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Ecole Normale Supérieure
de Lyon

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Unitary Group Representations in Fock Spaces with Generalized Exchange Properties

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Abstract

We investigate the notion of second R -quantization – a suitable deformation of the standard second quantization which properly takes into account the non-trivial exchange properties characterizing generalized statistics. We also perform the R -quantization of a class of unitary one-particle representations relevant for the description of symmetries. The Euclidean covariance of anyons is analyzed in this context.

ENSLAPP-A-490/94
IFUP-TH 51/94
September 1994

¹ URA 14-36 du CNRS, associée à l'E.N.S. de Lyon et au L.A.P.P.

1. Introduction

The present work pursues further our analysis of the (associative) algebra $\mathcal{A}_R$, whose generators $\{a(x), a^*(x) : x \in \mathbf{R}^s\}$ satisfy the quadratic relations

$$\begin{aligned} a(x_1) a(x_2) - R(x_2, x_1) a(x_2) a(x_1) &= 0 \quad , \\ a^*(x_1) a^*(x_2) - R(x_2, x_1) a^*(x_2) a^*(x_1) &= 0 \quad , \\ a(x_1) a^*(x_2) - R(x_1, x_2) a^*(x_2) a(x_1) &= \delta(x_1 - x_2) \quad . \end{aligned} \quad (1.1)$$

The exchange factor R is in general a measurable function on $\mathbf{R}^s \times \mathbf{R}^s$, obeying the consistency conditions

$$R(x_1, x_2) R(x_2, x_1) = 1 \quad , \quad (1.2)$$

$$\overline{R}(x_1, x_2) = R(x_2, x_1) \quad , \quad (1.3)$$

where the bar stands for complex conjugation. For $R = \pm 1$ one has the familiar algebra of canonical (anti)commutation relations $\mathcal{A}_\pm$.

Let us denote by $\mathcal{E}$ the set of all admissible exchange factors. For any $R \in \mathcal{E}$ we have explicitly constructed in [1,2] a Fock representation $\mathcal{F}_R$ of $\mathcal{A}_R$. Here we concentrate on the realization of certain unitary group representations on $\mathcal{F}_R$, which find application for describing the symmetries of the quantum systems associated with $\mathcal{A}_R$.

To the end of this section we recall some basic structures introduced in [1,2] and needed also in this paper. Setting $\mathcal{H}^0 = \mathbf{C}^1$ and $\mathcal{H}^n = L^2(\mathbf{R}^s)^{\otimes n}$, the Fock space over $L^2(\mathbf{R}^s)$ is given by (see e.g. [3])

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n \quad . \quad (1.4)$$

The standard Fock annihilation and creation operators (see e.g. [4]) are denoted by $\{b(f) : f \in L^2(\mathbf{R}^s)\}$ and $\{b^*(f) : f \in L^2(\mathbf{R}^s)\}$ respectively. One has

$$b^\sharp(f) : \mathcal{F}^0(\mathcal{H}) \longrightarrow \mathcal{F}^0(\mathcal{H}) \quad , \quad (1.5)$$

where $b^\sharp$ stands for b or b^* and $\mathcal{F}^0(\mathcal{H})$ is the finite particle subspace of $\mathcal{F}(\mathcal{H})$.

Let $R \in \mathcal{E}$. For $n \geq 2$ we introduce the operators $\{ S_i : i = 1, \dots, n-1 \}$ which act in $\mathcal{H}^n$ according to

$$[S_i \varphi](x_1, \dots, x_i, x_{i+1}, \dots, x_n) = R(x_i, x_{i+1}) \varphi(x_1, \dots, x_{i+1}, x_i, \dots, x_n) \quad . \quad (1.6)$$

From eqs.(1.2,3) it follows that:

- (i) $\{ S_i : i = 1, \dots, n-1 \}$ are bounded ($\|S_i\| = 1$) Hermitian operators on $\mathcal{H}^n$ and the mapping $S : \sigma_i \mapsto S_i$ is a representation of the permutation group $\mathcal{P}_n$ in $\mathcal{H}^n$;
- (ii) The operators

$$P_R^{(n)} = \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} S(\sigma) \quad , \quad n \geq 2 \quad , \quad (1.7)$$

are orthogonal projectors.

By means of $P_R^{(n)}$ we define the R -subspace $\mathcal{H}_R^n = P_R^{(n)} \mathcal{H}^n$ of $\mathcal{H}^n$ and, setting $\mathcal{H}_R^0 = \mathbb{C}^1$ and $\mathcal{H}_R^1 = \mathcal{H}^1 = L^2(\mathbf{R}^s)$, we construct the R -Fock space

$$\mathcal{F}_R(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_R^n \quad . \quad (1.8)$$

Let P_R be the orthogonal projection operator on $\mathcal{F}_R(\mathcal{H}) \subset \mathcal{F}(\mathcal{H})$ and let us introduce

$$a^\sharp(f) = P_R b^\sharp(f) P_R \quad . \quad (1.9)$$

By construction $a^\sharp(f)$ are densely defined (with domain $\mathcal{F}^0(\mathcal{H})$) linear operators, which satisfy

$$(\psi, a^*(f)\varphi) = (a(f)\psi, \varphi) \quad . \quad (1.10)$$

Moreover, $a^\sharp(f)$ are closable and map $\mathcal{F}^0(\mathcal{H})$ in the finite particle R -subspace $\mathcal{F}_R^0(\mathcal{H}) = P_R \mathcal{F}^0(\mathcal{H})$. In particular, $a^\sharp(f)$ leave invariant $\mathcal{F}_R^0(\mathcal{H})$, the explicit action being

$$[a(f)\varphi]^{(n)}(x_1, \dots, x_n) = \sqrt{n+1} \int d^s x \bar{f}(x) \varphi^{(n+1)}(x, x_1, \dots, x_n) \quad , \quad (1.11)$$

$$[a^*(f)\varphi]^{(n)}(x_1, \dots, x_n) = \frac{1}{\sqrt{n}} f(x_1) \varphi^{(n-1)}(x_2, \dots, x_n) +$$

$$\frac{1}{\sqrt{n}} \sum_{k=2}^n R(x_{k-1}, x_k) \dots R(x_1, x_k) f(x_k) \varphi^{(n-1)}(x_1, \dots, \widehat{x}_k, \dots, x_n) \quad , \quad (1.12)$$

where $\varphi \in \mathcal{F}_R^0(\mathcal{H})$ and $\widehat{x}$ indicates that the argument x must be omitted. One can directly verify that the operator-valued distributions $a^\sharp(x)$, defined by

$$a(f) = \int d^s x \bar{f}(x) a(x) \quad , \quad a^*(f) = \int d^s x f(x) a^*(x) \quad ,$$

satisfy (1.1). This is the Fock representation $\mathcal{F}_R$ of $\mathcal{A}_R$, which has been constructed in [1,2]. In particular, $\mathcal{F}_\pm$ corresponding to $R = \pm 1$ are the well known Fock representations of $\mathcal{A}_\pm$.

The notion of second R -quantization, developed in [1,2], plays a central role in what follows. Let A be a densely defined operator with domain $D \subset \mathcal{H}$. The second quantization $d\Gamma(A)$ of A is defined on the dense subset

$$D(A) = \{ \varphi \in \mathcal{F}^0(\mathcal{H}) : \varphi^{(n)} \in D^{\otimes n} \text{ for } n \geq 1 \} \subset \mathcal{F}(\mathcal{H})$$

in the standard way [4]:

$$[d\Gamma(A)\varphi]^{(0)} = 0 \quad ,$$

$$[d\Gamma(A)\varphi]^{(n)} = (A \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1} + \mathbf{1} \otimes A \otimes \dots \otimes \mathbf{1} + \dots + \mathbf{1} \otimes \mathbf{1} \otimes \dots \otimes A) \varphi^{(n)} \quad . \quad (1.13)$$

Provided that

$$P_R D(A) \subset D(A) \quad , \quad (1.14)$$

the operator

$$d\Gamma_R(A) = P_R d\Gamma(A) P_R \quad , \quad (1.15)$$

is well defined on the dense subset $D_R(A) = P_R D(A)$ of $\mathcal{F}_R(\mathcal{H})$ and is called the second R -quantization of A . Eq.(1.15) can be rewritten in the form

$$d\Gamma_R(A) = d\Gamma(A) \Big|_{D_R(A)} + [P_R, d\Gamma(A)] \Big|_{D_R(A)} \quad ,$$

which shows that the second R -quantization $d\Gamma_R$ is a non-trivial deformation of $d\Gamma$ when $[P_R, d\Gamma(A)] \neq 0$ on $D_R(A)$.

Now we are in the position to formulate more precisely the main subject of this letter. Let $V^{(1)}$ be a unitary representation of a continuous group G in $\mathcal{H}$, whose generators are X_i . Assuming that the domains $D(X_i)$ respect the condition (1.14), our aim will be to construct a unitary representation $\mathcal{U}_R$ of G in $\mathcal{F}_R(\mathcal{H})$, whose generators are $d\Gamma_R(X_i)$. In what follows we refer to $\mathcal{U}_R$ as the second R -quantization of the one-particle representation $V^{(1)}$. We have considered previously [1] the R -quantization of the time evolution $V^{(1)}(t)$ associated to the free one-particle hamiltonian – the laplacian Δ in $\mathbf{R}^s$. So, we know already that even in this relatively simple case, $\mathcal{U}_R(t)$ is highly complicated and actually not known for arbitrary $R \in \mathcal{E}$. For this reason we do not address below to the R -quantization of a generic $V^{(1)}$, but concentrate on one-particle representations generated by first-order differential operators. Such representations are important in the physical context because they implement in $\mathcal{H}$ groups of smooth transformations in $\mathbf{R}^s$. With the above assumption on $V^{(1)}$ we are going to show in sect.2 that $\mathcal{U}_R$ is unitary equivalent to some well-known representations in $\mathcal{F}_+(\mathcal{H})$ or $\mathcal{F}_-(\mathcal{H})$, provided that R is differentiable. Because of domain problems related essentially to the condition (1.14), this statement is in general not correct for non-differentiable R -factors. Since the last case is interesting from the physical point of view, we shall illustrate its fundamental aspects in sect.3 on the example of anyons.

We conclude this preliminary part of the paper by some technical remarks about the family $\{\mathcal{F}_R(\mathcal{H}) : R \in \mathcal{E}\}$ of subspaces of the total Fock space $\mathcal{F}(\mathcal{H})$. Being infinite dimensional separable Hilbert spaces, all elements of this family are isomorphic. In [1,2] we have described some particular isomorphisms, which we would like to recall because they will be essential also in what follows. Consider the functional equation

$$T_{\pm}(x_1, x_2) R(x_1, x_2) = \pm T_{\pm}(x_2, x_1) \quad (1.16)$$

with the supplementary condition

$$\bar{T}_{\pm}(x_1, x_2) T_{\pm}(x_1, x_2) = 1 \quad . \quad (1.17)$$

Eqs.(1.16,17) have solution [1,2] for any $R \in \mathcal{E}$ and one can define the operators

$$\left[U_{\pm}^{(n)}(R) \varphi \right] (x_1, \dots, x_n) = \left[\prod_{\substack{i,j=1 \\ i < j}}^n T_{\pm}(x_i, x_j) \right] \varphi(x_1, \dots, x_n) \quad , \quad n \geq 2 \quad . \quad (1.18)$$

By construction $U_{\pm}^{(n)}(R)$ are unitary operators on $\mathcal{H}^n$ which, setting $U_{\pm}^{(0)}(R) = \mathbf{1}$ and $U_{\pm}^{(1)}(R) = \mathbf{1}$, extend to unitary operators $U_{\pm}(R)$ on $\mathcal{F}(\mathcal{H})$. Employing eq.(1.16) one can verify that

$$U_{\pm}(R) \mathcal{F}_R(\mathcal{H}) = \mathcal{F}_{\pm}(\mathcal{H}) \quad . \quad (1.19)$$

Therefore the compositions $U_{\pm}(R_1)^{-1} U_{\pm}(R_2)$ provide two isomorphisms between $\mathcal{F}_{R_2}(\mathcal{H})$ and $\mathcal{F}_{R_1}(\mathcal{H})$. We would like to mention also that $U_{\pm}(R)$ satisfy on $\mathcal{F}(\mathcal{H})$ the identities

$$U_{\pm}(R) P_R = P_{\pm} U_{\pm}(R) \quad , \quad (1.20)$$

$P_{\pm}$ being the projectors on $\mathcal{F}_{\pm}(\mathcal{H})$.

Finally, it is useful to observe that any $R \in \mathcal{E}$ admits the parametrization

$$R(x_1, x_2) = \exp[ir(x_1, x_2)] \quad , \quad (1.21)$$

where r is real-valued and obeys

$$r(x_1, x_2) + r(x_2, x_1) = 2\pi k_r \quad , \quad k_r \in \mathbf{Z} \quad . \quad (1.22)$$

In terms of $r(x_1, x_2)$, the simplest solution of eqs.(1.16,17) reads

$$\exp \left[-\frac{i}{2} r(x_1, x_2) \right] = \begin{cases} T_+(x_1, x_2) & \text{if } k_r \in 2\mathbf{Z} ; \\ T_-(x_1, x_2) & \text{if } k_r \in 2\mathbf{Z} + 1 . \end{cases} \quad (1.23)$$

2. Unitary Representations in $\mathcal{F}_R(\mathcal{H})$ from R -Quantization

Our strategy in this section is as follows. We shall start with the familiar construction of unitary group representations $V_{\pm}$ obtained by standard second quantization in

$\mathcal{F}_\pm(\mathcal{H})$. By means of the operators $U_\pm(R)$ we shall then transfer $V_\pm$ to a representation $\mathcal{V}_R$ in $\mathcal{F}_R(\mathcal{H})$, arriving in this way to the central point of our discussion – the relationship of $\mathcal{V}_R$ with the second R -quantization $\mathcal{U}_R$ described in the introduction.

Let us consider a group G and a unitary one-particle representation $V^{(1)}$ of G in $\mathcal{H}^1$. We set

$$V^{(0)}(g) = \mathbf{1} \quad , \quad V^{(n)}(g) = \bigotimes_{k=1}^n V^{(1)}(g) \quad . \quad (2.1)$$

Moreover, we define $V(g)$ to be the unitary operator on $\mathcal{F}(\mathcal{H})$ which equals $V^{(n)}(g)$ when restricted to $\mathcal{H}^n$. Obviously $\{V(g) : g \in G\}$ is a unitary representation of G in $\mathcal{F}(\mathcal{H})$, which leaves invariant both $\mathcal{F}_\pm(\mathcal{H})$. We denote by $V_\pm$ the restrictions of V to $\mathcal{F}_\pm(\mathcal{H})$; using eq.(1.19), one has that the operators

$$\mathcal{V}_R(g) = \begin{cases} U_+(R)^{-1} V_+(g) U_+(R) & \text{if } k_r \in 2\mathbf{Z} ; \\ U_-(R)^{-1} V_-(g) U_-(R) & \text{if } k_r \in 2\mathbf{Z} + 1 , \end{cases} \quad (2.2)$$

leave invariant $\mathcal{F}_R(\mathcal{H})$, thus defining a unitary representation $\{\mathcal{V}_R(g) : g \in G\}$ of G in $\mathcal{F}_R(\mathcal{H})$. Notice that if G is a topological group, the continuity of the mapping $g \mapsto V^{(1)}(g)$ in the strong operator topology implies that the representation $\mathcal{V}_R$ is strongly continuous as well.

We now specify G , considering a topological group G with smooth (C^∞) action

$$g : x \longmapsto gx \quad , \quad (gx)^\mu = g^\mu(x) \quad , \quad \mu = 1, \dots, s \quad , \quad (2.3)$$

in $\mathbf{R}^s$. This case is interesting from the physical point of view because the continuous space symmetries of any quantum system associated with $\mathcal{A}_R$ are of this type. As a one-particle representation $V^{(1)}$ of G in $L^2(\mathbf{R}^s)$ we take

$$[V^{(1)}(g)\varphi](x) = u(g) \varphi(g^{-1}x) |\det \mathcal{J}[g^{-1}](x)|^{\frac{1}{2}} \quad . \quad (2.4)$$

Here u is a continuous one-dimensional unitary representation of G and $\mathcal{J}[g](x)$ denotes the Jacobian matrix of the mapping g at the point x . It is easily seen that $V^{(1)}$ is unitary and strongly continuous. Substituting eqs.(1.18) and (2.4) in eq.(2.2), one derives the explicit action of $\mathcal{V}_R(g)$ in $\mathcal{F}_R(\mathcal{H})$; for any $\varphi \in \mathcal{F}_R(\mathcal{H})$ one has

$$[\mathcal{V}_R(g)\varphi]^{(n)}(x_1, \dots, x_n) = \left[\prod_{\substack{i,j=1 \\ i < j}}^n T_{\pm}(g^{-1}x_i, g^{-1}x_j) \bar{T}_{\pm}(x_i, x_j) \right] u(g)^n \varphi^{(n)}(g^{-1}x_1, \dots, g^{-1}x_n) \prod_{k=1}^n |\det \mathcal{J}[g^{-1}](x_k)|^{\frac{1}{2}}, \quad (2.5)$$

where $+$ and $-$ correspond to $k_r \in 2\mathbb{Z}$ and $k_r \in 2\mathbb{Z} + 1$ respectively.

It is instructive to derive the transformation properties of $a^{\sharp}(x)$ under $\mathcal{V}_R(g)$. First of all we introduce the operators $\{\mathcal{S}^{(n)}(g; x) : n \geq 1\}$ acting in $\mathcal{H}^n$ according to

$$\left[\mathcal{S}^{(n)}(g; x) \varphi^{(n)} \right] (x_1, \dots, x_n) = \left[\prod_{i=1}^n T_{\pm}(x, x_i) \bar{T}_{\pm}(g^{-1}x, g^{-1}x_i) \right] \varphi(x_1, \dots, x_n) \quad . \quad (2.6)$$

Because of eq.(1.17), $\mathcal{S}^{(n)}(g; x)$ are unitary operators which, setting $\mathcal{S}^{(0)}(g; x) = 1$, extend to unitary operators $\mathcal{S}(g; x)$ on $\mathcal{F}(\mathcal{H})$. By construction $\mathcal{S}(g; x)$ leave invariant $\mathcal{F}_R^0(\mathcal{H})$ and it is easy to prove the following

Proposition 1: The operators $a^{\sharp}(x)$ have on $\mathcal{F}_R^0(\mathcal{H})$ the transformation properties:

$$\mathcal{V}_R(g) a(x) \mathcal{V}_R(g)^{-1} = |\det \mathcal{J}[g](x)|^{\frac{1}{2}} \mathcal{S}(g; gx) u(g)^{-1} a(gx) \quad , \quad (2.7)$$

$$\mathcal{V}_R(g) a^*(x) \mathcal{V}_R(g)^{-1} = |\det \mathcal{J}[g](x)|^{\frac{1}{2}} a^*(gx) u(g) \mathcal{S}(g; gx)^{-1} \quad . \quad (2.8)$$

The proof is straightforward and relies on eqs.(1.11,12) and eqs.(2.5,6).

Eqs.(2.7,8) express a sort of generalized G -covariance of $a^{\sharp}(x)$. One should keep in mind that in general $\mathcal{S}(g; x)$ and $a^{\sharp}(x)$ do not commute and that $\mathcal{S}(g; x)$ is not a representation of G because $\mathcal{S}(g_1; x)\mathcal{S}(g_2; x) \neq \mathcal{S}(g_1 g_2; x)$. In order to better clarify the role of $\mathcal{S}(g; x)$, it is instructive to look at the behavior of the correlation functions

$$w_n(x_1, \dots, x_n; y_1, \dots, y_n) = (a^*(x_1) \cdots a^*(x_n) \Omega, a^*(y_1) \cdots a^*(y_n) \Omega) \quad , \quad (2.9)$$

$$\Omega = (1, 0, \dots, 0, \dots) \in \mathcal{H}^0 \quad ,$$

under the G -transformations implemented by $\mathcal{V}_R$. The computation shows that the operators $\mathcal{S}(g; x)$ precisely compensate the G -variation of the exchange factor R , thus

ensuring the G -invariance of w_n . This feature of the correlation functions could be expected on general grounds. Notice that when $T_{\pm}$ (and consequently R) are G -invariant, $S(g; x) = 1$ and eqs.(2.7,8) take the familiar form.

Now we come to the main point of this section. As already explained, we are mainly interested in the second R -quantization $\mathcal{U}_R$ of $V^{(1)}$ defined by eq.(2.4). In this respect we are going to show below that

$$\mathcal{U}_R = \mathcal{V}_R \quad , \quad (2.10)$$

provided that R is differentiable. This statement can be equivalently reformulated in terms of generators. Let $\{g_{\tau} : \tau \in \mathbf{R}, g_0 = e\}$ be any strongly continuous one-parameter subgroup of G . According to a well known theorem by Stone (see e.g. [3]), there exist a unique densely defined self-adjoint operator X in $L^2(\mathbf{R}^s)$ such that

$$V^{(1)}(g_{\tau}) = \exp(-i\tau X) \quad . \quad (2.11)$$

Applying the same theorem to $\mathcal{V}_R(g_{\tau})$, one can deduce the existence of a unique self-adjoint operator X_R defined on a dense subspace of $\mathcal{F}_R(\mathcal{H})$ and satisfying

$$\mathcal{V}_R(g_{\tau}) = \exp(-i\tau X_R) \quad . \quad (2.12)$$

The point is to demonstrate that it is precisely the second R -quantization (1.15) which relates the generators X and X_R .

As a preliminary step towards the proof of the last statement, it is convenient to establish first a useful general result concerning the R -quantization of first order linear differential operators of the form

$$A = a^{\mu}(x)\partial_{\mu} + b(x) \quad , \quad (2.13)$$

where the summation over repeated upper and lower indices is hereafter always understood. Any such operator is defined on the dense subspace of differentiable functions with compact support $C_0^1(\mathbf{R}^s) \subset L^2(\mathbf{R}^s)$. Accordingly, the domain of $d\Gamma(A)$ is $D(C_0^1)$, where we define in general

$$D(C_0^k) = \{ \varphi \in \mathcal{F}^0(\mathcal{H}) : \varphi^{(n)} \in C_0^k(\mathbf{R}^s)^{\otimes n} \text{ for } n \geq 1 \} \quad . \quad (2.14)$$

Let us adopt the parametrization (1.21) and call $R \in \mathcal{E}$ differentiable if $r \in C^1(\mathbf{R}^s \times \mathbf{R}^s)$. The condition (1.14) is satisfied for any differentiable exchange factor and $d\Gamma_R(A)$ is defined on the dense subset $D_R(C_0^1) = P_R D(C_0^1)$ of $\mathcal{F}_R(\mathcal{H})$. For any $\varphi \in D_R(C_0^1)$ one gets

$$[d\Gamma_R(A)\varphi]^{(n)}(x_1, \dots, x_n) = P_R^{(n)} [d\Gamma(A)\varphi]^{(n)}(x_1, \dots, x_n) = P_R^{(n)} \left[\sum_{i=1}^n a^\mu(x_i) \partial_\mu^i + b(x_i) \right] \varphi^{(n)}(x_1, \dots, x_n) \quad , \quad (2.15)$$

where

$$\partial_\mu^i = \frac{\partial}{\partial x_i^\mu} \quad .$$

At this point, using the explicit form of the projector $P_R^{(n)}$ given by eqs.(1.6,7), we can evaluate the commutator of $P_R^{(n)}$ with the square brackets in (2.15); taking into account that $P_R^{(n)}\varphi^{(n)} = \varphi^{(n)}$, the result of this operation is

$$[d\Gamma_R(A)\varphi]^{(n)}(x_1, \dots, x_n) = \left[\sum_{i=1}^n a^\mu(x_i) \partial_\mu^i + b(x_i) \right] \varphi^{(n)}(x_1, \dots, x_n) - \frac{i}{2} \left[\sum_{\substack{i,j=1 \\ i < j}}^n a^\mu(x_i) (\partial_\mu^i r)(x_i, x_j) + a^\mu(x_j) (\partial_\mu^j r)(x_i, x_j) \right] \varphi^{(n)}(x_1, \dots, x_n) \quad . \quad (2.16)$$

We introduce furthermore the operator

$$A_R = \begin{cases} U_+(R)^{-1} d\Gamma(A) U_+(R) & \text{if } k_r \in 2\mathbf{Z} \ , \\ U_-(R)^{-1} d\Gamma(A) U_-(R) & \text{if } k_r \in 2\mathbf{Z} + 1 \ , \end{cases} \quad (2.17)$$

where $U_\pm(R)$ are obtained by inserting eq.(1.23) in (1.18). By construction A_R is a well defined operator on $D_R(C_0^1)$, which can be written in the form

$$A_R = d\Gamma(A) + \begin{cases} U_+(R)^{-1} [d\Gamma(A), U_+(R)] & \text{if } k_r \in 2\mathbf{Z} \ , \\ U_-(R)^{-1} [d\Gamma(A), U_-(R)] & \text{if } k_r \in 2\mathbf{Z} + 1 \ . \end{cases} \quad (2.18)$$

Clearly, when applied to $\varphi \in D_R(C_0^1)$, the first term in the right hand side of eq.(2.18) reproduces the first term in the right hand side of eq.(2.16). The commutator in eq.(2.18) can also be evaluated explicitly and gives precisely the second term in the right hand side of eq.(2.16). Therefore one has

Proposition 2: The equality

$$d\Gamma_R(A) = A_R \quad (2.19)$$

holds on $D_R(C_0^1)$ for any first order linear differential operator A , provided that $R \in \mathcal{E}$ is differentiable.

We stress that the requirement A to be a first order differential operator is essential; it is easily seen that in general the above statement is false for second order operators even for smooth $R \in \mathcal{E}$. One concrete example is provided by the laplacian Δ , whose R -quantization is discussed in [1].

Let us come back now to our original problem – the relationship between the generators X and X_R . From eq.(2.4) one derives

$$[\exp(-i\tau X)\varphi](x) = \exp(-i\tau\xi) \varphi(g_\tau^{-1}x) |\det \mathcal{J}[g_\tau^{-1}](x)|^{\frac{1}{2}} \quad , \quad \xi \in \mathbf{R} \quad . \quad (2.20)$$

Applying theorem VIII.7 of reference [3] to the one-parameter unitary group (2.20) one concludes that $C_0^1(\mathbf{R}^s)$ belongs to the domain of X . The explicit action of X in $C_0^1(\mathbf{R}^s)$ can be obtained by taking the derivative of both sides of eq.(2.16) with respect to τ and setting afterwards $\tau = 0$. For any $\varphi \in C_0^1(\mathbf{R}^s)$ one finds

$$[X\varphi](x) = -i \left[v^\mu(x) \partial_\mu + \frac{1}{2} (\partial_\mu v^\mu)(x) + i\xi \right] \varphi(x) \quad , \quad (2.21)$$

where

$$v^\mu(x) = \left. \frac{d}{d\tau} g_\tau^\mu(x) \right|_{\tau=0} \quad . \quad (2.22)$$

Applied to the operator X , Proposition 2 leads to the following

Corollary: For any differentiable exchange factor $R \in \mathcal{E}$, the generator X_R of $\mathcal{V}_R$ coincides on $D_R(C_0^1)$ with $d\Gamma_R(X)$, X being the generator of $V^{(1)}$. If in addition $R \in C^\infty(\mathbf{R}^s \times \mathbf{R}^s)$, according to theorem VIII.10 of [3] the subspace $D_R(C_0^\infty) \subset D_R(C_0^1)$ is a domain of essential self-adjointness for $d\Gamma_R(X)$. The transformation laws (2.7,8) remain valid also after replacing $\mathcal{V}_R$ by $\mathcal{U}_R$.

Summarizing, we have proven the validity of eq.(2.10) under the assumption of differentiability of $R \in \mathcal{E}$. If R is not differentiable, the domain of essential self-adjointness $D(X)$ of $d\Gamma(X)$ may not satisfy the condition (1.14). The resolution of this problem leads in general to a representation $\mathcal{U}_R$, which is not equivalent to $\mathcal{V}_R$. We shall describe this phenomenon in the next section, discussing the covariance properties of the Leinaas-Myrheim anyon field under rotations.

We conclude this section with some concrete examples for G . In physical applications one most frequently encounters the group of diffeomorphisms $\text{Diff}(\mathbf{R}^s)$ and some of its subgroups. Among them we would like to mention the group of conformal transformations $\mathcal{C}(\mathbf{R}^s)$ and isometries $E(\mathbf{R}^s)$ relative to the flat metric $\delta_{\mu\nu}$. We recall [5] that the vector field (2.22), corresponding to a one parameter group of diffeomorphisms, generates a conformal transformation if there exists a function $\lambda(x)$ such that

$$\partial_\mu v_\nu(x) + \partial_\nu v_\mu(x) = \lambda(x)\delta_{\mu\nu} \quad , \quad (2.23)$$

where $v_\mu = \delta_{\mu\nu}v^\nu$. If $\lambda(x) = 0$, the vector field v^μ gives rise to an isometry. By definition one has $E(\mathbf{R}^s) \subset \mathcal{C}(\mathbf{R}^s) \subset \text{Diff}(\mathbf{R}^s)$. Usually $E(\mathbf{R}^s)$ is called the Euclidean group of $\mathbf{R}^s$; for Euclidean transformations eq.(2.21) simplifies, taking the form

$$[X\varphi](x) = -i v^\mu(x)\partial_\mu\varphi(x) \quad . \quad (2.24)$$

3. Euclidean Covariance of the Anyon Field

As already mentioned, the R -quantization of $V^{(1)}$ in the case of non-differentiable R -factor is not straightforward because of domain problems. In this section we illustrate some characteristic features of this case, analyzing the transformation properties of the Leinaas-Myrheim (L-M) anyon field [6] under Euclidean transformations.

Let us recall first some basic facts about the L-M anyons in our formalism. Consider a two dimensional space ($s = 2$), and let u be an arbitrary but fixed unit vector in $\mathbf{R}^2$.

The exchange factor relative to the L-M anyon field reads [1]

$$R(x_1, x_2; u) = \exp\left(-i\pi \vartheta \operatorname{sgn}[(x_1 - x_2)^\mu \tilde{u}_\mu]\right) \quad , \quad \vartheta \in \mathbf{R} \quad . \quad (3.1)$$

Here “sgn” is the sign function and $\tilde{u}_\mu = \varepsilon_{\mu\nu} u^\nu$ is the vector dual to u (with the convention $\varepsilon_{12} = 1$). It is not hard to recognize that the correlation functions of $a(x)$ and $a^*(x)$ satisfying (1.1) with the exchange factor (3.1) have just the symmetry properties that L-M required for the wave functions of their anyons; the u vector that appears here as an external parameter, occurred implicitly in their work as a choice of a reference frame for measuring angles. In related works u appears in the form of a Dirac string or of a logarithm cut and its presence is an unavoidable characteristic feature of anyons [7,8]. It is worth mentioning also that the exchange factor (3.1) is discontinuous on the line

$$\{x_1 = x_2 - \lambda u : \lambda \in \mathbf{R}\} \quad , \quad (3.2)$$

except when the *statistical parameter* ϑ is an integer, anyons becoming in that case bosons or fermions.

In the process of the R -quantization we will need the operator $U_+(R)$. Following [1], we use for its construction (see eq.(1.18)) the solution

$$T_+(x_1, x_2) = \exp[-i\vartheta \arg(x_1 - x_2; u)] \quad (3.3)$$

of eqs.(1.16,17). The function $\arg(x; u) \in [-\pi, \pi)$ in (3.3) is the oriented angle between x and u and by using the identity

$$\arg(x; u) - \arg(-x; u) = -\pi \operatorname{sgn}(x^\mu \tilde{u}_\mu) \quad , \quad (3.4)$$

one can immediately check that (3.3) satisfies eq.(1.16). Notice that T_+ is discontinuous not on the whole line (3.2), but only for $\lambda \geq 0$.

We will be concerned below with the R -quantization of the following one-particle representation

$$[V^{(1)}(\alpha, a)\varphi](x) = \exp(-i\xi\alpha) \varphi(g_\alpha^{-1}x - a) \quad , \quad (3.5)$$

where $\alpha, \xi \in \mathbf{R}$, $a \in \mathbf{R}^2$ and

$$g_\alpha x = (x^1 \cos \alpha - x^2 \sin \alpha, x^1 \sin \alpha + x^2 \cos \alpha) \quad . \quad (3.6)$$

Eqs.(3.5,6) define a unitary strongly continuous representation of the universal covering $\tilde{E}(\mathbf{R}^2)$ of the two-dimensional Euclidean group, the parameter ξ describing the spin of the one-particle states. In what follows the α -transformations are called rotations, but one has to keep in mind that we are dealing actually with the universal covering of the rotation group in $\mathbf{R}^2$. The operators generating (3.5) are

$$\mathcal{M} = -i\varepsilon_{\mu\nu}x^\mu\partial^\nu + \xi \quad , \quad \mathcal{P}^\mu = -i\partial^\mu \quad (3.7)$$

and have $C_0^\infty(\mathbf{R}^2)$ as a common domain of essential self-adjointness. For further use, we introduce the generators $d\Gamma_+(\mathcal{M}) = P_+d\Gamma(\mathcal{M})P_+$ and $d\Gamma_+(\mathcal{P}^\mu) = P_+d\Gamma(\mathcal{P}^\mu)P_+$ in the bosonic Fock space $\mathcal{F}_+(\mathcal{H})$. The operators $d\Gamma_+(\mathcal{M})$ and $d\Gamma_+(\mathcal{P}^\mu)$ are essentially self-adjoint on

$$D_+(C_0^\infty) = P_+D(C_0^\infty)$$

and generate a unitary representation $V_+(\alpha, a)$ of $\tilde{E}(\mathbf{R}^2)$ in $\mathcal{F}_+(\mathcal{H})$.

We turn now to the construction of $d\Gamma_R(\mathcal{M})$ and $d\Gamma_R(\mathcal{P}^\mu)$. Let A be the operator (2.13). For discontinuous R -factors the projector $P_R^{(n)}$ defined by eqs.(1.6,7) is not differentiable everywhere, which in general creates domain problems in the R -quantization of A . One way to overcome these problems for anyons is to observe that in this concrete case the set

$$\{x_i - x_j = \lambda u : \lambda \in \mathbf{R}, i, j = 1, \dots, n, i \neq j\} \quad ,$$

where differentiability is lost, has measure zero. Therefore, the derivative exists almost everywhere and for a suitable domain (depending on A) gives a square-integrable function. In particular, one can easily verify in this framework that the operators $d\Gamma_R(\mathcal{M})$ and $d\Gamma_R(\mathcal{P}^\mu)$ are well-defined on

$$\tilde{D}_R(C_0^\infty) = U_+(R)^{-1}D_+(C_0^\infty) \quad .$$

We shall justify the choice of this domain a few lines below, analyzing first the dependence on the statistical parameter ϑ . The ϑ -dependence is carried in part by the domain $\tilde{D}_R(C_0^\infty)$ and in part by the operator $d\Gamma_R(A)$, where from now on A denotes $\mathcal{M}$ or $\mathcal{P}^\mu$. In order to simplify the domain, instead of studying $d\Gamma_R(A)$ it is more convenient and, at the same time equivalent, to investigate the operator

$$\tilde{A}_+ = U_+(R) d\Gamma_R(A) U_+(R)^{-1} \quad (3.8)$$

on $D_+(C_0^\infty)$. Since $D_+(C_0^\infty)$ does not depend on ϑ , in this way the ϑ -dependence is entirely shifted to the operator $\tilde{A}_+$ and can be isolated in a manifest form. Indeed, taking into account the identity (1.20), we get

$$\tilde{A}_+\varphi = P_+ U_+(R) d\Gamma(A) U_+(R)^{-1} \varphi \quad (3.9)$$

for any $\varphi \in D_+(C_0^\infty)$. The right hand side of eq.(3.9) can be computed explicitly. Using that

$$\partial_\mu^i \arg(x_i - x_j; u) = -\varepsilon_{\mu\nu} \frac{(x_i - x_j)^\nu}{(x_i - x_j)^2} \quad (3.10)$$

holds almost everywhere, one finds

$$\begin{aligned} & [\tilde{A}_+\varphi]^{(n)}(x_1, \dots, x_n) = \\ & \left\{ \sum_{i=1}^n [a^\mu(x_i) \partial_\mu^i + b(x_i)] - i\vartheta \sum_{\substack{i,j=1 \\ i < j}}^n [a^\mu(x_i) - a^\mu(x_j)] \varepsilon_{\mu\nu} \frac{(x_i - x_j)^\nu}{(x_i - x_j)^2} \right\} \varphi^{(n)}(x_1, \dots, x_n) = \\ & [d\Gamma(A)\varphi]^{(n)}(x_1, \dots, x_n) - i\vartheta \varphi^{(n)}(x_1, \dots, x_n) \sum_{\substack{i,j=1 \\ i < j}}^n [a^\mu(x_i) - a^\mu(x_j)] \varepsilon_{\mu\nu} \frac{(x_i - x_j)^\nu}{(x_i - x_j)^2} \quad . \quad (3.11) \end{aligned}$$

The term proportional to ϑ in eq.(3.11) is the genuine new contribution coming from the R -quantization. Notice also that neither $\tilde{A}_+$ nor its domain depend on the external vector u . Consequently, the spectrum of $d\Gamma_R(A)$ is u -independent.

Inserting in eq.(3.11) $a^\mu(x)$ and $b(x)$ determined by eq.(3.7), one gets

$$[\tilde{\mathcal{P}}_+^\mu \varphi]^{(n)}(x_1, \dots, x_n) = -i \sum_{i=1}^n \partial_i^\mu \varphi^{(n)}(x_1, \dots, x_n) \quad , \quad (3.12)$$

$$[\tilde{\mathcal{M}}_+ \varphi]^{(n)}(x_1, \dots, x_n) = \left\{ \left[\sum_{i=1}^n (-i \varepsilon_\mu{}^\nu x_i^\mu \partial_\nu^i + \xi) \right] + \frac{\vartheta}{2} n(n-1) \right\} \varphi^{(n)}(x_1, \dots, x_n) \quad , \quad (3.13)$$

for any $\varphi \in D_+(C_0^\infty)$. From eqs.(3.12,13) it follows that $D_+(C_0^\infty)$ is a domain of essential self-adjointness of $\tilde{\mathcal{P}}_+^\mu$ and $\tilde{\mathcal{M}}_+$. This fact, together with the independence of the spectrum of $d\Gamma_R(A)$ on u , justify a posteriori the choice of the domain $\tilde{D}_R(C_0^\infty)$.

Comparing $\tilde{\mathcal{P}}_+^\mu$ and $\tilde{\mathcal{M}}_+$ to the generators $d\Gamma_+(\mathcal{P}^\mu)$ and $d\Gamma_+(\mathcal{M})$ of $V_+(\alpha, a)$, one has on $D_+(C_0^\infty)$

$$\tilde{\mathcal{P}}_+^\mu = d\Gamma_+(\mathcal{P}^\mu) \quad , \quad \tilde{\mathcal{M}}_+ = d\Gamma_+(\mathcal{M}) + \frac{\vartheta}{2} N(N-1) \quad , \quad (3.14)$$

N being the number operator. Accordingly, the exponentiation of the unique self-adjoint extensions of $\tilde{\mathcal{P}}_+^\mu$ and $\tilde{\mathcal{M}}_+$ gives

$$\tilde{V}_+(a) = V_+(a) \quad , \quad \tilde{V}_+(\alpha) = \exp \left[-i \frac{\vartheta}{2} N(N-1) \alpha \right] V_+(\alpha) \quad . \quad (3.15)$$

By means of $U_+(R)$ these representations can be pulled back to $\mathcal{F}_R(\mathcal{H})$, giving rise to

$$\mathcal{U}_R(a) = U_+(R)^{-1} V_+(a) U_+(R) = \mathcal{V}_R(a) \quad , \quad (3.16)$$

and

$$\mathcal{U}_R(\alpha) = U_+(R)^{-1} \tilde{V}_+(\alpha) U_+(R) = \exp \left[-i \frac{\vartheta}{2} N(N-1) \alpha \right] \mathcal{V}_R(\alpha) \quad . \quad (3.17)$$

Therefore for anyons the translations respect eq.(2.10), but this is not the case of rotations. As already anticipated, violations of eq.(2.10) are indeed allowed for non-differentiable R -factors.

In order to better clarify the physical meaning of eq.(3.17), it is instructive to work out explicitly the transformation properties of the L-M anyon under rotations. This is

interesting also because as an extended object, the anyon field is expected to have a non-standard behavior under these transformations. The basic formulae for arbitrary $\alpha \in \mathbf{R}$ are collected in the appendix. Here we concentrate only on the case $\alpha = 2\pi$, which shows the presence of, what we have called in a previous work [9], an “operator-valued monodromy”. Directly from eq.(3.17) one finds

$$\mathcal{U}_R(2\pi) = \exp [-i\pi\vartheta N(N-1) - i2\pi\xi N] \quad . \quad (3.18)$$

Thus, under 2π -rotations anyons behave as follows:

$$\begin{aligned} \mathcal{U}_R(2\pi) a(x) \mathcal{U}_R(2\pi)^{-1} &= \exp [i2\pi(\vartheta N + \xi)] a(x) \quad , \\ \mathcal{U}_R(2\pi) a^*(x) \mathcal{U}_R(2\pi)^{-1} &= a^*(x) \exp [-i2\pi(\vartheta N + \xi)] \quad . \end{aligned} \quad (3.19)$$

Accordingly, anyons admit a sort of operator-valued monodromy, which is represented by the exponential factors in the right hand side of eq.(3.19). The N -independent term of these factors is proportional to the parameter ξ and is, in this sense, arbitrary. The term involving N however is completely fixed by the statistical parameter ϑ . Imposing the “spin-statistic” relation in the one-particle space, namely $\xi = \vartheta/2$, we derive

$$\mathcal{U}_R(2\pi) = \exp (-i\pi\vartheta N^2) \quad . \quad (3.20)$$

Eq.(3.20) entails the spin addition rule for anyons [10,11].

4. Conclusions

Our analysis [1,2] of the algebra $\mathcal{A}_R$ and its Fock representation $\mathcal{F}_R$ shows that the R -quantization is the natural generalization of the standard second quantization when passing from the algebra of canonical (anti)commutation relations $\mathcal{A}_\pm$ to $\mathcal{A}_R$. The explicit R -quantization of unitary one-particle representations is however a complicated open problem in general. In the present paper we have studied and solved this problem for representations generated by first order differential operators in the one-particle

space. This is a necessary initial step towards the description of the symmetry content of quantum systems associated with $\mathcal{A}_R$. Some concrete examples of such systems are currently under investigation in this framework.

Acknowledgements

A. Liguori and M. Mintchev would like to thank Prof. P. Sorba for the kind hospitality at L.A.P.P., where this work has been completed.

Appendix

We describe here the explicit action of rotations on both states and fields for L-M anyons. According to eq.(3.17), we first need to compute $\mathcal{V}_R(\alpha)$ defined in general by eq.(2.5). The main building block is the product

$$T_+(g_\alpha^{-1}x, g_\alpha^{-1}y) \bar{T}_+(x, y) = \exp \{ -i\vartheta [\arg(g_\alpha^{-1}x - g_\alpha^{-1}y; u) - \arg(x - y; u)] \} \quad . \quad (A.1)$$

It is not hard to check that

$$T_+(g_\alpha^{-1}x, g_\alpha^{-1}y) \bar{T}_+(x, y) = \exp \{ i\vartheta [\alpha - 2\pi I_\alpha(y - x)] \} \quad , \quad (A.2)$$

where $I_\alpha(x)$ is the characteristic function of the set swept by the rotation of the ray $\{x - \lambda u : \lambda \geq 0\}$ around x by the angle $\alpha \in [0, 2\pi)$, counting also the multiplicity for general $\alpha \in \mathbf{R}$. One can write formally

$$I_\alpha(x) = \int_0^\alpha d\phi \int_0^\infty \lambda d\lambda \delta(x - \lambda g_\phi u) \quad . \quad (A.3)$$

For any $\varphi \in \mathcal{F}_R(\mathcal{H})$ we then find

$$\begin{aligned} & [\mathcal{V}_R(\alpha)\varphi]^{(n)}(x_1, \dots, x_n) = \\ & \exp \left[i\frac{\vartheta}{2} \alpha n(n-1) - i\xi \alpha n - 2\pi i\vartheta \sum_{\substack{i,j=1 \\ i < j}}^n I_\alpha(x_j - x_i) \right] \varphi^{(n)}(g_\alpha^{-1}x_1, \dots, g_\alpha^{-1}x_n) \quad . \quad (A.4) \end{aligned}$$

Inserting (A.4) into (3.17) yields

$$\begin{aligned}
[\mathcal{U}_R(\alpha)\varphi]^{(n)}(x_1, \dots, x_n) &= \exp \left[-i\frac{\vartheta}{2}\alpha n(n-1) \right] [\mathcal{V}_R(\alpha)\varphi]^{(n)}(x_1, \dots, x_n) = \\
&\exp \left[-i\xi\alpha n - 2\pi i\vartheta \sum_{\substack{i,j=1 \\ i < j}}^n I_\alpha(x_j - x_i) \right] \varphi^{(n)}(g_\alpha^{-1}x_1, \dots, g_\alpha^{-1}x_n) \quad . \quad (A.5)
\end{aligned}$$

Eq.(A.5) defines the unitary operator implementing the rotations in the anyon Fock space.

Concerning the behavior of the anyon field under rotations, from eqs.(3.17) and (A.4) one has

$$\mathcal{U}_R(\alpha) a(x) \mathcal{U}_R(\alpha)^{-1} = \exp(i\vartheta N \alpha) \mathcal{S}(g_\alpha, g_\alpha x) \exp(i\xi\alpha) a(g_\alpha x) \quad , \quad (A.6)$$

$$\mathcal{U}_R(\alpha) a^*(x) \mathcal{U}_R(\alpha)^{-1} = a^*(g_\alpha x) \exp(-i\xi\alpha) \mathcal{S}(g_\alpha, g_\alpha x) \exp(-i\vartheta N \alpha) \quad , \quad (A.7)$$

where in view of eq.(A.2), $\mathcal{S}(g_\alpha, x)$ defined in general by eq.(2.6), takes the form

$$[\mathcal{S}(g_\alpha; x)\varphi]^{(n)}(x_1, \dots, x_n) = \exp \left[2\pi i\vartheta \sum_{j=1}^n I_\alpha(x_j - x) - i\vartheta\alpha n \right] \varphi^{(n)}(x_1, \dots, x_n) \quad . \quad (A.8)$$

A comparison to eqs.(2.7,8) reveals the presence of extra exponential factors depending on the number operator N in eqs.(A.6,7). Taking into account that $I_{2\pi}(x) = 1$, from eq.(A.8) one derives $\mathcal{S}(g_{2\pi}; x) = 1$ and recovers from (A.6,7) the transformation law under 2π -rotations given by eqs.(3.19).

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