ORDINARY MATTER IN NONLINEAR AFFINE GAUGE THEORIES OF GRAVITATION

by

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ABSTRACT

We present a general framework to include ordinary fermionic matter in the metric-affine gauge theories of gravity. It is based on a nonlinear gauge realization of the affine group, with the Lorentz group as the classification subgroup of the matter and gravitational fields.

1. Introduction

Several generalizations of the gauge theory of gravitation originally proposed by Utiyama, Sciama and Kibble⁽¹⁾ were developed in recent times⁽²⁾. The metric-affine gauge theories (MAGT's) propugated by Hehl et al.⁽³⁾ are particularly interesting, since they provide a dynamical foundation of the geometrical theory equipped with the most general connection, namely with the general linear connection Γ_{α}^{β} . The affine group is the transformation group of the geometry, thus being the main candidate to describe things in the absence of matter. According to Ne'eman and Šijački⁽⁴⁾, the adoption of the affine group as the dynamical group in a gauge theory of gravitation could solve certain renormalizability and unitarity problems in Quantum Gravity, with the help of the additional degrees of freedom contained in the linear connection.

A main difficulty for the common acceptance of the MAGT's of gravitation is the problem of including fermionic matter in this general framework, since no finite linear half-integer representations of the affine group exist. To avoid this difficulty, infinite dimensional representations were suggested by Ne'eman and Šijački⁽⁴⁾. They demonstrated that the groups $SL(n, R)$ admit half-integer representations, called *manifolds*,

suitable to describe fermionic matter, and they proposed such a group as the classification group for hadrons, which allows to study the corresponding phenomenological matter Lagrangians. But this does not solve the problem of explaining the coupling of the affine gravitational interactions to fundamental Dirac particles.

Nevertheless, the inclusion of ordinary matter in the affine scheme becomes possible in the context of nonlinear realizations of symmetry groups⁽⁵⁾. The nonlinear coset realization, initially proposed to treat internal symmetries, was soon extended to spacetime symmetries. In fact, several attempts were made to apply the nonlinear realizations to gravity⁽⁶⁾. In 1965, Ogievetskii and Polubarinov⁽⁷⁾ found out a nonlinear spinorial transformation law, which strongly suggested to explore the possibility of a nonlinear realization of the affine group. Borisov and Ogievetski⁽⁸⁾ considered this possibility with the Lorentz group as the classification subgroup. However, they only took into account the global case, in conjunction with a simultaneous nonlinear realization of the conformal group, since they were interested in reproducing local diffeomorphisms from the closure of the algebras of both groups. They did not construct a gauge theory with degrees of freedom associated to independent connections.

2. The affine and the Lorentz groups: geometry and matter

Einstein's aim was to construct a theory which guaranteed the undistinguishability of the states of movement of arbitrary observers. From the geometrical point of view, the most general spacetime transformations are those of the affine group. The affine tensor spaces⁽⁹⁾ involve the vector basis $e_\alpha := e_\alpha^i \partial_i$, representing a coordinate-independent local frame, and its dual 1-form basis ϑ^β called the coframe, defined by the requirement that the interior product of both bases is $e_\alpha \lrcorner \vartheta^\beta = \delta_\alpha^\beta$. In the gauge theories of the affine group, the coframe plays the role of a nonlinear translational connection, invariant under translations, see below. In addition, a linear connection Γ_α^β , necessary in view of the local covariance, is in order. The group transformations act on the anholonomic indices. In particular, the general linear group acts on the frame attached to the observer, rotating, shearing and dilatating it. Accordingly, the linear connections may be splitted up into an antisymmetric, a symmetric traceless and a trace contribution. In principle, these are, in the absence of matter, the dynamical degrees of freedom involved.

However, the inclusion of matter drastically modifies the situation. The role of the Lorentz group as the classification group of the elementary particles oblige us to consider it as the effective dynamical group. In so far, we described a pure affine geometry without a metric. The metrization takes place when we impose the condition that the speed of light remains a constant in every local reference frame. This means that we have to introduce an invariant Minkowski metric, so that the Lorentz group becomes the only admissible transformation group to act on the frames.

Matter fields must be placed on the linear representations of the Lorentz group. Nevertheless, this does not mean that we have to renounce to the original affine

symmetry. One can realize the affine group nonlinearly in such a way that the Lorentz group plays the role of the classification subgroup, without imposing constraints which reduce the number of degrees of freedom of the original general linear connection. In fact, this is the most natural way to match geometry and matter, preserving the simultaneous validity of the spacetime groups characteristic for each one.

There is no an *a priori* reason to suppose that a given physical symmetry must be realized linearly in nature. Indeed, studying the general formulation of nonlinear realizations of spacetime symmetry groups, the linear realizations manifest themselves as a very special case. Let us mention some arguments in favour of the nonlinear realization of the affine group. In the first place, this group does not posses natural invariants, as required to define the action of the theory. The conventional approach to overcome this difficulty consists in distinguishing two kinds of representations describing co- and contravariant objects, transforming with Λ and Λ^{-1} respectively, where Λ is a general linear transformation. This duplication of the representation space is related with the inclusion of a metric tensor in the MAGT's, with ten additional, dynamically (i.e. gauge-theoretically) not justified degrees of freedom, playing a role similar as in the geometrical approach. Invariants are then obtained saturating indices with the usual rules. The dynamical role played by the metric tensor in the MAGT's is obscure. An essential feature of the gauge theories of internal groups is that the interactions are mediated exclusively by connections, playing the role of gauge fields. Nevertheless, as far as gravity is concerned, a metric tensor is included by hand, involving dynamical degrees of freedom of unknown origin.

We claim that a more natural solution to get invariants is to project the action of the whole affine group into a subgroup which admits such invariants. In particular, due also to the dynamical reasons mentioned above, we choose the Lorentz group, which has the Minkowski metric as a natural invariant. Thus, as a consequence of the nonlinear realization, it becomes possible to construct invariants without increasing the number of degrees of freedom of the theory, since the Minkowski metric is a constant tensor. We will show bellow that the metric tensor of the MAGT's can be obtained from our approach as a degenerate case, by rearranging the hidden degrees of freedom contained in the nonlinear realization, in such a way that the general linear group becomes the classification subgroup.

The problem of the interpretation of the dynamical role played by the metric tensor is closely related to that of the correct treatment of the coframes. At this respect, a further argument to consider nonlinear realizations of spacetime symmetry groups is their success in explaining the link between the translational connections and the coframes. This problem⁽¹⁰⁾ is solved by means of a nonlinear realization of the translations. The coframes turn out to be nonlinear translational connections, with the right tensorial transformation properties⁽⁶⁾⁽¹¹⁾.

In spite of the geometrical interpretation of the dynamical objects, in particular of the translational connections, one has to be careful in distinguishing the true dynamics from the misleading manipulations allowed by the geometrical formalism, since this may lead to the misunderstandings of the physical meaning of both, the coframes and

the metric tensor. Let us consider an illustrative example. As we have pointed out, in a gauge theory the interactions are given by the connections. In our nonlinear approach, the coframes depend on the translational connection Γ^α in the form $\vartheta^\alpha := \overset{(T)}{\Gamma}^\alpha + D\xi^\alpha$, see Ref.(11) and below. Thus, the coframe cannot be brought into the value $\vartheta^\alpha = \delta_i^\alpha dx^i$ in the whole space, unless in the absence of interactions. Nevertheless, in practice one often makes this choice in order to work with holonomic frames (defined by the condition $d\vartheta^\alpha = 0$). This is not rigorous dynamically, but geometrically it makes sense, since it is equivalent to redefine the metric tensor. Indeed, being the coframes 1-forms, we can write them down as $\vartheta^\alpha = e^\alpha_i dx^i$. Thus, one only has to define the holonomic metric tensor $g_{ij} := o_{\alpha\beta} e^\alpha_i e^\beta_j$, in order to realize the invariant line element alternatively in the anholonomic and holonomic forms respectively as $o_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta = g_{ij} dx^i \otimes dx^j$, where dx^i in the r.h.s. plays the role of an holonomic coframe as discussed above. This coframe does not possess the meaning of a gauge field any more, since the dynamical information contained in the original coframes with gauge theoretical meaning has been transferred to the metric tensor.

A systematic approach to the nonlinear realization of groups is given by the so called *coset realizations*⁽⁵⁾, that we are going to discuss in the following section.

3. Coset realizations of symmetry groups

Let $G = \{g\}$ be a Lie group including a subgroup $H = \{h\}$ whose linear representations $\rho(h)$ are known, acting on functions ψ belonging to a representation space of H . The elements of the quotient space G/H are equivalence classes of the form $gH = \{gh_1, gh_2 \dots gh_n\}$, and they constitute a complete partition of the group space. We call the elements of the quotient space cosets to the left (right) of H with respect to g . Since we deal with Lie groups, the elements of G/H are labeled by continuous parameters, say ξ . We represent the elements of G/H by means of the coset indicators $c(\xi)$, parametrized by the coset parameters ξ , playing the role of a kind of coordinates. The nonlinear coset realizations⁽⁵⁾ are based on the action of the group on G/H , i.e., on a partition of its own space. An arbitrary element $g \in G$ acts on G/H transforming a coset into another, that is

$$\begin{aligned} g : G/H &\rightarrow G/H \\ c(\xi) &\rightarrow c(\xi'), \end{aligned} \tag{3.1}$$

according to the general law

$$g c(\xi) = c(\xi') h(\xi, g). \tag{3.2}$$

The elements $h(\xi, g)$ which appear in (3.2) belong to the subgroup H , that we will call in the following the *classification subgroup*, since the elements g of the whole group

G considered in (3.2) act nonlinearly on the representation space of the classification subgroup H according to

$$\psi' = \rho(h(\xi, g))\psi, \quad (3.3)$$

where ρ , as mentioned above, is a linear representation of H in the space of the matter fields ψ . Therefore, the action of the total group G projects on the representations of the subgroup H through the dependence of $h(\xi, g)$ in (3.2) on the group element g , as given by eq.(3.3). The action of the group is realized on the couples (ξ, ψ) . It reduces to the usual linear action of H when we take in particular for g in (3.2) an element of H .

In order to define a covariant differential transforming like (3.3) under local transformations, we need a suitable nonlinear connection. We define it as

$$\Gamma := c^{-1}\mathcal{D}c, \quad (3.4)$$

where the covariant differential on the coset space is defined as

$$\mathcal{D}c := (d + \Omega)c, \quad (3.5)$$

with the ordinary linear connection Ω of the whole group G transforming as

$$\Omega' = g\Omega g^{-1} + g dg^{-1}. \quad (3.6)$$

It is easy to prove that the nonlinear gauge field Γ defined in (3.4) transforms as

$$\Gamma' = h\Gamma h^{-1} + h dh^{-1}, \quad (3.7)$$

thus allowing to define the nonlinear covariant differential operator

$$\mathbf{D} := d + \Gamma. \quad (3.8)$$

One can read out from (3.7) that only the components of Γ related to the generators of H behave as true connections, transforming inhomogeneously, whereas the components of Γ over the generators associated with the cosets c transform as tensors with respect to the subgroup H notwithstanding their nature of connections.

4. Nonlinear gauge approach to the affine group

As we have discussed above, ordinary matter fields such as Dirac fields can only be included in an affine theory if the affine group is realized nonlinearly, with the Lorentz group as the classification subgroup. Thus, let us consider the affine group $A(4, R) = GL(4, R) \rtimes R^4$ in 4 dimensions, defined as the semidirect product of the translations and the general linear transformations, with the generators Λ^α_β of the linear transformations splitted up into the Lorentz generators L^α_β , plus the remaining generators S^α_β of the symmetric linear transformations, as $\Lambda^\alpha_\beta = L^\alpha_\beta + S^\alpha_\beta$. In

addition, we have the generators P_α of the translations. The commutation relations read

$$\begin{aligned}
[L_{\alpha\beta}, L_{\mu\nu}] &= -i (o_{\alpha[\mu} L_{\nu]\beta} - o_{\beta[\mu} L_{\nu]\alpha}) , \\
[L_{\alpha\beta}, S_{\mu\nu}] &= i (o_{\alpha(\mu} S_{\nu)\beta} - o_{\beta(\mu} S_{\nu)\alpha}) , \\
[S_{\alpha\beta}, S_{\mu\nu}] &= i (o_{\alpha(\mu} L_{\nu)\beta} + o_{\beta(\mu} L_{\nu)\alpha}) , \\
[L_{\alpha\beta}, P_\mu] &= i o_{\mu[\alpha} P_{\beta]} , \\
[S_{\alpha\beta}, P_\mu] &= i o_{\mu(\alpha} P_{\beta)} , \\
[P_\alpha, P_\beta] &= 0 .
\end{aligned} \tag{4.1}$$

In order to realize the group action on the coset space $A(4, R)/SO(1, 3)$, we make use of the general formula (3.2) which defines the nonlinear group action, choosing in particular for the cosets the parametrization

$$c := e^{-i\xi^\alpha P_\alpha} e^{i h^{\mu\nu} S_{\mu\nu}} , \tag{4.2}$$

where ξ^α and $h^{\mu\nu}$ are the coset parameters. The group elements of the whole affine group $A(4, R)$ are parametrized as

$$g = e^{i\epsilon^\alpha P_\alpha} e^{i\alpha^{\mu\nu} S_{\mu\nu}} e^{i\beta^{\mu\nu} L_{\mu\nu}} , \tag{4.3}$$

and those of the classification Lorentz subgroup are taken to be

$$h := e^{i u^{\mu\nu} L_{\mu\nu}} . \tag{4.4}$$

Other parametrizations are possible, leading to equivalent results.

In the Appendix A, we show in some detail the calculations which lead to the explicit form of the infinitesimal transformations of the coset parameters. The variation of the translational ones reads

$$\delta\xi^\alpha = -(\alpha_\beta{}^\alpha + \beta_\beta{}^\alpha) \xi^\beta - \epsilon^\alpha , \tag{4.5}$$

thus playing these parameters the role of coordinates, transforming as usual under the affine group. On the other hand, we obtain the variation of the matrix

$$r^{\alpha\beta} := e^{h^{\alpha\beta}} , \tag{4.6}$$

constructed from the coset parameters $h^{\alpha\beta}$ associated to the symmetric affine transformations. In the following, $r^{\alpha\beta}$ more than the coset parameters itself, will play the fundamental role. Its variation reads

$$\delta r^{\alpha\beta} = (\alpha^\alpha{}_\gamma + \beta^\alpha{}_\gamma) r^{\gamma\beta} + u^\beta{}_\gamma r^{\gamma\alpha} . \tag{4.7}$$

Since $r^{\alpha\beta}$ is symmetric, the antisymmetric part of (4.7) must vanish. From this symmetry condition, as shown in Appendix B, follows for the nonlinear Lorentz parameter the expression

$$u^{\alpha\beta} = \beta^{\alpha\beta} - \alpha^{\mu\nu} \tanh \left\{ \frac{1}{2} \log \left[r^\alpha{}_\mu (r^{-1})^\beta{}_\nu \right] \right\}, \quad (4.8)$$

which shows how it consists of the linear Lorentz parameter $\beta^{\alpha\beta}$ plus a contribution depending on the symmetric affine parameter $\alpha^{\alpha\beta}$ *cum* $r^{\alpha\beta}$. The action (3.3) of the whole affine group on arbitrary fields of a given representation space of the Lorentz group reads infinitesimally

$$\delta\psi = i u^{\alpha\beta} \rho(L_{\alpha\beta}) \psi, \quad (4.9)$$

being $u^{\alpha\beta}$ the nonlinear Lorentz parameter (4.8), and $\rho(L_{\alpha\beta})$ an arbitrary representation of the Lorentz group. This equation is the key which allows to include ordinary matter into an affine theory, since the action of the symmetric linear transformations on the fields occurs through the nonlinear Lorentz parameter, whereas the group generators are the ordinary generators of the Lorentz group. Thus, not only bosonic matter, but also Dirac fields, can be considered as the sources of the gravitational potentials.

Now we will define the suitable connection for the nonlinear gauge realization in two steps. We first introduce the ordinary linear affine connection Ω in (3.5) as

$$\Omega := -i \overset{(T)}{\Gamma}^\alpha P_\alpha - i \overset{(GL)}{\Gamma}_\alpha{}^\beta (S^\alpha{}_\beta + L^\alpha{}_\beta), \quad (4.10)$$

which includes the true translational potential $\overset{(T)}{\Gamma}^\alpha$, and the $GL(4, R)$ connection $\overset{(GL)}{\Gamma}_\alpha{}^\beta$. Their transformations (3.6) present the standard form

$$\delta \overset{(T)}{\Gamma}^\alpha = \overset{(GL)}{D} \epsilon^\alpha - (\alpha_\gamma{}^\alpha + \beta_\gamma{}^\alpha) \overset{(T)}{\Gamma}^\gamma, \quad (4.11)$$

and

$$\delta \overset{(GL)}{\Gamma}_\alpha{}^\beta = \overset{(GL)}{D} (\alpha_\alpha{}^\beta + \beta_\alpha{}^\beta), \quad (4.12)$$

with $\overset{(GL)}{D}$ as the covariant differential constructed with the $GL(4, R)$ connection. Eq.(4.12) shows that both, the symmetric and the antisymmetric parts of the linear connection $\overset{(GL)}{\Gamma}_\alpha{}^\beta$ transform as true connections, as expected. Making then use of definition (3.4), we get the nonlinear connection

$$\Gamma := c^{-1} (d + \Omega) c = -i \vartheta^\alpha P_\alpha - i \overset{(GL)}{\Gamma}_\alpha{}^\beta (S^\alpha{}_\beta + L^\alpha{}_\beta), \quad (4.13)$$

with the nonlinear translational connection ϑ^α and the nonlinear $GL(4, R)$ connection Γ_α^β respectively defined as

$$\vartheta^\alpha := r_\beta^\alpha \left(\Gamma_\beta^\alpha + {}^{(GL)}D \xi^\beta \right), \quad (4.14)$$

and

$$\Gamma_\alpha^\beta := (r^{-1})_\alpha^\gamma \Gamma_\gamma^\lambda r_\lambda^\beta - (r^{-1})_\alpha^\gamma dr_\gamma^\beta. \quad (4.15)$$

In the following, we will interpret (4.14) and (4.15) geometrically, identifying ϑ^α with the coframe and Γ_α^β with a *geometrical* connection respectively. Eqs.(4.14,15) thus establish the correspondence between the *geometrical* objects in the l.h.s. and the original groupal objects in the r.h.s. According to (3.7), we find that the coframe and the connection transform respectively as

$$\delta \vartheta^\alpha = -u_\beta^\alpha \vartheta^\beta. \quad (4.16)$$

and

$$\delta \Gamma_\alpha^\beta = D u_\alpha^\beta, \quad (4.17)$$

with D constructed with the nonlinear connection (4.15) itself. Eqs.(4.16,17) show that the coframe transforms as a Lorentz vector, with the nonlinear Lorentz parameter (4.8), and the connection behaves as a Lorentz connection. In view of (4.13), this connection is actually composed of two essentially different parts, defined on different elements of the Lie algebra of the affine generators. Eq.(4.17) shows that only the antisymmetric part of (4.15), defined on the Lorentz generators, behaves as a true connection with respect to the Lorentz subgroup, whereas the symmetric part is tensorial, transforming without any inhomogeneous contribution, since $u^{\alpha\beta}$ is antisymmetric. The geometrical meaning of the nonlinear connection associated to the symmetric affine generators will become clearer in terms of the Minkowski metric, as we are going to show.

Since the nonlinear realization we are studying consists in a projection of the action of the whole affine group on its Lorentz subgroup, i.e. on the pseudoorthogonal group which, by definition, leaves invariant the symmetric tensor

$$o_{\alpha\beta} := \text{diag}(+ \ - \ - \ -), \quad (4.18)$$

this tensor appears automatically in the theory as a consequence of the nonlinear treatment, being a natural invariant of the Lorentz group. We interpret it geometrically in the common way as the Minkowskian metric tensor. No degrees of freedom are related to it. Its inverse reads

$$o^{\alpha\beta} := \text{diag}(+ \ - \ - \ -), \quad (4.19)$$

and, as mentioned above, both are Lorentz invariant:

$$\delta o_{\alpha\beta} = 0 \quad , \quad \delta o^{\alpha\beta} = 0. \quad (4.20)$$

Defining the nonmetricity $Q_{\alpha\beta}$ as usual⁽³⁾, i.e. as minus the covariant differential of the metric tensor, we find that it is proportional to the symmetric part of the nonlinear $GL(4, R)$ connection, namely

$$Q_{\alpha\beta} := -Do_{\alpha\beta} = 2\Gamma_{(\alpha\beta)}. \quad (4.21)$$

Thus, the geometrical meaning of the nonlinear connection associated to the symmetric affine generators becomes apparent as the nonmetricity, which behaves as a Lorentz tensor. In order to show that, the transformation law (4.17) can be splitted up into two parts, corresponding to the Lorentz connection $\overset{(Lor)}{\Gamma}_{\alpha\beta} := \Gamma_{[\alpha\beta]}$, and to the nonmetricity respectively. We get

$$\delta \overset{(Lor)}{\Gamma}_{\alpha}{}^{\beta} = \overset{(Lor)}{D} u_{\alpha}{}^{\beta}, \quad (4.22)$$

and

$$\delta Q_{\alpha\beta} = 2u_{(\alpha}{}^{\gamma}Q_{\beta)\gamma}. \quad (4.23)$$

The tensorial character of the symmetric part of the connection is a result of the nonlinear realization of the symmetric affine transformations, but it is also evident from the definition (4.21) of the nonmetricity, since $do_{\alpha\beta} = 0$.

Let us now find the natural field strengths of the theory from the commutation of two covariant differentials operators (3.8). This yields

$$\mathbf{D} \wedge \mathbf{D} = -iT^{\alpha}P_{\alpha} - iR_{\alpha}{}^{\beta}(S^{\alpha}{}_{\beta} + L^{\alpha}{}_{\beta}), \quad (4.24)$$

with the torsion T^{α} and the curvature $R_{\alpha}{}^{\beta}$ respectively defined as

$$T^{\alpha} := D\vartheta^{\alpha}, \quad (4.25)$$

and

$$R_{\alpha}{}^{\beta} := d\Gamma_{\alpha}{}^{\beta} + \Gamma_{\gamma}{}^{\beta} \wedge \Gamma_{\alpha}{}^{\gamma}, \quad (4.26)$$

thus showing the groupal character of the torsion as the field strength of the translations. The curvature has a symmetric and an antisymmetric part, corresponding to the symmetric and the antisymmetric (Lorentz) generators of the affine group respectively. Both, the torsion and the curvature, transform as Lorentz tensors, namely

$$\delta T^{\alpha} = -u_{\beta}{}^{\alpha}T^{\beta}, \quad (4.27)$$

and

$$\delta R_\alpha{}^\beta = u_\alpha{}^\gamma R_\gamma{}^\beta - u_\gamma{}^\beta R_\alpha{}^\gamma. \quad (4.28)$$

The geometrization of the gauge theory is completed by introducing vector bases which represent the reference frames. We interpreted the nonlinear translational connections (4.14) as the coframes, which play the role of the 1-forms bases. In terms of ϑ^α , we define their dual vector bases e_α by means of the interior product, as fulfilling the general relation

$$e_\alpha \rfloor \vartheta^\beta = \delta_\alpha^\beta. \quad (4.29)$$

The vector bases thus transform as Lorentz vectors, namely

$$\delta e_\alpha = u_\alpha{}^\beta e_\beta. \quad (4.30)$$

Our nonlinear approach contains all the elements which appear in the standard MAGT's, with the main difference of their transformation properties. In particular, the metric tensor is Lorentz invariant, i.e. it is fixed to be the Minkowskian metric, which does not possess any degrees of freedom. This seemingly makes a difference between the dynamical contents of both theories, since the metric tensor in the MAGT's involves in general ten degrees of freedom, but we will show below that these can be factorized into the coframes and the connections.

5. Correspondence to the standard metric-affine case

It is usually assumed in the context of metric-affine gauge theories, that by means of a suitable affine transformation one can always choose the anholonomic metric tensor in such a way that it becomes locally the Minkowskian metric. This choice is reasonable since spacetime is admitted to be locally Minkowskian, and it is also useful for simplifying calculations when one looks for solutions of the field equations of a given metric-affine Lagrangian. But the deep meaning of the seemingly innocent choice $g_{\alpha\beta} = o_{\alpha\beta}$ first becomes clear in view of our results. When considered as globally valid, this choice reveals itself as the key which allows the transition from the standard metric-affine theories to the nonlinear case with the Lorentz group as the classification subgroup.

In principle, the nonlinear theory depends on the variables ξ^α , $h^{\alpha\beta}$, $\overset{(T)}{\Gamma}^\alpha$, and $\overset{(GL)}{\Gamma}_\alpha{}^\beta$, i.e., on $4 + 10 + 16 + 64 = 94$ degrees of freedom. However, in the nonlinear realization, these variables occur factorized in such a manner that only the particular combinations ϑ^α and $\Gamma_\alpha{}^\beta$, see (4.14,15), i.e. $16 + 64 = 80$ degrees of freedom, are relevant. Neither ξ^α nor $h^{\alpha\beta}$ appear explicitly as independent degrees of freedom, since the symmetries to which they correspond are realized nonlinearly.

We are interested in showing the correspondence between our theory and the standard MAGT, in which only the translations are realized nonlinearly, whereas the

whole general linear group acts linearly. In this case, only ξ^α remains hidden, whereas the ten degrees of freedom of $h^{\alpha\beta}$ must be rearranged in a different way. In particular, they become manifest through the metric tensor. Formally, the transition from the nonlinear to the usual metric-affine objects resembles a finite gauge transformation, with $r^{\alpha\beta}$ as the matrix of the symmetric affine transformations. We represent the standard metric-affine objects signed by a tilde. Then, the corresponding coframes and connections are those of a nonlinear realization of the affine group in which the role of the classification group is played by the $GL(4, R)$ subgroup⁽¹¹⁾, namely

$$\tilde{\vartheta}^\alpha := \overset{(T)}{\Gamma}^\alpha + \overset{(GL)}{D} \xi^\alpha = (r^{-1})_\beta{}^\alpha \vartheta^\beta, \quad (5.1)$$

and

$$\tilde{\Gamma}_\alpha{}^\beta := \overset{(GL)}{\Gamma}_\alpha{}^\beta = r_\alpha{}^\gamma \Gamma_\gamma{}^\lambda (r^{-1})_\lambda{}^\beta - r_\alpha{}^\gamma d(r^{-1})_\gamma{}^\beta. \quad (5.2)$$

In addition, the metric tensor, which is usually introduced by hand in the affine scheme, and whose gauge-theoretical origin is not manifest if one simply includes it in the nonlinear realization with $GL(4, R)$ as the classification subgroup—in which the coset parameters $h^{\alpha\beta}$ are absent—, reveals itself as a particular factorization of the degrees of freedom present in the nonlinear approach which takes the Lorentz group as the classification subgroup. Actually, we have

$$\tilde{g}_{\alpha\beta} := r_\alpha{}^\mu r_\beta{}^\nu o_{\mu\nu}, \quad \tilde{g}^{\alpha\beta} := (r^{-1})_\mu{}^\alpha (r^{-1})_\nu{}^\beta o^{\mu\nu}, \quad (5.3)$$

compare with Ref.(8). Thus, the metric-affine theory is equivalent to our own theory. One only has to fix the metric tensor to take globally the Minkowskian form, and the degrees of freedom contained in it automatically reorganize themselves in such a way that they reproduce our nonlinear approach. As a consequence, observe that invariants like the line element may be alternatively expressed in terms of the nonlinear or metric-affine objects respectively, namely

$$ds^2 = o_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta = \tilde{g}_{\alpha\beta} \tilde{\vartheta}^\alpha \otimes \tilde{\vartheta}^\beta, \quad (5.4)$$

where the transition from $o_{\alpha\beta}$ to $\tilde{g}_{\alpha\beta}$ or *vice versa* takes place by means of the suitable factorization of the coset parameters associated to the symmetric affine transformations. The gauge-theoretical origin of the metric tensor in the MAGT's is thus explained. Moreover, the possibility of factorizing the degrees of freedom of the metric into the coframes and the connections explains why the field equations found by varying with respect to the metric tensor in the framework of metric-affine theories are not essential. Hehl et al.⁽³⁾ showed that they are redundant, when the field equations found from the coframes and the linear connections are taken into account.

Making use of the transformation properties of $r^{\alpha\beta}$, see (4.7), we find that the variations of (5.1–3) are the expected ones, namely

$$\delta \tilde{\vartheta}^\alpha = -(\alpha_\beta{}^\alpha + \beta_\beta{}^\alpha) \tilde{\vartheta}^\beta, \quad (5.5)$$

$$\delta \tilde{\Gamma}_\alpha^\beta = \tilde{D} (\alpha_\alpha^\beta + \beta_\alpha^\beta) , \quad (5.6)$$

and

$$\delta \tilde{g}_{\alpha\beta} = 2 (\alpha_{(\alpha}{}^\gamma + \beta_{(\alpha}{}^\gamma) \tilde{g}_{\beta)\gamma} \quad , \quad \delta \tilde{g}^{\alpha\beta} = -2 (\alpha_\gamma^{(\alpha} + \beta_\gamma^{(\alpha}) \tilde{g}^{\beta)\gamma} . \quad (5.7)$$

The nonmetricity reads now

$$\tilde{Q}_{\alpha\beta} := - \overset{(GL)}{D} \tilde{g}_{\alpha\beta} = r_\alpha{}^\mu r_\beta{}^\nu Q_{\mu\nu} , \quad (5.8)$$

and still transforms as a tensor, namely

$$\delta \tilde{Q}_{\alpha\beta} = 2 (\alpha_{(\alpha}{}^\gamma + \beta_{(\alpha}{}^\gamma) \tilde{Q}_{\beta)\gamma} , \quad (5.9)$$

but due to the fact that $d\tilde{g}_{\alpha\beta} \neq 0$, the symmetric part of the connection is no more a tensor, but a true connection. On the other hand, the metric-affine torsion and curvature relate to the nonlinear ones through

$$\tilde{T}^\alpha := \overset{(GL)}{D} \tilde{\vartheta}^\alpha = (r^{-1})_\beta{}^\alpha T^\beta , \quad (5.10)$$

and

$$\tilde{R}_\alpha{}^\beta := d\Gamma_\alpha{}^\beta + \overset{(GL)}{\Gamma}_\gamma{}^\beta \wedge \overset{(GL)}{\Gamma}_\alpha{}^\gamma = r_\alpha{}^\mu (r^{-1})_\nu{}^\beta R_\mu{}^\nu , \quad (5.11)$$

and their variations read

$$\delta \tilde{T}^\alpha = - (\alpha_\beta{}^\alpha + \beta_\beta{}^\alpha) \tilde{T}^\beta , \quad (5.12)$$

and

$$\delta \tilde{R}_\alpha{}^\beta = (\alpha_\alpha{}^\gamma + \beta_\alpha{}^\gamma) \tilde{R}_\gamma{}^\beta - (\alpha_\gamma{}^\beta + \beta_\gamma{}^\beta) \tilde{R}_\alpha{}^\gamma . \quad (5.13)$$

The general law relating the variations of the fields ψ of our theory, see (4.9), and of those of the MAGT, can be written in an abstract way as

$$\delta \left(e^{i h^{\mu\nu} S_{\mu\nu}} \psi \right) = i (\alpha^{\alpha\beta} S_{\alpha\beta} + \beta^{\alpha\beta} L_{\alpha\beta}) \left(e^{i h^{\mu\nu} S_{\mu\nu}} \psi \right) , \quad (5.14)$$

which does not present any problems as far as the regular representations are concerned. How to interpret (5.14) in the case of fields ψ in a half-integer representation remains an open question. The same comment holds for the relation between the abstract differential operators, namely

$$\mathbf{D}\psi = e^{-i h^{\mu\nu} S_{\mu\nu}} \tilde{\mathbf{D}} \left(e^{i h^{\mu\nu} S_{\mu\nu}} \psi \right) . \quad (5.15)$$

In short, the standard MAGT's result from a particular factorization of the ten degrees of freedom of the nonlinear theory, which manifest themselves as the degrees of freedom of the metric tensor.

6. Conclusions

The standard MAGT is a nonlinear realization of the affine group in which only the translations are treated nonlinearly –the nonlinear connections of the translations being identified with the coframes–, whereas the general linear group plays the role of the classification subgroup. In addition, a metric tensor involving ten degrees of freedom is included. We have proven that this theory, metric tensor included, can be obtained from a more fundamental approach to gravity as a degenerate case, where ordinary fermionic matter is absent.

In fact, the choice of the Lorentz group as the classification subgroup, in a nonlinear realization, leads to a theory in which ordinary Dirac fields can be included. The essential difference between our approach and the standard metric–affine theories consists in the transformation properties of the physical fields, which behave as representations of the Lorentz group, and in that the symmetric part of the connection is strictly proportional to the nonmetricity, transforming as a tensor. Nothing has to be changed in the standard metric–affine formalism but the choice of the metric, and of course the interpretation. The metric tensor is taken as the Minkowski metric. Since it is invariant under Lorentz transformations, it is globally fixed once and forever, so that space and time are everywhere well defined as distinguished quantities. Being a constant tensor, the metric has not a dynamical meaning any more. The geometry is determined by the interactions, and in the absence of gravity, the geometrical background simplifies to a flat spacetime. But the effect of gravitation is, strictly speaking, affine more than metric. The whole information about the gravitational interactions is contained in the connections, as in the gauge theories of internal groups.

Let us finally point out that, having the Lorentz group natural invariants, this is also the case for the affine group when realized nonlinearly with the Lorentz group as the classification subgroup. In the context of MAGT's of gravitation, McCrea⁽¹²⁾ calculated the irreducible decomposition of the curvature, the torsion and the nonmetricity with respect to the Lorentz group. The utility of his work to construct all possible invariants becomes evident in view of the nonlinear realization presented here, since it enables us to work with an affine geometrical formalism whose objects are defined on representation spaces of the Lorentz group.

APPENDIX

A.– Let us show how the expressions (4.5) and (4.7) were deduced (cfr. Ref.(8)). We apply the general formula (3.2) defining the nonlinear action, namely

$$g c(\xi) = c(\xi') h(\xi, g), \quad (A.1)$$

with the particular choices

$$g = e^{i\epsilon^\alpha P_\alpha} e^{i\alpha^{\mu\nu} S_{\mu\nu}} e^{i\beta^{\mu\nu} L_{\mu\nu}}, \quad c := e^{-i\xi^\alpha P_\alpha} e^{i h^{\mu\nu} S_{\mu\nu}}, \quad h := e^{i u^{\mu\nu} L_{\mu\nu}}, \quad (A.2)$$

as given in (4.2–4). The equation we have to solve then reads

$$e^{i\epsilon^\alpha P_\alpha} e^{i\alpha^{\mu\nu} S_{\mu\nu}} e^{i\beta^{\mu\nu} L_{\mu\nu}} e^{-i\xi^\alpha P_\alpha} e^{ih^{\mu\nu} S_{\mu\nu}} = e^{-i\xi'^\alpha P_\alpha} e^{ih'^{\mu\nu} S_{\mu\nu}} e^{iu^{\mu\nu} L_{\mu\nu}}. \quad (\text{A.3})$$

We consider an affine transformation with infinitesimal group parameters ϵ^α , $\alpha^{\mu\nu}$ and $\beta^{\mu\nu}$. Thus, the transformed coset parameters in the r.h.s. of (A.3) reduce to $\xi'^\alpha = \xi^\alpha + \delta\xi^\alpha$ and $h'^{\mu\nu} = h^{\mu\nu} + \delta h^{\mu\nu}$, and $u^{\mu\nu}$ is also infinitesimal. Consequently, eq.(A.3) reduces to

$$\begin{aligned} e^{i\xi^\alpha P_\alpha} (1 + i\epsilon^\alpha P_\alpha + i\alpha^{\mu\nu} S_{\mu\nu} + i\beta^{\mu\nu} L_{\mu\nu}) e^{-i\xi^\alpha P_\alpha} \\ = (1 - i\delta\xi^\alpha P_\alpha) e^{i(h^{\mu\nu} + \delta h^{\mu\nu}) S_{\mu\nu}} (1 + iu^{\mu\nu} L_{\mu\nu}) e^{-ih^{\mu\nu} S_{\mu\nu}}. \end{aligned} \quad (\text{A.4})$$

In order to simplify the calculations, we make use of the generators of the whole linear transformations $\Lambda^\alpha_\beta := L^\alpha_\beta + S^\alpha_\beta$, in terms of which, the commutation relations (4.1) reduce to

$$\begin{aligned} [\Lambda^\alpha_\beta, \Lambda^\mu_\nu] &= i \left(\delta^\alpha_\nu \Lambda^\mu_\beta - \delta^\mu_\beta \Lambda^\alpha_\nu \right), \\ [\Lambda^\alpha_\beta, P_\mu] &= i \delta^\alpha_\mu P_\beta, \\ [P_\alpha, P_\beta] &= 0. \end{aligned} \quad (\text{A.5})$$

Making then use of the Campell–Hausdorff formula, we find the algebraic relations

$$e^{i\xi^\mu P_\mu} \omega_\alpha{}^\beta \Lambda^\alpha_\beta e^{-i\xi^\mu P_\mu} = \omega_\alpha{}^\beta \Lambda^\alpha_\beta + \omega_\alpha{}^\beta \xi^\alpha P_\beta, \quad (\text{A.6})$$

and

$$e^{i\zeta_\mu{}^\nu \Lambda^\mu_\nu} \kappa_\alpha{}^\beta \Lambda^\alpha_\beta e^{-i\zeta_\mu{}^\nu \Lambda^\mu_\nu} = e^{\zeta_\alpha{}^\mu} \kappa_\mu{}^\nu e^{-\zeta_\nu{}^\beta} \Lambda^\alpha_\beta. \quad (\text{A.7})$$

Applying (A.6) to the l.h.s. of (A.4), it simplifies to

$$1 + i \left[\epsilon^\alpha + (\alpha_\beta{}^\alpha + \beta_\beta{}^\alpha) \xi^\beta \right] P_\alpha + i\alpha^{\mu\nu} S_{\mu\nu} + i\beta^{\mu\nu} L_{\mu\nu}, \quad (\text{A.8})$$

whereas in the r.h.s. we have to make use of the fact that

$$\begin{aligned} e^{i(h^{\mu\nu} + \delta h^{\mu\nu}) S_{\mu\nu}} &= e^{ih^{\mu\nu} S_{\mu\nu}} \left(1 + e^{-ih^{\mu\nu} S_{\mu\nu}} \delta e^{ih^{\mu\nu} S_{\mu\nu}} \right) \\ &= e^{ih^{\mu\nu} S_{\mu\nu}} \left[1 + i e^{-h^\alpha{}_\gamma} \delta e^{h^\gamma{}_\beta} (S_{\alpha\beta} + L_{\alpha\beta}) \right], \end{aligned} \quad (\text{A.9})$$

in order to apply (A.7) particularized to the cases

$$e^{ih^{\mu\nu} S_{\mu\nu}} \sigma^{\alpha\beta} S_{\alpha\beta} e^{-ih^{\mu\nu} S_{\mu\nu}} = e^{h^\alpha{}_\mu} \sigma^{\mu\nu} e^{-h_\nu{}^\beta} (S_{\alpha\beta} + L_{\alpha\beta}), \quad (\text{A.10})$$

and

$$e^{ih^{\mu\nu} S_{\mu\nu}} \tau^{\alpha\beta} L_{\alpha\beta} e^{-ih^{\mu\nu} S_{\mu\nu}} = e^{h^\alpha{}_\mu} \tau^{\mu\nu} e^{-h_\nu{}^\beta} (S_{\alpha\beta} + L_{\alpha\beta}). \quad (\text{A.11})$$

Then, for ξ^α and $r^{\alpha\beta} := e^{h^{\alpha\beta}}$, it follows

$$\delta\xi^\alpha = -(\alpha_\beta{}^\alpha + \beta_\beta{}^\alpha) \xi^\beta - \epsilon^\alpha, \quad (A.12)$$

and

$$\delta r^{\alpha\beta} = (\alpha^\alpha{}_\gamma + \beta^\alpha{}_\gamma) r^{\gamma\beta} + u^\beta{}_\gamma r^{\gamma\alpha}. \quad (A.13)$$

Q.E.D.

B.—On the other hand, the nonlinear Lorentz parameter (4.8) is calculated as follows. First we rewrite the symmetry condition of (A.13), namely

$$u^{[\alpha}{}_\gamma r^{\beta]\gamma} = \left(\alpha^{[\alpha}{}_\gamma + \beta^{[\alpha}{}_\gamma \right) r^{\beta]\gamma}, \quad (B.1)$$

in the form

$$(u^{\alpha\beta} - \beta^{\alpha\beta}) \left[\delta_\alpha^\lambda \delta_\beta^\rho + (r^{-1})^{[\lambda}{}_\alpha r^{\rho]\beta} \right] = \alpha^{\mu\nu} (r^{-1})^{[\lambda}{}_\mu r^{\rho]\nu}, \quad (B.2)$$

and we resolve it formally as

$$u^{\alpha\beta} = \beta^{\alpha\beta} + \alpha^{\mu\nu} (r^{-1})^{[\lambda}{}_\mu r^{\rho]\nu} \left[\delta_\alpha^\lambda \delta_\beta^\rho + (r^{-1})^{[\lambda}{}_\alpha r^{\rho]\beta} \right]^{-1}. \quad (B.3)$$

Making then use of the fact that

$$(r^{-1})^{[\lambda}{}_\mu r^{\rho]\nu} = \frac{1}{2} \left[e^{(h^\rho{}_\nu - h^\lambda{}_\mu)} - e^{-(h^\rho{}_\mu - h^\lambda{}_\nu)} \right], \quad (B.4)$$

we recognize that

$$\begin{aligned} u^{\alpha\beta} &= \beta^{\alpha\beta} - \alpha^{\mu\nu} \left\{ \sinh(h^\lambda{}_\mu - h^\rho{}_\nu) \left[\delta_\alpha^\lambda \delta_\beta^\rho + \cosh(h^\lambda{}_\alpha - h^\rho{}_\beta) \right]^{-1} \right\} \\ &= \beta^{\alpha\beta} - \alpha^{\mu\nu} \tanh\left(\frac{h^\alpha{}_\mu - h^\beta{}_\nu}{2}\right) \\ &= \beta^{\alpha\beta} - \alpha^{\mu\nu} \tanh\left\{ \frac{1}{2} \log \left[r^\alpha{}_\mu (r^{-1})^\beta{}_\nu \right] \right\}. \end{aligned} \quad (B.5)$$

Q.E.D.

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