SCAN-9410168

SW 9442

Time in the Semiclassical Approximation to Quantum Gravity Coupled with a Background Scalar Field and Kaluza-Klein Theory

YOSHIKI OHKUWA

Department of Mathematics, Miyazaki Medical College
Kiyotake, Miyazaki 889-16, Japan

ABSTRACT

We consider the semiclassical approximation to quantum gravity coupled with a background scalar field and a quantum matter field. We define the semiclassical time variable and write down the Schrödinger equation for the quantum matter field. This can be rewritten into the form whose time variable is the original time coordinate, when the relation of Hamilton-Jacobi holds. It is shown that the 4-dimensional semiclassical time with a background scalar field is related to the 5-dimensional semiclassical time without it by means of the Kaluza-Klein dimensional reduction.

1. Introduction

In quantum gravity the wave function is time independent because of the constraints in the Dirac theory.^[1] Banks and others applied semiclassical methods to the problem of time in quantum gravity coupled with a quantum matter field. They wrote down a functional Schrödinger equation for the matter field whose time variable is defined by the semiclassical approximation to the wave function of pure gravity.^{[2]-[12]} [1]

In realistic cosmologies with inflation a background scalar field is necessary to explain the inflation.^[13] So we will consider the gravity coupled with a background scalar field and a quantum matter field and apply the semiclassical methods to define the time variable. From a consideration of the path integral, the wave function for the quantum matter field is expected to obey the Schrödinger equation whose time variable is the original time coordinate used in the original quantization.^[14] We will show that this holds when the relation of Hamilton-Jacobi is assumed.

On the other hand, it is known that the 4-dimensional gravity coupled with a scalar field can be derived from the 5-dimensional pure gravity by the dimensional reduction of Kaluza-Klein.^[15] [16] Therefore, we will investigate the relation between the 4-dimensional time variable in a gravity with a background scalar field and the 5-dimensional time variable in pure gravity.

In §2 we apply the semiclassical approximation to the quantum gravity coupled with a background scalar field and a quantum matter field, and we define the semiclassical time variable. We derive the Schrödinger equation in terms of this variable and rewrite this into the form whose time variable is the original time coordinate, when the relation of Hamilton-Jacobi is assumed. In §3 we find the relation between 4 and 5-dimensional semiclassical time variables by means of the Kaluza-Klein dimensional reduction. Our conclusions are presented in §4. Appendix A describes a consideration of the path integral.

2. Semiclassical Approximation to Quantum Gravity and the Time T

In this section we assume that the space-time is an $(n+1)$ -dimensional Lorentzian manifold. In reality $n=3$, but we consider the more general case. The $(n+1)$ -metric is assumed to be

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j). \quad (1)$$

Here $N^\perp$ and N^i are the lapse and shift functions, and h_{ij} is the induced n -metric on a spatial n -manifold which is the sole boundary B of an $(n+1)$ -manifold. Let us consider the action of gravity coupled with a background scalar field and a quantum matter field,

$$S = S_B + S_Q,$$

$$S_B = \int d^{n+1}x \sqrt{-g} \left[\frac{1}{16\pi G} \left({}^{(n+1)}R(g_{\mu\nu}) - 2\Lambda \right) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi_b \partial_\nu \phi_b - V_b(\phi_b) \right], \quad (2)$$

where S_B is the action of a gravitational field $g_{\mu\nu}$ coupled with a background scalar field ϕ_b , S_Q is the action of a quantum matter field ϕ_q (Here ϕ_q symbolically stands for any matter field.), $V_b(\phi_b)$ is a potential, and we have ignored the surface term. Later we will regard the quantum matter field ϕ_q is the order of $\sqrt{\hbar}$. The momentum conjugates to h_{ij} and ϕ_b can be derived as

$$p^{ij} = -\frac{1}{16\pi G} \sqrt{h} (K^{ij} - h^{ij} K), \quad p_b = \frac{\sqrt{h}}{N^\perp} \left(\frac{\partial \phi_b}{\partial t} - N^i \partial_i \phi_b \right), \quad (3)$$

$$K_{ij} = \frac{1}{2N^\perp} \left(-\frac{\partial h_{ij}}{\partial t} + \nabla_i N_j + \nabla_j N_i \right) \quad \text{and} \quad K = h^{ij} K_{ij}.$$

Here ∇_i is the covariant derivative in the n -surface. According to the Dirac quantization procedure, we quantize this theory with respect to the time coordinate

t . The wave functional $\Psi[h_{ij}, \phi_b, \phi_q]$ satisfies the Wheeler-DeWitt equation and the momentum constraint,

$$\begin{aligned} \mathcal{H}_\perp \Psi &= (\mathcal{H}_{B_\perp} + \mathcal{H}_{Q_\perp}) \Psi, \\ &= \left[-16\pi G \hbar^2 \mathcal{G}_{ijkl} \frac{\delta^2}{\delta h_{ij} \delta h_{kl}} - \frac{1}{16\pi G} \sqrt{h} \left({}^{(n)}R(h_{ij}) - 2\Lambda \right) \right. \\ &\quad \left. - \frac{\hbar^2}{2\sqrt{h}} \frac{\delta^2}{\delta \phi_b^2} + \frac{\sqrt{h}}{2} h^{ij} \partial_i \phi_b \partial_j \phi_b + \sqrt{h} V_b(\phi_b) + \mathcal{H}_{Q_\perp} \right] \Psi = 0, \end{aligned} \quad (4)$$

$$\mathcal{G}_{ijkl} = \frac{1}{2\sqrt{h}} \left(h_{ik} h_{jl} + h_{il} h_{jk} - \frac{2}{n-1} h_{ij} h_{kl} \right), \quad (5)$$

$$\begin{aligned} \mathcal{H}^i \Psi &= (\mathcal{H}_B^i + \mathcal{H}_Q^i) \Psi, \\ &= \left[2i\hbar \nabla_j \frac{\delta}{\delta h_{ij}} - i\hbar h^{ij} \partial_j \phi_b \frac{\delta}{\delta \phi_b} + \mathcal{H}_Q^i \right] \Psi = 0, \end{aligned} \quad (6)$$

where we have ignored ordering problems.

In order to look for semiclassical solutions to Eqs. (4),(5), write

$$\Psi[h_{ij}, \phi_b, \phi_q] = \Phi[h_{ij}, \phi_b, \phi_q] e^{\frac{i}{\hbar} S_0[h_{ij}, \phi_b]}. \quad (7)$$

Substituting (7) to Eq. (4) and equating powers of $\hbar$, we obtain

$$\begin{aligned} 16\pi G \mathcal{G}_{ijkl} \frac{\delta S_0}{\delta h_{ij}} \frac{\delta S_0}{\delta h_{kl}} - \frac{\sqrt{h}}{16\pi G} \left({}^{(n)}R - 2\Lambda \right) \\ + \frac{1}{2\sqrt{h}} \left(\frac{\delta S_0}{\delta \phi_b} \right)^2 + \frac{\sqrt{h}}{2} h^{ij} \partial_i \phi_b \partial_j \phi_b + \sqrt{h} V_b(\phi_b) = 0, \end{aligned} \quad (8)$$

$$\begin{aligned} -i16\pi G \hbar \mathcal{G}_{ijkl} \left(\frac{\delta^2 S_0}{\delta h_{ij} \delta h_{kl}} \Phi + 2 \frac{\delta S_0}{\delta h_{ij}} \frac{\delta \Phi}{\delta h_{kl}} \right) \\ - \frac{i\hbar}{2\sqrt{h}} \left(\frac{\delta^2 S_0}{\delta \phi_b^2} \Phi + 2 \frac{\delta S_0}{\delta \phi_b} \frac{\delta \Phi}{\delta \phi_b} \right) + \mathcal{H}_{Q_\perp} \Phi = 0, \end{aligned} \quad (9)$$

where we have regarded ϕ_q is the order of $\sqrt{\hbar}$. Eq. (8) is the Hamilton-Jacobi equation for the gravity coupled with the background scalar field, and S_0 is a solution of this equation. Now we write the solution Φ of Eq. (9) as Φ_0 , when there is not the quantum matter field ϕ_q . Namely

$$-i16\pi G\hbar\mathcal{G}_{ijkl}\left(\frac{\delta^2 S_0}{\delta h_{ij}\delta h_{kl}}\Phi_0 + 2\frac{\delta S_0}{\delta h_{ij}}\frac{\delta\Phi_0}{\delta h_{kl}}\right) - \frac{i\hbar}{2\sqrt{h}}\left(\frac{\delta^2 S_0}{\delta\phi_b^2}\Phi_0 + 2\frac{\delta S_0}{\delta\phi_b}\frac{\delta\Phi_0}{\delta\phi_b}\right) = 0. \quad (10)$$

Let us write

$$\Psi[h_{ij}, \phi_b, \phi_q] = \Phi_0[h_{ij}, \phi_b] e^{\frac{i}{\hbar}S_0[h_{ij}, \phi_b]} \psi[h_{ij}, \phi_b, \phi_q]. \quad (11)$$

Using Eqs. (9),(10) and (11), we have

$$-2i\hbar16\pi G\mathcal{G}_{ijkl}\frac{\delta S_0}{\delta h_{ij}}\frac{\delta\psi}{\delta h_{kl}} - \frac{i\hbar}{\sqrt{h}}\frac{\delta S_0}{\delta\phi_b}\frac{\delta\psi}{\delta\phi_b} + \mathcal{H}_{q\perp}\psi = 0. \quad (12)$$

Now define $T(x; h_{ij}, \phi_b)$ which is a function of x and a functional of h_{ij} and ϕ_b as

$$32\pi G\mathcal{G}_{ijkl}\frac{\delta S_0}{\delta h_{ij}(x)}\frac{\delta T(y)}{\delta h_{kl}(x)} + \frac{1}{\sqrt{h}}\frac{\delta S_0}{\delta\phi_b(x)}\frac{\delta T(y)}{\delta\phi_b(x)} = \delta(x, y). \quad (13)$$

Note that T is a solution of this equation, so T depends on the specific choice of the solution S_0 . Then Eq. (12) can be written as

$$i\hbar\frac{\delta\psi}{\delta T} = \mathcal{H}_{q\perp}\psi. \quad (14)$$

Similarly from the momentum constraint (6) and Eq. (11), the momentum constraint to ψ can be calculated as

$$2i\hbar\nabla_j\frac{\delta\psi}{\delta h_{ij}} - i\hbar h^{ij}\partial_j\phi_b\frac{\delta\psi}{\delta\phi_b} + \mathcal{H}_q^i\psi = 0. \quad (15)$$

The equation (14) is a generalization of the equation derived by Banks and others.^{[2]–[12]} [1] From the consideration in appendix A, we expect that Eq. (14)

can be rewritten into the form whose time variable is the original time coordinate. Let us confirm this quickly.

Following Refs. [3], [4], [17], we assume

$$p^{ij} = \frac{\delta S_0}{\delta h_{ij}}, \quad p_b = \frac{\delta S_0}{\delta\phi_b}. \quad (16)$$

Note that these are classically well known equations of Hamilton-Jacobi,^[17] but they are not so trivial in quantum mechanics. It was shown in Refs. [3], [4] that the wave function is peaked about the correlations between p^{ij} and h_{ij} of the form (16) in the minisuperspace case, when there is not the scalar field ϕ_b . Though Eqs. (16) are valid exactly only in the classical limit, we take the assumption (16), because we consider the case that the semiclassical approximation is good. In this case we obtain from Eqs. (3), (5) and (16) that

$$16\pi G\mathcal{G}_{ijkl}\frac{\delta S_0}{\delta h_{ij}} = -K_{kl}, \quad \frac{1}{\sqrt{h}}\frac{\delta S_0}{\delta\phi_b} = \frac{1}{N^\perp}\left(\frac{\partial\phi_b}{\partial t} - N^i\partial_i\phi_b\right). \quad (17)$$

If we multiply $N^\perp$ to Eq. (12) and integrate with respect to x^i , we have

$$2i\hbar\int d^n x N^\perp 16\pi G\mathcal{G}_{ijkl}\frac{\delta S_0}{\delta h_{ij}}\frac{\delta\psi}{\delta h_{kl}} + i\hbar\int d^n x \frac{N^\perp}{\sqrt{h}}\frac{\delta S_0}{\delta\phi_b}\frac{\delta\psi}{\delta\phi_b} = \int d^n x N^\perp \mathcal{H}_{q\perp}\psi. \quad (18)$$

Using Eqs. (3),(15) and (17), we can calculate the left-hand side of Eq. (18) as

$$\begin{aligned} & i\hbar\int d^n x \left[\frac{\partial h_{kl}}{\partial t} - (\nabla_k N_l + \nabla_l N_k)\right]\frac{\delta\psi}{\delta h_{kl}} + i\hbar\int d^n x \left(\frac{\partial\phi_b}{\partial t} - N^i\partial_i\phi_b\right)\frac{\delta\psi}{\delta\phi_b} \\ &= i\hbar\int d^n x \left[\frac{\partial h_{kl}}{\partial t}\frac{\delta\psi}{\delta h_{kl}} + \frac{\partial\phi_b}{\partial t}\frac{\delta\psi}{\delta\phi_b} + 2N_k\nabla_l\frac{\delta\psi}{\delta h_{kl}} - N^i\partial_i\phi_b\frac{\delta\psi}{\delta\phi_b}\right] \\ &= i\hbar\int d^n x \left(\frac{\partial h_{kl}}{\partial t}\frac{\delta\psi}{\delta h_{kl}} + \frac{\partial\phi_b}{\partial t}\frac{\delta\psi}{\delta\phi_b}\right) - \int d^n x N_k \mathcal{H}_q^k \psi, \end{aligned} \quad (19)$$

where we have done integration by parts. Therefore, Eqs. (18) and (19) mean

that

$$\begin{aligned} i\hbar \frac{\delta\psi}{\delta t} &= H_Q \psi, \\ \frac{\delta}{\delta t} &= \int d^n x \left(\frac{\partial h_{kl}}{\partial t} \frac{\delta}{\delta h_{kl}} + \frac{\partial \phi_b}{\partial t} \frac{\delta}{\delta \phi_b} \right), \\ H_Q &= \int d^n x (N^\perp \mathcal{H}_{Q\perp} + N^k \mathcal{H}_{Qk}). \end{aligned} \quad (20)$$

This equation is the desired Schrödinger equation for the matter field whose time variable is nothing but the original time coordinate.

3. Relation from Kaluza-Klein Theory

It is known that 4-dimensional gravity coupled with a scalar field can be derived from 5-dimensional pure gravity by the Kaluza-Klein dimensional reduction.^{[15][16]} Therefore, the time variable T in 4 dimension must be related to the 5 dimensional time variable $T^{(5)}$ for pure gravity. In this section we will consider this relation.

Let us start from the action,

$$\begin{aligned} S^{(5)} &= S_B^{(5)} + S_Q^{(5)}, \\ S_B^{(5)} &= \int d^5 x \frac{1}{16\pi G^{(5)}} \sqrt{-g^{(5)}} \left(R^{(5)}(g_{MN}) - 2\Lambda^{(5)} \right), \end{aligned} \quad (21)$$

where $S^{(5)}$, $G^{(5)}$, $g^{(5)}$, $R^{(5)}$, $\Lambda^{(5)}$ are 5-dimensional quantities, and $S^{(5)}_Q$ is the action of a quantum matter field $\phi_q^{(5)}$. We assume that the fifth dimensional space is a circle S^1 and $0 \leq x_5 \leq 1$. Take the following ansatz,

$$g_{MN}^{(5)}(x, x_5) = \begin{pmatrix} \varphi^{-\frac{1}{3}}(x) g_{\mu\nu}(x) & 0 \\ 0 & l^2 \varphi^{\frac{2}{3}}(x) \end{pmatrix}. \quad (22)$$

Here $g_{\mu\nu}(x)$ and $\varphi(x)$ are 4-dimensional metric and scalar fields, and l represents the size of the fifth dimensional space. From the ansatz (22) the action can be

drived as^{[15][16]*}

$$S_B^{(5)} = \int d^4 x \frac{l}{16\pi G^{(5)}} \sqrt{-g} \left[R^{(5)}(g_{\mu\nu}) - \frac{g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi}{6\varphi^2} - 2\Lambda^{(5)} \varphi^{-\frac{1}{3}} \right]. \quad (23)$$

Let us introduce a new scalar field $\phi_b(x)$ as

$$\varphi(x) = e^{k\phi_b(x)} \quad (24)$$

with a constant k . Then the action becomes

$$S_B^{(5)} = \int d^4 x \sqrt{-g} \left[\frac{1}{16\pi G} \left(R^{(5)}(g_{\mu\nu}) - 2\Lambda \right) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi_b \partial_\nu \phi_b - V_b(\phi_b) \right], \quad (25)$$

where we have set

$$\begin{aligned} \frac{l}{16\pi G^{(5)}} &= \frac{1}{16\pi G}, & \Lambda^{(5)} &= \Lambda, \\ k &= \sqrt{48\pi G}, & V_b(\phi_b) &= \frac{2\Lambda}{16\pi G} \left(e^{-\frac{1}{3}\phi_b} - 1 \right). \end{aligned} \quad (26)$$

This is a special case of the action (2), when $n=3$ and the potential V_b is given in Eq. (26).

Using Eqs. (1) and (22), we obtain

$$\begin{aligned} dS^{(5)2} &= g_{MN}^{(5)} dx^M dx^N, \\ &= -N^{\perp 2} dt^2 + h_{IJ}^{(5)} (N^I dt + dx^I) (N^J dt + dx^J), \\ &= \varphi^{-\frac{1}{3}} \left[-N^{\perp 2} dt^2 + h_{ij} (N^i dt + dx^i) (N^j dt + dx^j) \right] \\ &\quad + l^2 \varphi^{\frac{2}{3}} dx_5^2, \end{aligned} \quad (27)$$

where $N^{(5)\perp}$, $N^{(5)I}$, $h^{(5)}_{IJ}$ are 5-dimensional quantities and $N^\perp$, N^i , h_{ij} are 4-

* Note that $R^{(5)}(g^{(5)}_{MN}) = \varphi^{\frac{1}{3}} \left[R^{(4)}(g_{\mu\nu}) - \frac{\varphi^{-2}}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{\varphi^{-1}}{3} g^{\mu\nu} \nabla_\mu \partial_\nu \varphi \right]$.

dimensional quantities. Eqs. (27) mean that

$$\begin{aligned} N^\perp &= \varphi^{-\frac{1}{2}} N^\perp, & N^i &= N^i, & N^5 &= 0, \\ h_{ij} &= \varphi^{-\frac{1}{2}} h_{ij}, & h_{i5} &= 0, & h_{55} &= l^2 \varphi^{\frac{2}{3}}, \\ \sqrt{h} &= l \varphi^{-\frac{1}{6}} \sqrt{h}, \end{aligned} \quad (28)$$

where $h^{(s)} = \det h^{(s)}_{IJ}$, $h = \det h_{ij}$.

The Wheeler-DeWitt equation and the momentum constraint for the wave functional $\Psi^{(s)}[h^{(s)}_{IJ}, \phi_q^{(s)}]$ read

$$\begin{aligned} \left[-16\pi G \hbar^2 \mathcal{G}_{IJKL} \frac{\delta^2}{\delta h^{(s)}_{IJ} \delta h^{(s)}_{KL}} - \frac{1}{16\pi G^{(s)}} \sqrt{h} \left({}^{(4)}R(h_{IJ}) - 2 {}^{(s)}\Lambda \right) \right. \\ \left. + \mathcal{H}_{Q\perp} \right] \Psi^{(s)} = 0, \end{aligned} \quad (29)$$

$$\mathcal{G}_{IJKL} = \frac{1}{2\sqrt{h^{(s)}}} \left(h_{IK} h_{JL} + h_{IL} h_{JK} - \frac{2}{3} h_{IJ} h_{KL} \right), \quad (30)$$

$$\left[2i\hbar \nabla_J \frac{\delta}{\delta h^{(s)}_{IJ}} + \mathcal{H}^I_Q \right] \Psi^{(s)} = 0. \quad (31)$$

Here we have ignored operator ordering. So as to make the semiclassical approximation, define

$$\Psi^{(s)}[h_{IJ}, \phi_q] = \Phi_0[h_{IJ}] e^{\frac{i}{\hbar} S^{(s)}_0[h^{(s)}_{IJ}]} \psi[h_{IJ}, \phi_q]. \quad (32)$$

Substituting (32) to Eq. (29), we can derive the following equations:

$$16\pi G^{(s)} \mathcal{G}_{IJKL} \frac{\delta S^{(s)}_0}{\delta h^{(s)}_{IJ}} \frac{\delta S^{(s)}_0}{\delta h^{(s)}_{KL}} - \frac{\sqrt{h^{(s)}}}{16\pi G^{(s)}} \left({}^{(4)}R - 2 {}^{(s)}\Lambda \right) = 0, \quad (33)$$

$$-i16\pi G \hbar \mathcal{G}_{IJKL} \left(\frac{\delta^2 S^{(s)}_0}{\delta h^{(s)}_{IJ} \delta h^{(s)}_{KL}} \Phi_0 + 2 \frac{\delta S^{(s)}_0}{\delta h^{(s)}_{IJ}} \frac{\delta \Phi^{(s)}_0}{\delta h^{(s)}_{KL}} \right) = 0, \quad (34)$$

$$-2i\hbar 16\pi G^{(s)} \mathcal{G}_{IJKL} \frac{\delta S^{(s)}_0}{\delta h^{(s)}_{IJ}} \frac{\delta \psi^{(s)}}{\delta h^{(s)}_{KL}} + \mathcal{H}_{Q\perp} \psi^{(s)} = 0. \quad (35)$$

If we define $T^{(s)}(x; h^{(s)}_{IJ})$ as

$$32\pi G^{(s)} \mathcal{G}_{IJKL} \frac{\delta S^{(s)}_0}{\delta h^{(s)}_{IJ}(x)} \frac{\delta T^{(s)}(y)}{\delta h^{(s)}_{KL}(x)} = \delta(x, y), \quad (36)$$

we obtain from Eqs. (35) and (36) that

$$i\hbar \frac{\delta \psi^{(s)}}{\delta T^{(s)}} = \mathcal{H}_{Q\perp} \psi^{(s)}. \quad (37)$$

Since we want to find a relation between 5 and 4-dimensional theories, we will express these 5-dimensional equations by 4-dimensional quantities. Consider the following form:

$$\mathcal{G}_{IJKL} \frac{\delta A}{\delta h^{(s)}_{IJ}} \frac{\delta B}{\delta h^{(s)}_{KL}}. \quad (38)$$

Here A and B are arbitrary functionals of $h^{(s)}_{IJ}$. From Eqs. (28) and (30) we can derive

$$\begin{aligned} \mathcal{G}_{ijkl} &= \frac{\varphi^{-\frac{1}{2}}}{l} \left(\mathcal{G}_{ijkl} + \frac{1}{6\sqrt{h}} h_{ij} h_{kl} \right), \\ \mathcal{G}_{ij55} &= -\frac{l\varphi^{\frac{1}{2}}}{3\sqrt{h}} h_{ij}, & \mathcal{G}_{i5k5} &= \frac{l\varphi^{\frac{1}{2}}}{2\sqrt{h}} h_{ik}, \\ \mathcal{G}_{5555} &= \frac{2l^3}{3\sqrt{h}} \varphi^{\frac{3}{2}}, & \text{others} &= 0, \quad \text{and} \end{aligned} \quad (39)$$

$$\frac{\delta}{\delta h^{(s)}_{ij}} = \varphi^{\frac{1}{2}} \frac{\delta}{\delta h_{ij}}, \quad \frac{\delta}{\delta h^{(s)}_{i5}} = 0,$$

$$\frac{\delta}{\delta h^{(s)}_{55}} = \frac{\varphi^{-\frac{3}{2}}}{2l^2} h_{ij} \frac{\delta}{\delta h_{ij}} + \frac{3\varphi^{\frac{1}{2}}}{2l^2} \frac{\delta}{\delta \varphi}.$$

Here $\mathcal{G}_{ijkl}$ is given in Eq. (5) with $n=3$. Substituting Eqs. (39) to the form (38),

we can calculate

$$\begin{aligned} & \mathcal{G}_{IJKL}^{(s)} \frac{\delta A}{\delta h^{(s)}_{IJ}} \frac{\delta B}{\delta h^{(s)}_{KL}} \\ &= \frac{e^{\frac{\kappa\phi_b}{6}}}{l} \left(\mathcal{G}_{ijkl} \frac{\delta A}{\delta h_{ij}} \frac{\delta B}{\delta h_{kl}} + \frac{1}{32\pi G \sqrt{h}} \frac{\delta A}{\delta \phi_b} \frac{\delta B}{\delta \phi_b} \right), \end{aligned} \quad (40)$$

where we have used Eq. (24). From Eqs. (24), (26), (28), (29), (36) and (40), we obtain

$$\begin{aligned} & \left[-16\pi G \hbar^2 \mathcal{G}_{ijkl} \frac{\delta^2}{\delta h_{ij} \delta h_{kl}} - \frac{1}{16\pi G} \sqrt{h} \left(R^{(3)}(h_{ij}) - 2\Lambda \right) \right. \\ & \left. - \frac{\hbar^2}{2\sqrt{h}} \frac{\delta^2}{\delta \phi_b^2} + \frac{\sqrt{h}}{2} h^{ij} \partial_i \phi_b \partial_j \phi_b + \sqrt{h} V_b(\phi_b) + e^{-\frac{\kappa\phi_b(x)}{6}} \mathcal{H}_{Q_\perp}^{(s)} \right] \Psi^{(s)} = 0, \end{aligned} \quad (41)$$

$$e^{\frac{\kappa\phi_b(x)}{6}} \left[32\pi G \mathcal{G}_{ijkl} \frac{\delta S^{(s)}_0}{\delta h_{ij}(x)} \frac{\delta T^{(s)}(y)}{\delta h_{kl}(x)} + \frac{1}{\sqrt{h}} \frac{\delta S^{(s)}_0}{\delta \phi_b(x)} \frac{\delta T^{(s)}(y)}{\delta \phi_b(x)} \right] = \delta(x, y). \quad (42)$$

In Eq. (41) we have used $R^{(4)}(h^{(s)}_{IJ}) = \varphi^\dagger \left[R^{(3)}(h_{ij}) - \frac{\varphi^{-2}}{6} h^{ij} \partial_i \varphi \partial_j \varphi \right]$ and have ignored ordering problems. When we compare Eq. (41) with Eq. (4), we find

$$\begin{aligned} & \Psi^{(s)} = \Psi, \quad S_0 = S_0, \quad \Phi_0 = \Phi_0, \quad \psi = \psi, \\ & \mathcal{H}_{Q_\perp}^{(s)} = e^{-\frac{\kappa\phi_b}{6}} \mathcal{H}_{Q_\perp}. \end{aligned} \quad (43)$$

Then we also obtain Eqs. (8),(10) from Eqs.(33),(34). The last equation of (43) is reasonable, because the part of the Hamiltonian for the quantum matter field that contains lapse function is

$$\overset{(s)}{H}_{Q_0} = \int d^4x N^\perp \overset{(s)}{N}^\perp \overset{(s)}{\mathcal{H}}_{Q_\perp} = \int d^3x N^\perp \overset{(s)}{\mathcal{H}}_{Q_\perp}, \quad (44)$$

where we have used the first equation of (28) and the last equation of (43). Therefore, we can see that the Schrödinger equation (37) corresponds to Eq. (14), and the relation between $T^{(s)}$ and T are given by Eqs. (42) and (13).

We have applied the semiclassical approximation to the gravity coupled with a background scalar field and a quantum matter field and defined the semiclassical time variable. It was shown that, as expected from the consideration of the path integral, the wave function for the quantum matter field is described by the Schrödinger equation whose time variable is nothing but the original time coordinate when the relation of Hamilton-Jacobi holds.

Using the dimensional reduction of Kaluza-Klein, we have derived the relation between the 4-dimensional semiclassical time variable with a background scalar field and the 5-dimensional semiclassical time variable without it. We think that these considerations are useful for further investigations of time in the semiclassical approximation to quantum cosmology.^[18]

ACKNOWLEDGEMENTS

The author would like to thank Professor T. Kitazoe and Professor Y. Mizumoto for giving him motivation for the first part of this paper. He is also grateful to Professor H. Itoyama and Professor T. Kubota for helpful discussions and encouragement.

APPENDIX A Consideration in Terms of Path Integral

To simplify the discussion in this appendix, we consider the gravity $g_{\mu\nu}$ coupled with a quantum matter field ϕ_q . Namely we ignore the background scalar field ϕ_b . The problem of time in quantum gravity may be roughly understood in terms of path integral as follows. In classical gravity our space-time is described by a manifold, so usually we can consider time coordinate of this manifold as the time variable. However, in quantum gravity all the configuration of $g_{\mu\nu}$ must

be summed up in the path integral. Therefore it is difficult to define the time variable in quantum gravity. (See Fig. 1.)

Fig. 1

The reason that we can consider time dependence in the semiclassical approximation to quantum gravity may be that we fix one classical solution (dominant configuration) and consider the quantum theory of the matter field in this background geometry. That is, roughly speaking, there is the correspondence written in Fig. 2.

Fig. 2

Of course this correspondence is not exact, because we can not factorize ψ owing to the path integral with respect to $g_{\mu\nu}$. However, this correspondence holds in the approximation that we are considering, which is described in the following paragraphs.

Let us assume $g_{\mu\nu} = g_{\mu\nu}^{cl} + \sqrt{\hbar}\xi_{\mu\nu}$, where $g_{\mu\nu}^{cl}$ is the classical solution of the Einstein equation, when the energy-momentum tensor is zero, $T_{\mu\nu} = 0$, and $\xi_{\mu\nu}$ is the quantum fluctuation. If we substitute this assumption into the gravitational action S_g , in the order of $\hbar^0$ we have the classical action S_g^{cl} which is equal to S_0 , and the term which is the order of $\hbar^{\frac{1}{2}}$ disappears due to the Euler-Lagrange (Einstein) equation. In the order of $\hbar$ we have quadratic terms of $\xi_{\mu\nu}$, and Φ_0 is the Van-Vleck determinant which is obtained by the path integral of these quadratic terms with respect to $\xi_{\mu\nu}$.

For example, let us take a massless scalar field as the matter field and consider $S_q = -\frac{1}{2} \int d^{n+1}x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi_q \partial_\nu \phi_q$. Then energy momentum tensor becomes $T_{\mu\nu} = \partial_\mu \phi_q \partial_\nu \phi_q - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho \phi_q \partial_\sigma \phi_q$, so $T_{\mu\nu} = 0$ means that $\partial_\mu \phi_q = 0$ and

$\phi_q = \phi_q^{cl} = constant$. This is also a solution of the field equation for the scalar field, $\square \phi_q = g^{\mu\nu} \nabla_\mu \partial_\nu \phi_q = 0$, where ∇_μ is the covariant derivative. If we expand the scalar field as $\phi_q = \phi_q^{cl} + \sqrt{\hbar}\zeta$, we obtain $S_q = -\frac{\hbar}{2} \int d^{n+1}x \sqrt{-g^{cl}} g_{cl}^{\mu\nu} \partial_\mu \zeta \partial_\nu \zeta + o(\hbar)$. Therefore in S_q the gravitational quantum fluctuation $\xi_{\mu\nu}$ is contained only in the higher order of $\hbar$, and consequently approximated matter action S_q contains only the classical solution of gravitational field $g_{\mu\nu}^{cl}$. This means that the path integral with respect to $\xi_{\mu\nu}$ does not have any effect on the wave function for the matter field ψ in the approximation which we are considering, so ψ can be factorized. (Note that we have neglected the back-reaction which is the effect of the quantum fluctuation on the background geometry.^{[8]–[10]})

This result also holds, when we take a massive scalar field as the matter field and consider $S_q = -\frac{1}{2} \int d^{n+1}x \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi_q \partial_\nu \phi_q + m^2 \phi_q^2)$. In this case energy-momentum tensor becomes $T_{\mu\nu} = \partial_\mu \phi_q \partial_\nu \phi_q - \frac{1}{2} g_{\mu\nu} (g^{\rho\sigma} \partial_\rho \phi_q \partial_\sigma \phi_q + m^2 \phi_q^2)$, so $T_{\mu\nu} = 0$ and the field equation for the scalar field, $\square \phi_q = m^2 \phi_q$, mean $\phi_q = \phi_q^{cl} = 0$. Expanding the scalar field as $\phi_q = 0 + \sqrt{\hbar}\zeta$, we obtain $S_q = -\frac{\hbar}{2} \int d^{n+1}x \sqrt{-g^{cl}} (g_{cl}^{\mu\nu} \partial_\mu \zeta \partial_\nu \zeta + m^2 \zeta^2) + o(\hbar)$. Hence in S_q the gravitational quantum fluctuation $\xi_{\mu\nu}$ is contained only in the higher order of $\hbar$, so again ψ can be factorized. The situation is the same, when we take other fields as the matter field.

So we can consider the correspondence in Fig. 2 holds in our approximation. Therefore the wave function ψ seems to obey the Schrödinger equation whose time variable is the original time t . It is known that this Schrödinger equation holds when the relation of Hamilton-Jacobi is assumed.^{[3][4][14]} We confirm this Schrödinger equation in general superspaces when there is a background scalar field ϕ_b at the end of §2.

REFERENCES

1. See, for example, C.J. Isham, preprint Imperial/TP/91-92/25 (1992); K.V. Kuchar, in *Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics*, eds. G. Kunstatter, D.E. Vincent and J.G. Williams (World Scientific, Singapore, 1992), and references therein.
2. T. Banks, Nucl. Phys. **B249** (1985) 332.
3. J.J. Halliwell, Phys. Rev. **D36** (1987) 3626.
4. J.J. Halliwell, in *Quantum Cosmology and Baby Universes*, eds. S. Coleman, J.B. Hartle, T. Piran and S. Weinberg (World Scientific, Singapore, 1991).
5. J.J. Halliwell and S.W. Hawking, Phys. Rev. **D31** (1985) 1777.
6. S. Wada, Nucl. Phys. **B276** (1986) 729; erratum, *ibid.* **B284** (1987) 747.
7. J.B. Hartle, in *Gravitation in Astrophysics*, eds. B. Carter and J.B. Hartle (Plenum, New York, 1986).
8. A. Vilenkin, Phys. Rev. **D39** (1989) 1116.
9. T.P. Singh and T. Padmanabhan, Ann. Phys. (N.Y.) **196** (1989) 296.
10. T. Padmanabhan, Class. Quantum Grav. **6** (1989) 533.
11. T. Padmanabhan, Pramāna Jour. Phys. **35** (1990) L199.
12. C. Kiefer and T.P. Singh, Phys. Rev. **D44** (1991) 1067.
13. See, for example, A. Linde, *Particle Physics and Inflationary Cosmology* (Harwood Academic Publishers, Chur, Switzerland, 1990).
14. Y. Ohkuwa, in *Evolution of the Universe and its Observational Quest*, ed. by K. Sato (Universal Academy Press, Tokyo, 1994).
15. T. Appelquist and A. Chodos, Phys. Rev. Lett. **50** (1983) 141.
16. Y. Ohkuwa, Soryushiron Kenkyu **69** (1984) 221.
17. See, for example, L.D. Landau and E.M. Lifshitz, *Mechanics* (Addison Wesley, Reading, 1960).
18. T. Kitazoe, Y. Mizumoto and Y. Ohkuwa, Miyazaki preprint MMC-M-6 (1994).

FIGURE CAPTIONS

1. The path integral of gravity with the boundary B .
2. The correspondence between wave functions of the path integral and the canonical quantization.

Fig. 1

$$\Psi = \underbrace{\int_B \mathcal{D}g_{\mu\nu} e^{\frac{i}{\hbar} S_B}}_{\Phi_0 e^{\frac{i}{\hbar} S_0}} \underbrace{\mathcal{D}\phi_q e^{\frac{i}{\hbar} S_q}}_{\psi},$$

Fig. 2