAMM-DANCOFF INTEGRAL EQUATION

Our next aim is to explore the behaviour of the model at weak couplings, and see how the various terms in the light-cone Hamiltonian contribute, using analytical techniques as far as possible. Once the behaviour in the weak-coupling limit is known, the results can be used as a comparison and check for numerical DLCQ calculations. For this purpose, we turn to the Tamm-Dancoff integral equation approach, aiming to calculate the bound-state spectrum of “positronium”.

A. Derivation of the equation

The light-cone Tamm-Dancoff approach has been applied to QED in (3+1) dimensions by Krautgärtner *et al.* [9]. We set out to follow their procedure in the (2+1)-dimensional case. Our notation will conform as far as possible with Krautgärtner *et al.* [9]. The starting point is the light-cone Hamiltonian H_{LC} , whose eigenvalues give the invariant mass squared:

$$H_{LC} |\psi_i\rangle = \hat{M}_i^2 |\psi_i\rangle . \quad (3.1)$$

If we restrict ourselves to the positron-electron (e^+e^-) and positron-electron-photon ($e^+e^-\gamma$) Fock state sectors, then Eq. (3.1) can be decomposed into block-matrix form:

$$H_{00} |\psi\rangle_0 + H_{01} |\psi\rangle_1 = \omega |\psi\rangle_0 , \quad (3.2)$$

$$H_{10} |\psi\rangle_0 + H_{11} |\psi\rangle_1 = \omega |\psi\rangle_1 , \quad (3.3)$$

where $\omega = \hat{M}_i^2$, and the subscripts 0 and 1 refer to the (e^+e^-) and ($e^+e^-\gamma$) sectors, respectively. Equation (3.3) gives formally

$$|\psi\rangle_1 = \frac{1}{\omega - H_{11}} H_{10} |\psi\rangle_0 \quad (3.4)$$

and inserting this into Eq. (3.2) one gets an “effective Hamiltonian” acting only in the (e^+e^-) sector:

$$H_{\text{eff}}(\omega) |\psi\rangle_0 = \hat{M}_i^2 |\psi\rangle_0 . \quad (3.5)$$

where

$$H_{\text{eff}}(\omega) = H_{00} + H_{01} \frac{1}{\omega - H_{11}} H_{10} . \quad (3.6)$$

The object is now to obtain an approximate solution of this system.

Any restriction to a finite number of gauge bosons violates gauge invariance. To prevent this violation from occurring already at the tree level, Tang [17,4] proposed a “gauge cutoff” rule, namely that diagrams containing “instantaneous” exchanged particles should only be

included when the corresponding diagram with a real dynamical particle is also included. In accordance with this rule, the only diagrams to be included in the effective Hamiltonian are those shown in Fig. 2, which involve no more than a single extra photon in the intermediate state: any such photon is not allowed to undergo any further interactions. Thus the operator H_{11} in the denominator of (3.6) is replaced by the expectation value of the diagonal free-particle Hamiltonian H_0 in the $(e^+e^-\gamma)$ intermediate state:

$$H_{11} \longrightarrow \tilde{M}_{e^+e^-\gamma}^2. \quad (3.7)$$

To proceed further, let us consider matrix elements of H_{eff} between initial and final (e^+e^-) states, whose kinematic variables are defined as in Fig. 3. Here $\underline{k} = (k^+, k^\perp)$ is the light-cone momentum of the initial electron, $\underline{k}'$ is that of the final electron, and σ_{e-} and σ'_{e-} are their respective “spin” states. The conserved light-cone momentum of the whole system is taken as $\underline{P} = (P^+, 0)$. The longitudinal momentum fraction of the electron is defined as $x = k^+/P^+$ and its (unperturbed) light-cone energy is $k^- = (m^2 + k^{\perp 2})/k^+$.

To make the problem more tractable, a further ad hoc approximation is made [9], namely to replace the eigenvalue ω in the denominator of (3.6) by the “symmetrized mass (squared)” ω^* , which is the average of the expectation values of H_0 in the initial and final states:

$$\begin{aligned} \omega &= \omega^* \equiv \frac{P^+}{2} \left(k_{e-}^- + k_{e+}^- + k'_{e-}^- + k'_{e+}^- \right) \\ &= \frac{1}{2} \left[\frac{m^2 + k^{\perp 2}}{x(1-x)} + \frac{m^2 + k'^{\perp 2}}{x'(1-x')} \right]. \end{aligned} \quad (3.8)$$

The importance of this choice will be discussed later.

In the continuum, the matrix equation (3.5) becomes an integral equation:

$$\begin{aligned} \sum_{\sigma'_{e-}, \sigma'_{e+}} \int dx' dk'^{\perp} \langle x, k^{\perp}; \sigma_{e-}, \sigma_{e+} | H_{\text{eff}} | x', k'^{\perp}; \sigma'_{e-}, \sigma'_{e+} \rangle \\ \times \psi(x', k'^{\perp}; \sigma'_{e-}, \sigma'_{e+}) \\ = \tilde{M}_i^2 \psi(x, k^{\perp}; \sigma_{e-}, \sigma_{e+}). \end{aligned} \quad (3.9)$$

The contributions of the diagrams in Fig. 2 to this equation can then be found by inserting the operators of Eqs. (2.30) – (2.35) for each corresponding vertex, and transcribing back from the discretized system to a continuous system using the correspondences (2.14) and (2.15) for the momenta, together with:

$$\frac{2\pi}{L} \sum_n \longrightarrow \int dk^+ , \quad \frac{\pi}{M} \sum_l \longrightarrow \int dk^\perp ,$$

$$\frac{L}{2\pi} \delta(n - n') \longrightarrow \delta(k^+ - k'^+) ,$$

$$\frac{M}{\pi} \delta(l - l') \longrightarrow \delta(k^\perp - k'^\perp) ,$$

$$\sqrt{\frac{LM}{2\pi^2}} |n, l; \sigma_{e-}, \sigma_{e+}\rangle \longrightarrow |x, k^\perp; \sigma_{e-}, \sigma_{e+}\rangle .$$

The contributions of the individual diagrams in Fig. 2 to the left-hand side of (3.9) are then as follows.

Diagrams 2(a)–(c) are diagonal. Figure 2(a) corresponds to the matrix element of H_0 , and contributes a term

$$\hat{M}_{e^+e^-}^2 = \frac{m^2 + k^{\perp 2}}{x(1-x)} . \quad (3.10)$$

Figure 2(b) corresponds to the matrix elements of V_4 , i.e. the contraction terms or “self-induced inertias”, and contributes

$$C = \frac{g^2}{8\pi^2} \int \frac{dx' dk'^\perp}{x'} \left[\mathcal{C}_x^{(1)}(x') + \mathcal{C}_{1-x}^{(1)}(x') \right] , \quad (3.11a)$$

where

$$\mathcal{C}_x^{(1)}(x') = \frac{x}{x^2 - x'^2} \left(1 + \frac{8x'^2}{x^2 - x'^2} \right) . \quad (3.11b)$$

Figure 2(c) consists of the self-mass diagrams, and contributes terms

$$\delta m = \frac{g^2}{8\pi^2} \int \frac{dx' dk'^\perp}{x'} \left[\mathcal{S}_{x,k^\perp}^{(1)}(x', k'^\perp) + \mathcal{S}_{1-x,k^\perp}^{(1)}(x', k'^\perp) \right] , \quad (3.12a)$$

where

$$\mathcal{S}_{x,k^\perp}^{(1)}(x', k'^\perp) = \Theta(x - x') \frac{m^2 \left(\frac{1}{x} - \frac{1}{x - x'} \right)^2 + \left(\frac{2k'^\perp}{x'} - \frac{k^\perp}{x} - \frac{(k - k')^\perp}{x - x'} \right)^2}{\frac{m^2 + k^{\perp 2}}{x} - \frac{k'^{\perp 2}}{x'} - \frac{m^2 + (k - k')^{\perp 2}}{x - x'}}. \quad (3.12b)$$

Diagrams 2(d) and 2(e) are off diagonal. Figure 2(d) is the instantaneous photon or “seagull” diagram, and contributes a term

$$-\frac{g^2}{2\pi^2} \int \frac{dx' dk'^\perp}{(x - x')^2} \psi(x', k'^\perp; \sigma_{e-}, \sigma_{e+}). \quad (3.13)$$

Figure 2(e) consists of the physical one-photon exchange diagrams, and contributes a term

$$\frac{g^2}{8\pi^2} \int \frac{dx' dk'^\perp}{x - x'} \psi(x', k'^\perp; \sigma_{e-}, \sigma_{e+}) \left[\mathcal{E}_{x,k^\perp}(x', k'^\perp) + \mathcal{E}_{1-x,-k^\perp}(1 - x', -k'^\perp) \right], \quad (3.14a)$$

where

$$\begin{aligned} \mathcal{E}_{x,k^\perp}(x', k'^\perp) = & \Theta(x - x') \left[\frac{2(k - k')^\perp}{x - x'} - \frac{k^\perp}{x} - \frac{k'^\perp}{x'} + im(-1)^{\sigma_{e-}} \left(\frac{1}{x'} - \frac{1}{x} \right) \right] \\ & \times \left[\frac{-2(k - k')^\perp}{x - x'} + \frac{k^\perp}{1 - x} + \frac{k'^\perp}{1 - x'} - im(-1)^{\sigma_{e+}} \left(\frac{1}{1 - x'} - \frac{1}{1 - x} \right) \right] \\ & \times \left[\frac{1}{2} \left(\frac{m^2 + k^{\perp 2}}{x(1 - x)} + \frac{m^2 + k'^{\perp 2}}{x'(1 - x')} \right) - \frac{m^2 + k'^{\perp 2}}{x'} - \frac{m^2 + k^{\perp 2}}{1 - x} - \frac{(k - k')^{\perp 2}}{x - x'} \right]. \end{aligned} \quad (3.14b)$$

Note that all terms are diagonal in the “spins” σ_{e-} and σ_{e+} . We suppress the spin indices henceforth.

B. Singularity structure

The individual terms (3.11) – (3.14) are all singular, and must be regulated before meaningful results can be obtained. Divergences at large light-cone momenta (“ultraviolet” divergences), for instance, can be cut off by imposing upper bounds on the momentum integrals $0 < k^+ \leq \Lambda_+$ and $|k^\perp| \leq \Lambda_M$, while divergences at small light-cone momenta (“infrared” divergences, though this is partly a misnomer) can be cut off by imposing a lower bound on the longitudinal momentum $x = k^+/P^+ \geq \mu$, as the discretized version does automatically.

In the end one finds that these divergences generally cancel between different diagrams, as we shall see.

Consider first the diagonal terms. The self-induced inertia term (3.11) diverges as $k'^\perp \rightarrow \infty$, and has a quadratic infrared divergence as $(x - x') \rightarrow 0$. With the cutoffs $|k'^\perp| \leq \Lambda_M$ and $|x - x'| \geq \mu$, the result is

$$C = \frac{g^2}{\pi^2} \left[\mathcal{C}_x^{(2)} + \mathcal{C}_{1-x}^{(2)} \right] . \quad (3.15a)$$

where

$$\mathcal{C}_x^{(2)} = \Lambda_M \left(\frac{1}{\mu} - \frac{1}{x} + \frac{1}{4x} \ln \frac{x}{\mu} \right) . \quad (3.15b)$$

The self-mass terms (3.12) have similar divergences as $k'^\perp \rightarrow \infty$ and $x' \rightarrow 0$, and a careful evaluation with corresponding cutoffs gives

$$\delta m = \frac{g^2}{\pi^2} \left[\mathcal{S}_x^{(2)} + \mathcal{S}_{1-x}^{(2)} \right] . \quad (3.16a)$$

where

$$\mathcal{S}_x^{(2)} = \Lambda_M \left(\frac{1}{x} - \frac{1}{\mu} - \frac{1}{4x} \ln \frac{x}{\mu} \right) + \frac{m\pi}{2x} \ln \frac{x}{\mu} . \quad (3.16b)$$

The ultraviolet divergent terms proportional to Λ_M cancel between (3.15) and (3.16) leaving a net contribution

$$C + \delta m = \frac{mg^2}{2\pi} \left(\frac{1}{x} \ln \frac{x}{\mu} + \frac{1}{1-x} \ln \frac{1-x}{\mu} \right) \quad (3.17)$$

which contains logarithmic infrared divergences which are, in fact, characteristic of QED in (2+1) dimensions [13,15]. The cancellation of any ultraviolet divergences is precisely what one would expect in this super-renormalizable theory. The role of the self-induced inertia terms seems to be simply the cancellation of “spurious” ultraviolet divergences in the self-mass diagrams in the light-cone formulation. This was not so evident in the treatments of the (3+1)-dimensional model [4,5,9], where the self-induced inertias were “swept under the carpet” to some extent by the renormalization process.

Next, consider the off-diagonal terms. The seagull diagram (3.13) contains a quadratic infrared divergence as $|x - x'| \rightarrow 0$: this is the notorious “spurious light-cone singularity” which afflicts the light-cone approach. The real photon exchange diagrams (3.14), however, are found to display an equal and opposite singularity as $|x - x'| \rightarrow 0$, so that the quadratic divergences cancel. The next-to-leading terms of order $|x - x'|^{-1}$ also cancel, but only if one adopts the ad hoc “symmetrized mass” prescription (3.8) for ω (we shall comment further on this in the conclusions). Putting the two terms (3.13) and (3.14) together, one finally obtains a new form of the integral Eq. (3.9):

$$\begin{aligned} \tilde{M}_i^2 \psi(x, k^\perp) &= \left[\frac{m^2 + k^{\perp 2}}{x(1-x)} + \frac{mg^2}{2\pi} \left(\frac{1}{x} \ln \frac{x}{\mu} + \frac{1}{1-x} \ln \frac{1-x}{\mu} \right) \right] \psi(x, k^\perp) \\ &\quad - \frac{g^2}{8\pi^2} \int \frac{dx' dk'^\perp}{\mathcal{D}} \left[\mathcal{N}_{x, k^\perp}(x', k'^\perp) + \mathcal{N}_{1-x, -k^\perp}(1-x', -k'^\perp) \right] \psi(x', k'^\perp) , \end{aligned} \quad (3.18a)$$

where

$$\begin{aligned} \mathcal{N}_{x, k^\perp}(x', k'^\perp) &= \Theta(x - x') \left\{ m^2 \left[2 \left(\frac{1}{xx'} + \frac{1}{(1-x)(1-x')} \right) - \frac{(-1)^{\sigma_{e-}}(-1)^{\sigma_{e+}}(x-x')^2}{xx'(1-x)(1-x')} \right] \right. \\ &\quad - im \frac{(-1)^{\sigma_{e+}}}{xx'} \left[2(k-k')^\perp + (x-x') \left(\frac{k^\perp}{1-x} + \frac{k'^\perp}{1-x'} \right) \right] \\ &\quad + im \frac{(-1)^{\sigma_{e-}}}{(1-x)(1-x')} \left[2(k-k')^\perp - (x-x') \left(\frac{k^\perp}{x} + \frac{k'^\perp}{x'} \right) \right] \\ &\quad + k^\perp k'^\perp \left[\frac{1}{x'(1-x)} + \frac{1}{x(1-x')} + \frac{2}{xx'} + \frac{2}{(1-x)(1-x')} \right] \\ &\quad \left. + \frac{k^{\perp 2}}{x(1-x)} + \frac{k'^{\perp 2}}{x'(1-x')} \right\} \end{aligned} \quad (3.18b)$$

and

$$\begin{aligned} \mathcal{D} &= \frac{m^2}{2} (x-x')^2 \left[\frac{1}{xx'} + \frac{1}{(1-x)(1-x')} \right] + (k-k')^{\perp 2} \\ &\quad - \frac{x-x'}{2} \left[k^{\perp 2} \left(\frac{1}{x} - \frac{1}{1-x} \right) + k'^{\perp 2} \left(\frac{1}{1-x'} - \frac{1}{x'} \right) \right] . \end{aligned} \quad (3.18c)$$

IV. THE NON-RELATIVISTIC SCHRÖDINGER EQUATION

A. Deriving the equation

In the non-relativistic limit when the coupling is weak ($\frac{g^2}{m} \ll 1$) and the relative momenta are small [$k^{\perp 2} \ll m^2$, $(x - \frac{1}{2})^2 \ll 1$], Eq. (3.18) reduces to the equation:

$$\begin{aligned} & \left\{ 4 \left[m^2 + k^{\perp 2} + 4m^2 \left(x - \frac{1}{2} \right)^2 \right] - \frac{2mg^2}{\pi} \ln 2\mu \right\} \\ & \times \psi \left(x, k^{\perp} \right) \\ & - \frac{g^2}{8\pi^2} \int dx' dk'^{\perp} \frac{16m^2 \psi \left(x', k'^{\perp} \right)}{4m^2(x - x')^2 + (k - k')^{\perp 2}} \\ & = \hat{M}^2 \psi \left(x, k^{\perp} \right) . \end{aligned} \quad (4.1)$$

We define a 2-vector momentum $\mathbf{k} = (k_{\parallel}, k^{\perp})$ where $k_{\parallel} = 2m(x - \frac{1}{2})$, together with a non-relativistic energy E given by the formula

$$\hat{M}^2 = 4m^2 + 4mE . \quad (4.2)$$

then (4.1) takes the form

$$\left(\frac{k^2}{m} - \frac{g^2}{2\pi} \ln 2\mu \right) \psi(\mathbf{k}) - \frac{g^2}{4\pi^2} \int d^2\mathbf{k}' \frac{\psi(\mathbf{k}')}{(\mathbf{k} - \mathbf{k}')^2} = E \psi(\mathbf{k}) \quad (4.3)$$

which is the desired non-relativistic Schrödinger equation in momentum space, with the reduced mass of the system equal to $\frac{1}{2}m$.

Now Eq. (4.3) still retains a dependence on the infrared cutoff μ in the second, logarithmic term. The integral, however, also contains a logarithmic infrared divergence as $(\mathbf{k} - \mathbf{k}')^2 \rightarrow 0$, i.e. a physical Coulomb singularity. If the longitudinal momentum integral is cut off at $|\mathbf{k} - \mathbf{k}'| \geq 2m\mu$, matching the previous cutoff, then the logarithmically divergent terms are found to cancel precisely, eliminating any cutoff dependence. We thus find that the self-mass diagrams and the one-photon exchange diagrams are separately infrared divergent, but the infrared divergences cancel out in the bound-state equation.

Equation (4.3) may be converted to a position-space Schrödinger equation by performing a standard Fourier transformation in two dimensions, including the cutoff on “longitudinal” momentum as mentioned above. The result is

$$\left[-\frac{1}{m} \nabla_r^2 + \frac{g^2}{2\pi} (\gamma + \ln mr) \right] \psi(\mathbf{r}) = E \psi(\mathbf{r}) , \quad (4.4)$$

where γ is Euler’s constant. The classical logarithmic behaviour of the Coulomb potential in two dimensions has emerged, with the bare fermion mass acting as a distance scale.

The same result can be reached from the standard equal-time field theory formalism, following Cornwall and Tiktopoulos [15], Sen [13], and Yung and Hamer [14], where the infrared divergences were regulated by giving a mass ν to the photon. Inserting some factors of $\ln 2$ which were neglected by these authors, one finds that the one-loop renormalized self-mass of the fermion is

$$m_R = m + \frac{g^2}{4\pi} \ln \frac{2m}{\nu} . \quad (4.5)$$

while the potential due to one-photon exchange between the electron and positron is

$$\begin{aligned} V(r) &= -g^2 \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{k^2 + \nu^2} \\ &= -\frac{g^2}{2\pi} K_0(\nu r) \\ &= \frac{g^2}{2\pi} \left(\gamma + \ln \frac{\nu r}{2} \right) + O(\nu^2 r^2) . \end{aligned} \quad (4.6)$$

Schrödinger’s equation is then

$$\begin{aligned} E \psi(\mathbf{r}) &= \left[-\frac{1}{m} \nabla_r^2 + 2(m_R - m) + V(r) \right] \psi(\mathbf{r}) \\ &= \left[-\frac{1}{m} \nabla_r^2 + \frac{g^2}{2\pi} (\gamma + \ln mr) \right] \psi(\mathbf{r}) \end{aligned} \quad (4.7)$$

which agrees with Eq. (4.4). Again we see that the self-mass and the potential are separately infrared divergent, but these divergences cancel in the bound-state equation.

B. Numerical solution

We now set out to obtain numerical solutions for the momentum-space Schrödinger equation (4.3), once again following Krautgärtner *et al.* [9]. For the rotationally invariant s -states, we can immediately integrate over angles. With $p = |\mathbf{k}|$ Eq. (4.3) becomes

$$\left(\frac{p^2}{m} - \frac{g^2}{2\pi} \ln 2\mu \right) \psi(p) - \frac{g^2}{2\pi} \int_0^\infty \frac{p' dp'}{|p^2 - p'^2|} \psi(p') = E \psi(p) , \quad (4.8)$$

where the non-relativistic momentum cutoff is $|p - p'| \geq 2m\mu$.

Before this equation can be solved numerically, the mutually-cancelling infrared divergences must be dealt with. One technique is to add and subtract from (4.8) a counter-term which must satisfy the following requirements:

- It must have a form similar to the integral term in (4.8);
- The integrand must have an infrared divergence as $|p - p'| \rightarrow 0$ matching that in (4.8), and no other divergences;
- It should be integrable analytically.

A simple choice for this counter-term is

$$\int_0^\infty \frac{p' dp'}{|p^2 - p'^2|} \frac{2p}{p + p'} \psi(p) = \ln \left(\frac{p}{m\mu} \right) \psi(p) \quad (4.9)$$

using the cutoff mentioned above.

Using this counter-term, the infrared divergences of (4.8) are explicitly removed. One may further simplify the equation by the removal of the dimensional quantities g , m , and E . Let

$$E = \frac{g^2}{2\pi} \left(\lambda - \frac{1}{2} \ln \frac{2g^2}{m\pi} \right) , \quad (4.10)$$

$$p = \sqrt{\frac{mg^2}{2\pi}} q , \quad (4.11)$$

then the end result is

$$\begin{aligned} (q^2 - \ln q) \psi(q) - \int_0^\infty \frac{q' dq'}{|q^2 - q'^2|} \\ \times \left[\psi(q') - \frac{2q}{q + q'} \psi(q) \right] = \lambda \psi(q) . \end{aligned} \quad (4.12)$$

Note that $\psi(0) \equiv 0$, otherwise Eq. (4.12) is ill-defined at $q = 0$.

One can numerically solve (4.12) using a procedure similar to the so-called Nyström method [25,26], whereby an integral equation is turned into a matrix equation. The method consists of approximating the integral by a quadrature and evaluating the whole equation at the same quadrature points. For a given n -point quadrature formula, let us denote the weights by w_i and the points by q_i , where $i = 1, \dots, n$. The domain of integration requires that $0 < q_1 < \dots < q_n < \infty$. Evaluation of (4.12) at these quadrature points gives a system of n linear equations:

$$\begin{aligned} (q_i^2 - \ln q_i) \psi^{(n)}(q_i) - \sum_{\substack{j=1 \\ j \neq i}}^n w_j \frac{q_j}{|q_i^2 - q_j^2|} \\ \times \left[\psi^{(n)}(q_j) - \frac{2q_i}{q_i + q_j} \psi^{(n)}(q_i) \right] \\ = \lambda^{(n)} \psi^{(n)}(q_i) , \quad i = 1, \dots, n . \end{aligned} \quad (4.13)$$

It is now straightforward to find the approximate eigenvalues $\lambda^{(n)}$ and eigenfunctions $\psi^{(n)}(q)$.

Various quadrature formulae were tried out. The best convergence was obtained with a Gaussian-Legendre quadrature [26] for the segment $[0, \infty)$. The major numerical difficulty is the fact that the integrand in (4.12) still has a finite discontinuity at $q = q'$. We estimate that this produces an overall error of order 0.05% for $n = 200$. The exact solution should be obtained in the $n \rightarrow \infty$ limit. The results of the calculation are presented below and in Figs. 4-6.

The two lowest eigenfunctions are shown in Figs. 4 and 5. They were calculated with $n = 200$ and are unnormalized. Both eigenfunctions exhibit the expected behaviour; in particular, $\psi(q) \rightarrow 0$ logarithmically as $q \rightarrow 0$, although this is not visible on the scale used in the figures.

The five lowest eigenvalues were calculated with n up to 200 and are plotted against $1/n$ in Fig. 6. The convergence is good, so that a relatively small matrix gives a reliable calculation. We present here the $n \rightarrow \infty$ extrapolated eigenvalues:

$$\begin{aligned}\lambda_0 &= 1.7969, \quad \lambda_1 = 2.9316, \\ \lambda_2 &= 3.4483, \quad \lambda_3 = 3.7985, \\ \lambda_4 &= 4.0585.\end{aligned}\tag{4.14}$$

These results may be compared with those of Yung and Hamer [14], who solved the position-space Schrödinger equation (4.7). Translating their results to our notation, they obtained:

$$\begin{aligned}\lambda_0 &= 1.9690, \quad \lambda_1 = 2.9873, \\ \lambda_2 &= 3.4807.\end{aligned}\tag{4.15}$$

which do not agree with (4.14). We tried to confirm the position-space results by two different methods. Firstly, Eq. (4.7) was replaced by a matrix equation, writing derivatives as finite differences, and solved. Secondly, a “discretized” version of the position-space Hamiltonian was treated using a crude Monte Carlo method. In both cases, the convergence was extremely slow as the number of quadrature points was increased, so that no accurate final result was obtained; but the results did appear to be trending towards the values (4.14). We conclude, therefore, that the previous position-space calculation [14] was inaccurate, due to the slow, logarithmic behaviour of the potential, and the cutoff at finite radius used in that calculation. The momentum-space calculation is much more rapidly convergent.

V. DISCUSSION AND CONCLUSIONS

In this paper, we have set up the discretized light-cone formulation of QED in (2+1) dimensions with 4-component spinors, and used it to derive a Tamm-Dancoff integral equation for the “positronium” bound states, following Krautgärtner *et al.* [9]. The effective “light-cone Hamiltonian” for the system involves several diagrams, as shown in Fig. 2. These

diagrams individually involve unphysical or spurious divergences, which is one of the complicating features of the light-cone formalism; but when they are combined together, these divergent terms cancel out. The pattern is that of Tang's "gauge cutoff" rule [4,17]: diagrams containing an "instantaneous" particle exchange should be paired with the corresponding diagrams containing a real particle exchange. Thus, if the self-induced inertia terms [Fig. 2(b)] are combined with the self-mass diagrams [Fig. 2(c)] the unphysical singularities cancel out, leaving no ultraviolet singularities, as we expect in this super-renormalizable theory. A logarithmic infrared divergence does remain in the self-mass term, just as in the equal-time formalism: this may be taken as a signal of charge confinement in the theory [15,13].

Similarly, when the instantaneous photon exchange term [Fig. 2(d)] is combined with the real photon exchange diagrams [Fig. 2(e)], the spurious light-cone singularities cancel, leaving only a physical Coulomb singularity at low relative momentum. One slightly worrying feature is that the ad hoc "symmetrized mass" replacement in the denominator of the effective Hamiltonian is crucial to the cancellation of the spurious light-cone singularities, as noted by Krautgärtner *et al.* [9]. A more satisfactory procedure may be to use the Leibbrandt-Mandelstam prescription [27] for the gauge-field singularities, whose application to the bound-state equation has been discussed by Liu and Soper [28].

In the weak coupling limit, we have shown that the Tamm-Dancoff integral equation reduces to a non-relativistic integral equation which matches the Schrödinger equation obtained from the equal-time formalism [13–15]. The infrared divergences in the bound-state equation cancel, as in the equal-time case, so that the bound-state masses remain finite in units of the bare fermion mass and coupling constant. We have obtained numerical solutions for the bound-state energies which are more accurate than those previously available [14]. It has thus been demonstrated that the light-cone approach gives correct results in the weak-coupling limit, and these can be used for comparison with later DLCQ calculations.

A major snag arises when one attempts to extend the Tamm-Dancoff integral equation to relativistic cases, and larger couplings. We have found that the infrared divergences no longer cancel in the bound-state equation. In Eq. (3.18), for example, the integral on the

right-hand side exhibits a logarithmic divergence as $\mathbf{k} \rightarrow \mathbf{k}'$, but the coefficient depends on $k^\perp$, and does not exactly cancel with the explicit logarithmic terms.

We expect that the infrared divergences should continue to cancel through higher orders in this model if a careful and correct order-by-order perturbative treatment of the bound-state problem was carried out, although no explicit proof of this exists as far as we are aware. The infrared problem outlined above presumably exposes a defect in the current approach. It is known [1] that the Tamm-Dancoff cutoff violates gauge invariance, and Tang's "gauge cutoff" rule [4,17] was introduced to prevent this violation from occurring already at the tree level. The infrared problem is thus very likely to be a manifestation of the violation of gauge invariance at higher order. The problem has been highlighted by the fact that infrared divergences are more severe in the (2+1)-dimensional model than in the (3+1)-dimensional one.

Whether or not the above ideas are correct, it is certainly true that in order to study larger couplings one will have to include new sectors of multi-particle states, containing more photons in particular, into the treatment. Some ideas along these lines have been discussed by Krautgärtner *et al.* [9], for instance. The integral equation approach is then likely to become extremely complicated, however, and little more has been done in this direction. The DLCQ technique may be better adapted to handle such complications.

Our conclusions, then, are somewhat gloomy. The light-cone approach gives correct results in the weak-coupling limit; but these can probably be obtained more easily by conventional equal-time methods. In order to move to stronger couplings, one will have to confront head-on the problem of dealing with states containing many particles, and in particular many gauge bosons, and the huge proliferation of basis states which that will produce. It is not clear that this can be done successfully in a quantum Hamiltonian framework. Hiller [29] has discussed the use of Lanczos techniques for this purpose; and Burkardt [30] has explored a Monte Carlo technique, but has not yet investigated a fermion model, where the infamous "minus sign" problem is likely to rear its head. The goal of useful calculations for QCD appears still some distance away.

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FIGURES

FIG. 1. Graphical representation of the interaction terms in the light-cone Hamiltonian. Similar diagrams with the external particles “permuted” between initial and final states are not shown. A vertical line with the stroke through it represents an “instantaneous” interaction, and a cross represents a contraction.

FIG. 2. Diagrams to be included in the effective Hamiltonian H_{eff} .

FIG. 3. Kinematic variables in the initial and final states.

FIG. 4. Unnormalized momentum-space eigenfunction for the ground state, calculated with a 200-point quadrature.

FIG. 5. Unnormalized momentum-space eigenfunction for the first excited state, calculated with a 200-point quadrature.

FIG. 6. The convergence of the five lowest eigenvalues λ versus $1/n$, where n is the number of quadrature points.

TABLES

TABLE I. Definitions of light-cone variables.

Variables	<p>light-cone time $= x^+ = x^0 + x^1$,</p> <p>light-cone position $= x^- = x^0 - x^1$,</p> <p>transverse position $= x^\perp = x^2$.</p>
Covariant notation	$A^\mu = (A^+, A^-, A^\perp)$.
Metric	$g^{\mu\nu} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$
Dot product	$x \cdot y = x^\mu g_{\mu\nu} y^\nu$ $= \frac{1}{2} (x^+ y^- + x^- y^+) - x^\perp y^\perp .$
Mass shell condition	$p^+ p^- - p^{\perp 2} = \tilde{M}^2$.
Derivatives	$\partial_\mu = \frac{\partial}{\partial x^\mu} ,$ $\partial^+ = 2\partial_- , \partial^- = 2\partial_+ , \partial^\perp = -\partial_\perp .$
Underscore notation	$\underline{k} = (k^+, k^\perp)$.

TABLE II. Dirac algebra, projection operators and spinors.

Dirac algebra	$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} .$
Projection operators	$\Lambda^\pm = \frac{1}{4} \gamma^\mp \gamma^\pm ,$
	$(\Lambda^\pm)^2 = \Lambda^\pm ,$
	$\Lambda^+ + \Lambda^- = 1 ,$
	$\Lambda^+ \Lambda^- = 0 .$
Useful relations	$\gamma^\pm = \Lambda^\mp \gamma^\pm \Lambda^\pm ,$
	$\Lambda^\pm \gamma^0 = \gamma^0 \Lambda^\mp .$
Projected spinors	$\psi_\pm = \Lambda^\pm \psi ,$
	$\Lambda^\pm u_\pm^\sigma = u_\pm^\sigma ,$
	$(\sigma = 1 \text{ or } 2) .$
	$u_\mu^{\sigma\dagger} u_\nu^\tau = \delta_{\mu\nu} \delta_{\sigma\tau} ,$
Normalization	$(\mu, \nu = + \text{ or } -) ,$
	$(\sigma, \tau = 1 \text{ or } 2) .$

a) V_1 :

b) V_2 :

c) V_3 :

d) V_4 :

e) V_5 :

Figure 1.

a)

b)

c)

d)

e)

Figure 2.

Figure 3.

Figure 4

Figure 5

Figure 6