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FLUID-DYNAMICAL MODEL OF HOT NUCLEI

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Abstract

A finite temperature fluid-dynamical model of the nucleus has been investigated. The model allows for surface modes with an excitation energy which is directly related to the surface tension. The currents associated with these modes have strong transverse components. We show that, as a result of thermal fluctuations, a new sound velocity appears, which is not present at zero temperature and, as a consequence, new eigenmodes also appear. While the energies are stable, when we take the limit $T \rightarrow 0$, the strength corresponding to these new eigenmodes tends to zero. Electric and magnetic modes are studied. The spectrum of excited energies and the currents are calculated for a nucleus with mass number $A=208$.

The present model predicts a sizable redistribution of transition strength when the temperature is increased.

1. Introduction

Fluid-dynamical models allow for a description of nuclear collective states and have the advantage of being physically intuitive. These models are rather versatile since it is possible to describe in their context the effect of temperature [1, 2, 3, 4] rotations [5], collisions [6, 7] and vibrations [1, 2, 3, 4, 8, 9, 10, 11, 12, 13]. Recently the same methods have also been used to calculate the bulk plasmon in a metal [14] and the vibrational modes of the valence electrons in a metal cluster [15, 16]. Nuclear fluid-dynamical models are related with the Vlasov equation. As is well known, the Vlasov equation describes the motion of an assembly of classical particles in the independent particle approximation.

In refs. [9, 10] a nuclear fluid-dynamical model was proposed which incorporates, at the same time, monopole distortions and quadrupole distortions of the Fermi surface and still allows for a description of surface dynamics [4, 9, 10]. Monopole distortions of the Fermi surface are implied by the time-dependent Thomas-Fermi approximation. On the other hand, quadrupole distortions are of importance for a correct description of surface dynamics. This model, which we wish to investigate in an extended version, may be solved analitically, allows for a rather wide strength distribution and leads to a description of giant resonances and also of low-lying collective modes which include important rotational components in the current distribution.

Since the models presented in refs. [1, 2, 3, 8] do not take into account the degrees of freedom responsible for the surface dynamics, they may not provide an adequate description of surface modes.

Our aim is to study the extension to a finite temperature of the method proposed in refs. [9, 10]. We will follow closely ref. [4] where a simpler version of the model of refs. [9, 10] was studied and extended to a finite temperature. It should be borne in mind that the model is classical in nature except that the Pauli principle is taken into account. Since shell effects are neglected, the predictions of the model apply only to the average behaviour, at which one arrives after shell fluctuations have been smoothed out, or for temperatures above $T=3$ MeV.

2. Lagrangian, equations of motion and boundary conditions

The Vlasov equation may be derived from a variational principle which is useful in order to obtain approximate solutions (see ref. [9]). We start from the classical limit of a variational formulation [17] of the Liouville-von Neumann

equation of statistical mechanics in order to obtain a mean field approximation appropriate to mixed states. The lagrangian

$$L = i\hbar \operatorname{tr}(\dot{U}D_0U^\dagger) - \operatorname{tr}(UD_0U^\dagger H), \quad (1)$$

determines the time-evolution of the density matrix

$$D = UD_0U^\dagger, \quad (2)$$

where D_0 stands for the density matrix describing a state of equilibrium and U is a unitary operator. If we write

$$U = \exp\left(\frac{i}{\hbar}\hat{\mathcal{G}}\right), \quad (3)$$

the lagrangian L reduces, for small amplitude oscillations, to the following harmonic lagrangian

$$L^{(2)} = \operatorname{tr}\left(D_0\left(\frac{i}{2\hbar}[\hat{\mathcal{G}}, \dot{\hat{\mathcal{G}}}] - \frac{1}{2\hbar^2}[\hat{\mathcal{G}}, [\hat{H}, \hat{\mathcal{G}}]]\right)\right). \quad (4)$$

The Vlasov equation is the classical limit of TDHF. In this limit the distribution function in phase space is the Wigner transform of the one-body density matrix. The Poisson bracket $\{, \}$ is the classical limit of the commutator $[\cdot, \cdot]$. The Vlasov equation is purely classical, but still compatible with the Pauli principle. We consider the classical limit of the lagrangian (4), retaining only the leading order terms in a Wigner-Kirkwood expansion in powers of $\hbar$. We assume density dependent Skyrme forces. In the energy functional characterizing the equilibrium state we do not include terms with derivatives of the density. These terms tend to compress the central density and determine the surface profile. We replace such terms by a temperature dependent surface tension $\sigma(T)$ which decreases with increasing temperature in order to mock up the fact that with increasing temperature the surface profile of the nucleus becomes smoother. We note that a sharp surface profile arises in our calculations because we are neglecting terms with derivatives of the density in the equilibrium energy density. In the equilibrium state (selfconsistency is important) the energy is,

$$E^{(0)} = \frac{4\pi}{3}R_0^3\left(\tau_0 + \sum_{\nu}a_{\nu}\rho_0^{\nu}\right) + \sigma 4\pi R_0^2, \quad (5)$$

where R_0 stands for the radius of the nucleus and ρ_0 and τ_0 stand respectively for the equilibrium density and the equilibrium kinetic energy density. The equilibrium distribution function is a Fermi type distribution function

$$f_0 = \frac{\Theta(R_0 - r)}{1 + \exp[(\frac{p^2}{2m} - \epsilon_0)/T]}, \quad (6)$$

where R_0 and ς_0 are determined by the particle number A and by the minimization of the free energy (see ref. [4]) which leads to the equilibrium condition

$$\frac{2}{3}\tau_0 + \sum_{\nu} a_{\nu}(\nu - 1)\rho_0^{\nu} - \frac{2\sigma}{R_0} = 0, \quad (7)$$

where

$$\rho_0 = g \int \frac{d^3p}{(2\pi\hbar)^3} f_0, \quad (8)$$

$$\tau_0 = g \int \frac{d^3p}{(2\pi\hbar)^3} f_0 \frac{p^2}{2m}, \quad (9)$$

and g stands for the spin-isospin multiplicity. Clearly it follows from eqs. (6), (8) and (9) that ρ_0 and τ_0 depend on the temperature and are step functions in r , $\tau_0(r, T) = \tau_0(0, T)\Theta(R_0 - r)$ and $\rho_0(r, T) = \rho_0(0, T)\Theta(R_0 - r)$. We note that $\varsigma_0 = \mu - U_0$, where μ stands for the chemical potential and $U_0 = \sum_{\nu} a_{\nu}\nu\rho_0^{\nu-1}$ is the equilibrium selfconsistent potential. It should be borne in mind that, although the equilibrium density is a step function, the density may be compressed so that, for the electric eigenmodes, the density fluctuates inside the nucleus and the nuclear surface oscillates.

In refs. [1, 4] we have started by determining the equilibrium state (the state of stable equilibrium is an useful concept). The distribution function $f_0(\mathbf{x}, \mathbf{p}, 0)$ describes a state of stable equilibrium if $F[f] \leq F[f_0]$, $\forall f \in \Lambda$ where Λ stands for the set of distribution functions which may be generated from f_0 by means of a canonical transformation (see ref. [9]) and $F = E - TS$ is the free energy. Suppose now that there is some initial condition which causes the system to deviate from the equilibrium state. Given the initial condition $f(\mathbf{x}, \mathbf{p}, 0)$, if $\{f(\mathbf{x}, \mathbf{p}, 0), h(\mathbf{x}, \mathbf{p}, 0)\} \neq 0$ (h is the single particle hamiltonian), the system will evolve in time. It will oscilate around $f_0(\mathbf{x}, \mathbf{p}, 0)$. We are interested in small amplitude oscillations. From the lagrangian (4) we will calculate the eigenmodes. If we do not allow for a completely general variation of the generator $\mathcal{G}$ then the dynamics of the system will be affected by the restrictive parametrization we have chosen for the generator $\mathcal{G}$. With respect to the variational parametrization of the distribution function there are many choices possible. Each one implicitly assumes a choice of relevant and irrelevant degrees of freedom. Therefore the parametrization we impose for $\mathcal{G}$ is a way of selecting the relevant collective variables for the dynamics.

In the classical limit we have

$$\hat{\mathcal{G}} = \sum_{i=1}^A \mathcal{G}(\mathbf{x}_i, \mathbf{p}_i, t), \quad (10)$$

where the generator $\mathcal{G}(\mathbf{x}, \mathbf{p}, t)$ may be decomposed into a time-even term Q and a time-odd term P ($\mathcal{G} = Q + P$). A parametrization by means of a scalar

field $\psi(\mathbf{x}, t)$ and a tensor field $\phi_{\alpha\beta}(\mathbf{x}, t)$ is chosen for the generator Q ,

$$Q = \psi(\mathbf{x}, t) + \frac{1}{2} p_\alpha p_\beta \phi_{\alpha\beta}(\mathbf{x}, t). \quad (11)$$

P is not actually given but is disguised in the definition of the distribution function $\tilde{f}$. We assume P is the generator of a canonical transformation such that the following equation is approximately satisfied

$$\begin{aligned} & f_0 + \{f_0, P\} + \frac{1}{2} \{\{f_0, P\}, P\} + \dots = \\ & \frac{\Theta(R_0 + R_1(\theta, \phi, t) - r)}{1 + \exp[(\frac{p^2}{2m} - \varsigma_0 - \varsigma_1(\mathbf{x}, t) + \frac{p_\alpha p_\beta}{2m} \chi_{\alpha\beta}(\mathbf{x}, t))/T]} (\equiv \tilde{f}). \end{aligned} \quad (12)$$

The distribution function $\tilde{f}$ may be decomposed in a regular part, for which the nuclear boundary does not change and an irregular part which accounts for a pure displacement of the surface and is parametrized by the function $R_1(\theta, \phi, t)$.

The time dependent distribution function is parametrized by the fields ψ , ς_1 , R_1 , $\phi_{\alpha\beta}$ and $\chi_{\alpha\beta}$. The field ς_1 is responsible for density fluctuations. The field $\chi_{\alpha\beta}$ gives rise to distortions of the Fermi surface. Displacements of the nuclear surface are acceptable motions. They originate from discontinuous components of ς_1 and may be parametrized by appropriate boundary conditions.

Equation (12) generalizes to a finite temperature the generator P given in refs. [9, 10]. We remark that R_1 describes the surface displacements. At a finite temperature no generator P exists such that eq. (12) holds exactly. This means that, at a finite temperature, R_1 , ς_1 and $\chi_{\alpha\beta}$ are not an adequate parametrization since these quantities can not be generated by a canonical transformation. However the errors involved in the deviation from canonicity will be minimized if we require that the fields R_1 , ς_1 and $\chi_{\alpha\beta}$ are such that the entropy remains constant.

Starting from the distribution function $\tilde{f}$, the associated density $\tilde{\rho}$ is easily computed and one obtains

$$\begin{aligned} \tilde{\rho} = & \left(\rho_0 + \mathcal{A}_1 \varsigma_1 + \frac{1}{2} \mathcal{A}_2 \varsigma_1^2 + \dots \right) \\ & \times \left(1 - \frac{1}{2} \chi_{\alpha\alpha} + \frac{1}{4} \chi_{\alpha\beta} \chi_{\alpha\beta} + \frac{1}{8} \chi_{\alpha\alpha} \chi_{\beta\beta} + \dots \right), \end{aligned} \quad (13)$$

where $\mathcal{A}_1 = \partial \rho_0 / \partial \varsigma_0$ and $\mathcal{A}_2 = \partial^2 \rho_0 / \partial \varsigma_0^2$. In an analogous way we also obtain the kinetic energy density associated with the distribution function $\tilde{f}$

$$\begin{aligned} \tilde{\tau} = & \left(\tau_0 + \frac{3}{2} \rho_0 \varsigma_1 + \frac{3}{4} \mathcal{A}_1 \varsigma_1^2 + \dots \right) \\ & \times \left(1 - \frac{5}{6} \chi_{\alpha\alpha} + \frac{7}{12} \chi_{\alpha\beta} \chi_{\alpha\beta} + \frac{7}{24} \chi_{\alpha\alpha} \chi_{\beta\beta} + \dots \right). \end{aligned} \quad (14)$$

The entropy is calculated from the grand potential Ω

$$\begin{aligned}\Omega &= -T \int d\Gamma \left\{ 1 + \exp \left[\left(\varsigma - \frac{1}{2m} p_\alpha p_\beta (\delta_{\alpha\beta} + \chi_{\alpha\beta}) \right) / T \right] \right\} \\ &= - \int d^3x \left[\Lambda \left(\frac{4}{15} \varsigma^{5/2} + \frac{\pi^2}{6} T^2 \varsigma^{1/2} - \frac{7\pi^4}{1440} T^4 \varsigma^{-3/2} + \dots \right) \right. \\ &\quad \left. \times \left(1 - \frac{1}{2} \chi_{\alpha\alpha} + \frac{1}{4} \chi_{\alpha\beta} \chi_{\alpha\beta} + \frac{1}{8} \chi_{\alpha\alpha} \chi_{\beta\beta} + \dots \right) \right] \quad (15)\end{aligned}$$

where $d\Gamma = g d^3x d^3p / (2\pi\hbar)^3$ is the volume element in phase space. We have

$$\begin{aligned}S &= - \left(\frac{\partial \Omega}{\partial T} \right) = \int d^3x \left[\left(s_0 + s_1 \varsigma_1 + \frac{1}{2} s_2 \varsigma_1^2 + \dots \right) \right. \\ &\quad \left. \times \left(1 - \frac{1}{2} \chi_{\alpha\alpha} + \frac{1}{4} \chi_{\alpha\beta} \chi_{\alpha\beta} + \frac{1}{8} \chi_{\alpha\alpha} \chi_{\beta\beta} + \dots \right) \right], \quad (16)\end{aligned}$$

where $s_0 = \Lambda \left(\frac{\pi^2}{3} T \varsigma_0^{1/2} - \frac{7\pi^4}{360} T^3 \varsigma_0^{-3/2} + \dots \right)$, $s_1 = \partial s_0 / \partial \varsigma_0$, $s_2 = \partial^2 s_0 / \partial \varsigma_0^2$, and $\Lambda = 2\pi g \left(\frac{\sqrt{2m}}{2\pi\hbar} \right)^3 \Theta(R_0 + R_1(\theta, \phi, t) - r)$. We note that the quantities in eqs. (13), (14) as well as the integrands in eqs. (15) and (16) refer to a domain D defined by the step function $\Theta(R_0 + R_1(\theta, \phi, t) - r)$, since these quantities are evaluated with the distribution function $\tilde{f}$ defined in eq. (12). The kinetic energy corresponding to the distribution function $\tilde{f}$ is $\langle \tilde{t} \rangle \equiv \int d^3x \tilde{t}$, and the potential energy corresponding to the distribution function $\tilde{f}$ is $\langle \tilde{v} \rangle \equiv \int d^3x \sum_\nu a_\nu \tilde{p}^\nu$.

From the distribution function f ,

$$f = f_0 + \{f_0, \mathcal{G}\} + \frac{1}{2} \{\{f_0, \mathcal{G}\}, \mathcal{G}\} + \dots, \quad (17)$$

we derive the physical quantities, such as the density, the current and the energy. The distribution function is parametrized by the scalar fields ς_1 , ψ and R_1 and by two tensor fields $\phi_{\alpha\beta}$ and $\chi_{\alpha\beta}$, canonically conjugate of each other. Up to first order in the variational fields we have the following expressions for the density

$$\rho = \rho_0 + \mathcal{A}_1 \varsigma_1 - \frac{\rho_0}{2} \chi_{\alpha\alpha}, \quad (18)$$

and for the current [3]

$$j_\alpha = \frac{\rho_0}{m} \partial_\alpha \psi + \frac{2\tau_0}{3} \left(\frac{1}{2} \partial_\alpha \phi_{\mu\mu} + \partial_\beta \phi_{\alpha\beta} \right). \quad (19)$$

In order to arrive at the expression (19) we have imposed the boundary condition [3]

$$x_\alpha \phi_{\alpha\beta}|_{r=R_0} = 0, \quad (20)$$

which ensures that the current is not singular at the surface.

We ensure that the boundary condition (20) is fulfilled and that the entropy S and the particle number A are conserved on the average by means of the Lagrange multipliers ξ_β , T and μ . Note that μ is determined by the equilibrium condition (eqs. (10) and (11) in ref. [4]). From the classical limit of the lagrangian (4), with the choices for the generators Q and P implied by eqs. (11) and (12) and taking into account the subsidiary conditions which ensure the boundary condition (20), the conservation of the entropy S and of the particle number A , the following effective lagrangian is obtained

$$\mathcal{L}^{(2)} = (L + \mu A + TS)^{(2)} + \oint_{\Sigma_0} d\Sigma \xi_\beta n_\alpha \phi_{\alpha\beta}, \quad (21)$$

where $n_\beta = x_\beta/r$. For the chosen parametrization and small amplitude oscillations, the lagrangian is given by

$$\begin{aligned} \mathcal{L}^{(2)} = & - \int_{D_0} d^3x \left\{ \dot{\psi} \left(\rho_0 + \mathcal{A}_1 \varsigma_1 - \frac{\rho_0}{2} \chi_{\mu\mu} \right) \right. \\ & + m \dot{\phi}_{\alpha\beta} \left[\delta_{\alpha\beta} \left(\frac{\tau_0}{3} + \frac{\rho_0}{2} \varsigma_1 \right) - \frac{\tau_0}{3} \left(\chi_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \chi_{\mu\mu} \right) \right] \left. \right\} + \oint_{\Sigma_0} d\Sigma \xi_\alpha n_\beta \phi_{\alpha\beta} \\ & - \oint_{\Sigma_0} d\Sigma R_1 \left(\rho_0 \dot{\psi} + \frac{m\tau_0}{3} \dot{\phi}_{\mu\mu} \right) - T^{(2)}[\psi, \phi_{\alpha\beta}] - \mathcal{E}^{(2)}[\varsigma_1, \chi_{\alpha\beta}, R_1], \end{aligned} \quad (22)$$

where

$$\begin{aligned} T^{(2)}[\psi, \phi_{\alpha\beta}] = & \int_{D_0} d^3x \left\{ \frac{\rho_0}{2m} (\nabla \psi) \cdot (\nabla \psi) + \frac{2\tau_0}{3} (\partial_\alpha \psi) \left(\frac{1}{2} \partial_\alpha \phi_{\mu\mu} + \partial_\beta \phi_{\alpha\beta} \right) \right. \\ & \left. + \frac{2mG}{15} \left[\left(\frac{1}{2} \partial_\alpha \phi_{\mu\mu} + \partial_\beta \phi_{\alpha\beta} \right)^2 + \frac{1}{6} (\partial_\alpha \phi_{\beta\gamma} + \partial_\beta \phi_{\gamma\alpha} + \partial_\gamma \phi_{\alpha\beta})^2 \right] \right\}, \end{aligned} \quad (23)$$

(see ref. [3]),

$$G = g \int \frac{d^3p}{(2\pi\hbar)^3} f_0 \left(\frac{p^2}{2m} \right)^2 \quad (24)$$

and $\mathcal{E}^{(2)}[\varsigma_1, R_1, \chi_{\alpha\beta}]$ stands for the second order terms appearing in $\langle \tilde{t} \rangle + \langle \tilde{v} \rangle - \mu A - TS$. We find

$$\begin{aligned} \mathcal{E}^{(2)}[\varsigma_1, R_1, \chi_{\alpha\beta}] = & \int_{D_0} d^3x \left\{ \frac{1}{2} \varsigma_1^2 \left[T \left(\frac{\mathcal{A}_2 \varsigma_1}{\mathcal{A}_1} - s_2 \right) + \frac{3}{2} \mathcal{A}_1 - \frac{3}{2} \frac{\mathcal{A}_2 \rho_0}{\mathcal{A}_1} \right] \right. \\ & + \left(\chi_{\alpha\beta} \chi_{\alpha\beta} + \frac{1}{2} \chi_{\mu\mu} \chi_{\nu\nu} \right) \left[\frac{T}{4} \left(\frac{s_1 \rho_0}{\mathcal{A}_1} - s_0 \right) + \frac{7}{12} \tau_0 - \frac{3}{8} \frac{\rho_0^2}{\mathcal{A}_1} \right] \\ & \left. - \frac{\rho_0}{2} \chi_{\nu\nu} \varsigma_1 + \frac{\mathcal{C}_3}{2} \left(\mathcal{A}_1 \varsigma_1 - \frac{1}{2} \rho_0 \chi_{\nu\nu} \right)^2 \right\} + \frac{\sigma}{2R_0^2} \oint_{\Sigma_0} d\Sigma R_1^2 [\ell(\ell+1) - 2]. \end{aligned} \quad (25)$$

where

$$\mathcal{C}_3 = \sum_\nu a_\nu \nu (\nu - 1) \rho_0^{\nu-2}, \quad (26)$$

D_0 is the domain occupied by the nucleus, in the state of equilibrium, and Σ_0 is the boundary of D_0 .

Taking into account the relations,

$$T \left(\frac{s_1 \rho_0}{\mathcal{A}_1} - s_0 \right) + \frac{5}{3} \tau_0 - \frac{3}{2} \frac{\rho_0^2}{\mathcal{A}_1} = 0, \quad (27)$$

and

$$T \left(\frac{\mathcal{A}_2 s_1}{\mathcal{A}_1} - s_2 \right) + \frac{1}{2} \mathcal{A}_1 - \frac{3}{2} \frac{\mathcal{A}_2 \rho_0}{\mathcal{A}_1} = 0, \quad (28)$$

we arrive at

$$\begin{aligned} \mathcal{E}^{(2)}[\varsigma_1, R_1, \chi_{\alpha\beta}] = & \int_{D_0} d^3x \left[\frac{1}{2} \mathcal{A}_1 \varsigma_1^2 + \frac{\tau_0}{6} \left(\chi_{\alpha\beta} \chi_{\alpha\beta} + \frac{1}{2} \chi_{\mu\mu} \chi_{\nu\nu} \right) \right. \\ & \left. - \frac{\rho_0}{2} \chi_{\nu\nu} \varsigma_1 + \frac{\mathcal{C}_3}{2} \left(\mathcal{A}_1 \varsigma_1 - \frac{1}{2} \rho_0 \chi_{\nu\nu} \right)^2 \right] + \frac{\sigma}{2} \oint_{\Sigma_0} d\Sigma \frac{R_1^2}{R_0^2} [\ell(\ell+1) - 2]. \end{aligned} \quad (29)$$

The equations of motion, eigenmodes, orthogonality relations and useful sum rules will be obtained. In what follows we derive the equations of motion and boundary conditions starting from the action integral. The action integral should be stationary for arbitrary variations of ψ , leading to

$$\begin{aligned} \delta I = \int dt \left\{ \int_{D_0} d^3x \delta\psi \left[\mathcal{A}_1 \dot{\varsigma}_1 - \frac{\rho_0}{2} \dot{\chi}_{\mu\mu} + \frac{\rho_0}{m} \Delta\psi + \frac{2\tau_0}{3} \left(\frac{1}{2} \Delta\phi_{\nu\nu} + \partial_\alpha \partial_\beta \phi_{\alpha\beta} \right) \right] \right. \\ \left. + \oint_{\Sigma_0} d\Sigma \delta\psi \left[\rho_0 \dot{R}_1 - n_\alpha \left(\frac{\rho_0}{m} \partial_\alpha \psi + \frac{2\tau_0}{3} \left(\frac{1}{2} \partial_\alpha \phi_{\mu\mu} + \partial_\gamma \phi_{\alpha\gamma} \right) \right) \right] \right\} = 0. \end{aligned} \quad (30)$$

The requirement of stationarity with respect to variations of $\phi_{\alpha\beta}$, gives

$$\begin{aligned} \delta I = \int dt \left\{ \int_{D_0} d^3x \delta\phi_{\alpha\beta} \left[\delta_{\alpha\beta} \frac{m\rho_0}{2} \dot{\varsigma}_1 \right. \right. \\ \left. - \frac{m\tau_0}{3} \left(\delta_{\alpha\beta} \frac{1}{2} \dot{\chi}_{\nu\nu} + \dot{\chi}_{\alpha\beta} \right) + \frac{2\tau_0}{3} \left(\delta_{\alpha\beta} \frac{1}{2} \Delta\psi + \partial_\alpha \partial_\beta \psi \right) \right. \\ \left. + \frac{2mG}{15} \left(\delta_{\alpha\beta} \left(\frac{1}{2} \Delta\phi_{\mu\mu} + \partial_\mu \partial_\nu \phi_{\nu\mu} \right) + \Delta\phi_{\alpha\beta} + \partial_\alpha \partial_\beta \phi_{\mu\mu} + 2\partial_\alpha \partial_\gamma \phi_{\gamma\beta} + 2\partial_\beta \partial_\gamma \phi_{\gamma\alpha} \right) \right] \\ \left. + \oint_{\Sigma_0} d\Sigma \delta\phi_{\alpha\beta} \left[\delta_{\alpha\beta} \left(\frac{m\tau_0}{3} \dot{R}_1 - n_\gamma \left(\frac{\tau_0}{3} \partial_\gamma \psi + \frac{2mG}{15} \left(\frac{1}{2} \partial_\gamma \phi_{\mu\mu} + \partial_\mu \phi_{\mu\gamma} \right) \right) \right) \right. \right. \\ \left. - n_\alpha \left(\frac{\tau_0}{3} \partial_\beta \psi + \frac{2mG}{15} \left(\frac{1}{2} \partial_\beta \phi_{\mu\mu} + \partial_\mu \phi_{\mu\beta} \right) - \frac{1}{2} \xi_\beta \right) \right. \\ \left. - n_\beta \left(\frac{\tau_0}{3} \partial_\alpha \psi + \frac{2mG}{15} \left(\frac{1}{2} \partial_\alpha \phi_{\mu\mu} + \partial_\mu \phi_{\mu\alpha} \right) - \frac{1}{2} \xi_\alpha \right) \right. \\ \left. \left. + n_\gamma \frac{2mG}{15} (\partial_\alpha \phi_{\beta\gamma} + \partial_\beta \phi_{\gamma\alpha} + \partial_\gamma \phi_{\alpha\beta}) \right] \right\} = 0. \end{aligned} \quad (31)$$

The requirement of stationarity with respect to variations of ς_1 , leads to

$$\delta I = - \int dt \int_{D_0} d^3x \delta \varsigma_1 \left[\mathcal{A}_1 \dot{\psi} + \frac{m\rho_0}{2} \dot{\phi}_{\alpha\alpha} + (1 + \mathcal{A}_1 \mathcal{C}_3) \left(\mathcal{A}_1 \varsigma_1 - \frac{\rho_0}{2} \chi_{\nu\nu} \right) \right] = 0. \quad (32)$$

The requirement of stationarity with respect to variations of $\chi_{\alpha\beta}$ implies

$$\delta I = \int dt \int_{D_0} d^3x \delta \chi_{\alpha\beta} \left\{ \delta_{\alpha\beta} \frac{\rho_0}{2} \left[\dot{\psi} + \varsigma_1 + \mathcal{C}_3 \left(\mathcal{A}_1 \varsigma_1 - \frac{1}{2} \rho_0 \chi_{\nu\nu} \right) \right] + \frac{\tau_0}{3} \left[m \dot{\phi}_{\alpha\beta} - \chi_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} (m \dot{\phi}_{\mu\mu} - \chi_{\mu\mu}) \right] \right\} = 0. \quad (33)$$

Finally, from the requirement of stationarity with respect to variations of R_1 , we obtain

$$\delta I = \int dt \oint_{\Sigma_0} d\Sigma \delta R_1 \left\{ -\rho_0 \dot{\psi} - \frac{m\tau_0}{3} \dot{\phi}_{\mu\mu} - \frac{\sigma R_1}{R_0^2} [\ell(\ell+1) - 2] \right\} = 0. \quad (34)$$

The dots over the variational fields stand for time-derivative. From the above conditions (30)-(34), for arbitrary variations of the fields ψ , $\phi_{\alpha\beta}$, ς_1 , $\chi_{\alpha\beta}$ and R_1 , the equations of motion and boundary conditions are obtained

$$\mathcal{A}_1 \dot{\varsigma}_1 - \frac{\rho_0}{2} \dot{\chi}_{\mu\mu} + \frac{\rho_0}{m} \Delta \psi + \frac{2\tau_0}{3} \left(\frac{1}{2} \Delta \phi_{\nu\nu} + \partial_\alpha \partial_\beta \phi_{\alpha\beta} \right) = 0, \quad (35)$$

$$\left\{ \rho_0 \dot{R}_1 - n_\alpha \left[\frac{\rho_0}{m} \partial_\alpha \psi + \frac{2\tau_0}{3} \left(\frac{1}{2} \partial_\alpha \phi_{\mu\mu} + \partial_\gamma \phi_{\alpha\gamma} \right) \right] \right\}_{\tau=R_0} = 0, \quad (36)$$

$$\begin{aligned} \delta_{\alpha\beta} \frac{m\rho_0}{2} \dot{\varsigma}_1 - \frac{m\tau_0}{3} \left(\delta_{\alpha\beta} \frac{1}{2} \dot{\chi}_{\nu\nu} + \dot{\chi}_{\alpha\beta} \right) + \frac{2\tau_0}{3} \left(\delta_{\alpha\beta} \frac{1}{2} \Delta \psi + \partial_\alpha \partial_\beta \psi \right) \\ + \frac{2mG}{15} \left[\delta_{\alpha\beta} \left(\frac{1}{2} \Delta \phi_{\mu\mu} + \partial_\mu \partial_\nu \phi_{\nu\mu} \right) + \Delta \phi_{\alpha\beta} \right. \\ \left. + \partial_\beta \partial_\alpha \phi_{\mu\mu} + 2\partial_\alpha \partial_\gamma \phi_{\gamma\beta} + 2\partial_\beta \partial_\gamma \phi_{\gamma\alpha} \right] = 0, \end{aligned} \quad (37)$$

$$\begin{aligned} \left\{ \delta_{\alpha\beta} \left[-\frac{m\tau_0}{3} \dot{R}_1 + n_\gamma \left(\frac{\tau_0}{3} \partial_\gamma \psi + \frac{2mG}{15} \left(\frac{1}{2} \partial_\gamma \phi_{\mu\mu} + \partial_\mu \phi_{\mu\gamma} \right) \right) \right] \right. \\ + n_\alpha \left[\frac{\tau_0}{3} \partial_\beta \psi + \frac{2mG}{15} \left(\frac{1}{2} \partial_\beta \phi_{\mu\mu} + \partial_\mu \phi_{\mu\beta} \right) - \frac{1}{2} \xi_\beta \right] \\ + n_\beta \left[\frac{\tau_0}{3} \partial_\alpha \psi + \frac{2mG}{15} \left(\frac{1}{2} \partial_\alpha \phi_{\mu\mu} + \partial_\mu \phi_{\mu\alpha} \right) - \frac{1}{2} \xi_\alpha \right] \\ \left. + n_\gamma \frac{2mG}{15} (\partial_\alpha \phi_{\beta\gamma} + \partial_\beta \phi_{\gamma\alpha} + \partial_\gamma \phi_{\alpha\beta}) \right\}_{\tau=R_0} = 0, \end{aligned} \quad (38)$$

$$\mathcal{A}_1 \dot{\psi} + \frac{m\rho_0}{2} \dot{\phi}_{\alpha\alpha} + (1 + \mathcal{A}_1 \mathcal{C}_3) \left(\mathcal{A}_1 \varsigma_1 - \frac{1}{2} \rho_0 \chi_{\nu\nu} \right) = 0, \quad (39)$$

$$\delta_{\alpha\beta} \frac{\rho_0}{2} \left[\dot{\psi} + \varsigma_1 + \mathcal{C}_3 \left(\mathcal{A}_1 \varsigma_1 - \frac{\rho_0}{2} \chi_{\nu\nu} \right) \right] + \frac{\tau_0}{3} \left[\delta_{\alpha\beta} \frac{1}{2} (m \dot{\phi}_{\mu\mu} - \chi_{\mu\mu}) + m \dot{\phi}_{\alpha\beta} - \chi_{\alpha\beta} \right] = 0, \quad (40)$$

$$\left\{ \rho_0 \dot{\psi} + \frac{m\tau_0}{3} \dot{\phi}_{\mu\mu} + \frac{\sigma R_1}{R_0^2} [\ell(\ell+1) - 2] \right\}_{r=R_0} = 0. \quad (41)$$

Eqs. (35) and (36) correspond to the continuity equation and ensure the conservation of the particle number up to first order in R_1 , ς_1 and $\chi_{\alpha\beta}$. Eq. (36) relates the radial current (at the surface) with the displacement of the surface. The particle number is exactly conserved, and for $\ell > 0$, the entropy is conserved not only on the average but also exactly. It follows from eqs. (39) and (40) that $\chi_{\alpha\beta}$ and $\phi_{\alpha\beta}$ are canonically conjugate variables,

$$\chi_{\alpha\beta} = m \dot{\phi}_{\alpha\beta}. \quad (42)$$

The boundary condition (41) corresponds to the boundary condition (30) of ref. [4] and is responsible for the existence of surface modes with the energy proportional to the surface tension σ and also approximately proportional to $A^{-1/2}$. If we did not allow for surface vibrations, as was done in ref. [13] we would have one equation less (41) and also one variable less, R_1 . Then all the energies of the eigenmodes would be proportional to $A^{-1/3}$.

We will now be concerned with the dispersion relations implied by this model. We first consider the electric modes. With respect to the transverse sound velocities we consider solutions of the form

$$\phi_{\alpha\beta}^{(1)} = [\partial_\alpha (\nabla \times \mathcal{L})_\beta + \partial_\beta (\nabla \times \mathcal{L})_\alpha] B(\mathbf{x}, t), \quad (43)$$

and

$$\begin{aligned} \phi_{\alpha\beta}^{(2)} = & [(\nabla \times \mathcal{L})_\alpha (\nabla \times \mathcal{L})_\beta + (\nabla \times \mathcal{L})_\beta (\nabla \times \mathcal{L})_\alpha - \partial_\alpha (\nabla \times \mathcal{L})_\beta \\ & - \partial_\beta (\nabla \times \mathcal{L})_\alpha + (\partial_\alpha \partial_\beta - \delta_{\alpha\beta} \nabla^2) \mathcal{L}^2] C(\mathbf{x}, t), \end{aligned} \quad (44)$$

where $\mathcal{L} = \mathbf{x} \times \nabla$. At this point it is convenient to notice that $\phi_{\alpha\beta}^{(2)}$ was constructed in such a way that $\phi_{\alpha\alpha}^{(2)} = 0$ and $\partial_\alpha \phi_{\alpha\beta}^{(2)} = 0$. Therefore the component $\phi_{\alpha\beta}^{(2)}$ does not contribute directly to the current (see eq. (19)), although it does contribute in an indirect way through the equations of motion and boundary conditions.

From eqs. (37) and (42) we obtain the transverse sound velocity previously derived in ref. [3],

$$c_{1T} = \sqrt{\frac{6G}{5m\tau_0}}, \quad (45)$$

which is associated to the solution (43), and additionally we obtain another sound velocity

$$c_{2T} = \sqrt{\frac{2G}{5m\tau_0}}, \quad (46)$$

associated to the solution (44). Concerning the longitudinal sound velocities we consider the solution of the form

$$\phi_{\alpha\beta}^{(3)} = \delta_{\alpha\beta}X(\mathbf{x}, t) + \partial_\alpha\partial_\beta Z(\mathbf{x}, t). \quad (47)$$

For each multipolarity, the scalar fields B , C , X , Z , ψ and ς_1 are all expressed as a product of the form $f(t)j_\ell(kr)Y_{\ell 0}$, where $f(t)$ is a time-dependent function appearing later in eqs. (62) or (63), $j_\ell(kr)$ is a spherical Bessel function and $Y_{\ell 0}$ is a spherical harmonic. From eqs. (37) and (42) we obtain the equations of motion

$$-\frac{m}{3}\tau_0\ddot{Z} + \frac{2\tau_0}{3m}\psi + \frac{2G}{15}(6\Delta Z + 7X) = 0, \quad (48)$$

$$\frac{\rho_0}{2}\dot{\varsigma}_1 - \frac{m\tau_0}{6}(5\ddot{X} + \Delta\ddot{Z}) + \frac{\tau_0}{3m}\Delta\psi + \frac{G}{15}(7\Delta X + 3\Delta\Delta Z) = 0. \quad (49)$$

From eq. (39) we find

$$\dot{\psi} + (1 + \mathcal{A}_1\mathcal{C}_3)\varsigma_1 - \mathcal{C}_3\frac{m\rho_0}{2}(3\dot{X} + \Delta\dot{Z}) = 0, \quad (50)$$

and from eq. (35) it follows that

$$\mathcal{A}_1\dot{\varsigma}_1 - \frac{m\rho_0}{2}(3\ddot{X} + \Delta\ddot{Z}) + \frac{\rho_0}{m}\Delta\psi + \frac{1}{3}\tau_0(5\Delta X + 3\Delta\Delta Z) = 0. \quad (51)$$

Starting from eqs. (48)-(51) we arrive, for a finite temperature, at a cubic equation for the longitudinal sound velocities

$$\begin{aligned} & -3\mathcal{O}_1\rho_0(mc^2)^3 + (mc^2)^2\left(\frac{10}{3}\mathcal{O}_3 + 6\mathcal{O}_1\mathcal{C}_5\rho_0\right) \\ & + mc^2\left[-\frac{20}{3}\mathcal{O}_3\left(\frac{\tau_0}{\rho_0} + \frac{\mathcal{C}_3\rho_0}{2}\right) - \frac{12\mathcal{O}_5G}{5\rho_0} + 6\rho_0\mathcal{O}_1\mathcal{C}_3\mathcal{C}_7\right] \\ & + \frac{8G\mathcal{O}_3}{3\rho_0}(1 + \mathcal{A}_1\mathcal{C}_3) = 0, \end{aligned} \quad (52)$$

where

$$\mathcal{O}_1 = 1 - \frac{10\tau_0\mathcal{A}_1}{9\rho_0^2}, \quad (53)$$

$$\mathcal{O}_3 = \tau_0 - \frac{21\rho_0G}{25\tau_0}, \quad (54)$$

$$\mathcal{O}_5 = \rho_0 - \frac{14\mathcal{A}_1 G}{15\tau_0}, \quad (55)$$

$$\mathcal{C}_5 = \frac{6G}{5\tau_0} + \frac{\rho_0 \mathcal{C}_3}{2}, \quad (56)$$

and

$$\mathcal{C}_7 = \tau_0 - \frac{6\rho_0 G}{5\tau_0}. \quad (57)$$

This is a remarkable result. We see that while at zero temperature this model leads to two longitudinal sound velocities $c_1^{(L)}$ and $c_2^{(L)}$, which were previously obtained at zero temperature in refs. [9, 10, 13], at a finite temperature an additional longitudinal sound velocity $c_3^{(L)}$ is obtained so that an additional degree of freedom arises. This happens due to the fact that, at finite temperature, the profile of the Fermi surface in an infinite medium acquires a finite width. We remark that information concerning the profile of the Fermi surface is contained in the moments of f_0 , namely, ρ_0 , τ_0 and G , which appear in the effective lagrangian. In particular we notice that, at zero temperature eq. (35) is identical to the trace of eq. (37). However, at finite temperature this is not true. The continuity equation (35) is not implied by the tensor equation (37) and thus it gives additional information. In fact, in the equations of motion at zero temperature, we always find the combination $\psi + \frac{1}{2}p_F^2 X$, and therefore we obtain an analytical expression for this sum and not for the two fields ψ and X separately. At a finite temperature this is no longer true and we find analytical solutions for both fields ψ and X . In this way this model leads to a larger fragmentation of the strength at a finite temperature than the one which is obtained at zero temperature.

Therefore, taking into account the fields R_1 , ψ , ς_1 , $\chi_{\alpha\beta}$ and $\phi_{\alpha\beta}$, leads to the appearance of 4 ($T=0$) or 5 ($T > 0$) sound velocities in contrast with the simplest model of ref. [4] which only considers the scalar fields R_1 , ψ and ς_1 , leading just to one sound velocity. As we will see the appearance of a larger number of sound velocities implies also a larger number of eigenmodes.

For a given multipolarity the general solution of the tensor $\phi_{\alpha\beta}$ corresponding to each eigenmode is expressed as a linear combination of three solutions $\phi_{\alpha\beta}^{(3)}(c_i^{(L)})$ corresponding to each of the longitudinal sound velocities and (if $\ell > 0$) of the two transverse solutions $\phi_{\alpha\beta}^{(1)}(c_1^{(T)})$ and $\phi_{\alpha\beta}^{(2)}(c_2^{(T)})$,

$$\phi_{\alpha\beta} = a_1^{(T)} \phi_{\alpha\beta}^{(1)}(c_1^{(T)}) + a_2^{(T)} \phi_{\alpha\beta}^{(2)}(c_2^{(T)}) + \sum_{i=1}^3 a_i^{(L)} \phi_{\alpha\beta}^{(3)}(c_i^{(L)}). \quad (58)$$

The expressions of all the other variational fields are then related to $\phi_{\alpha\beta}$ using the equations of motion. Finally the frequency ω , the coefficients $a_1^{(T)}$, $a_2^{(T)}$, and $a_i^{(L)}$ ($i=1, 2, 3$) and the wavevectors $k_1^{(T)}$, $k_2^{(T)}$ and $k_i^{(L)}$ ($i=1, 2$,

3) are obtained using the equations of motion, the boundary conditions, the following relations

$$\omega = k_1^{(T)} c_1^{(T)} = k_2^{(T)} c_2^{(T)} = k_1^{(L)} c_1^{(L)} = k_2^{(L)} c_2^{(L)} = k_3^{(L)} c_3^{(L)}, \quad (59)$$

and the normalization condition (eq. (66)). The $c_i^{(L)}$ correspond to the longitudinal sound velocities and the $k_i^{(L)}$ are the corresponding wavevectors for a given eigenmode.

For magnetic modes, we have solutions of the form

$$\phi_{\alpha\beta}^{(1)} = -i (\partial_\alpha \mathcal{L}_\beta + \partial_\beta \mathcal{L}_\alpha) B(\mathbf{x}, t), \quad (60)$$

and

$$\phi_{\alpha\beta}^{(2)} = -i [(\nabla \times \mathcal{L})_\alpha \mathcal{L}_\beta + (\nabla \times \mathcal{L})_\beta \mathcal{L}_\alpha - \partial_\alpha \mathcal{L}_\beta - \partial_\beta \mathcal{L}_\alpha] C(\mathbf{x}, t), \quad (61)$$

which correspond respectively to the sound velocities $c_1^{(T)}$ and $c_2^{(T)}$ given by eqs. (45) and (46). For each eigenmode the tensor $\phi_{\alpha\beta}$ is a linear combination of $\phi_{\alpha\beta}^{(1)}(c_1^{(T)})$ and $\phi_{\alpha\beta}^{(2)}(c_2^{(T)})$. Also, for the magnetic modes, $\phi_{\alpha\beta}^{(2)}$ satisfies the equation $\partial_\alpha \phi_{\alpha\beta}^{(2)} = 0$.

3. Orthogonality condition and sum rules

We seek solutions of the form

$$A(\mathbf{x}, t) = \overline{A}^{(0)}(\mathbf{x})t + \sum_n \overline{A}^{(n)}(\mathbf{x}) \sin(\omega_n t + \gamma_n), \quad (62)$$

for the fields ς_1 , R_1 and $\chi_{\alpha\beta}$ and of the form

$$B(\mathbf{x}, t) = \overline{B}^{(0)}(\mathbf{x}) + \sum_n \overline{B}^{(n)}(\mathbf{x}) \cos(\omega_n t + \gamma_n), \quad (63)$$

for the fields ψ and $\phi_{\alpha\beta}$. The eigenmodes corresponding to $\overline{A}^{(0)}$ and $\overline{B}^{(0)}$ are zero frequency modes, which, as explained in refs. [4, 10], appear for $\ell = 1$ and for $\ell > 1$ (electric eigenmodes) if $\sigma = 0$.

Making use of eqs. (20) and (35)-(38) we obtain

$$\begin{aligned}
T^{(2)}[\psi^{(i)}, \phi_{\alpha\beta}^{(i)}, \psi^{(j)}, \phi_{\alpha\beta}^{(j)}] &\equiv \frac{1}{2} \int_{D_0} d^3x \left\{ \frac{\rho_0}{m} (\nabla \psi^{(i)}) \cdot (\nabla \psi^{(j)}) \right. \\
&+ \frac{2\tau_0}{3} \left[(\partial_\alpha \psi^{(i)}) \left(\frac{1}{2} \partial_\alpha \phi_{\mu\mu} + \partial_\beta \phi_{\alpha\beta} \right)^{(j)} + (\partial_\alpha \psi^{(j)}) \left(\frac{1}{2} \partial_\alpha \phi_{\mu\mu} + \partial_\beta \phi_{\alpha\beta} \right)^{(i)} \right] \\
&+ \frac{4mG}{15} \left[\left(\frac{1}{2} \partial_\alpha \phi_{\mu\mu} + \partial_\mu \phi_{\alpha\mu} \right)^{(i)} \left(\frac{1}{2} \partial_\alpha \phi_{\nu\nu} + \partial_\nu \phi_{\alpha\nu} \right)^{(j)} \right. \\
&\left. \left. + \frac{1}{6} (\partial_\alpha \phi_{\beta\gamma} + \partial_\beta \phi_{\gamma\alpha} + \partial_\gamma \phi_{\alpha\beta})^{(i)} (\partial_\alpha \phi_{\beta\gamma} + \partial_\beta \phi_{\gamma\alpha} + \partial_\gamma \phi_{\alpha\beta})^{(j)} \right] \right\} \\
&= \frac{1}{2} \int_{D_0} d^3x \left\{ \psi^{(j)} \left(\mathcal{A}_1 \dot{\zeta}_1 - \frac{\rho_0}{2} \dot{\chi}_{\mu\mu} \right)^{(i)} + m \phi_{\alpha\beta}^{(j)} \left[\delta_{\alpha\beta} \frac{\rho_0}{2} \dot{\zeta}_1 - \frac{\tau_0}{3} \left(\dot{\chi}_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \dot{\chi}_{\mu\mu} \right) \right]^{(i)} \right\} \\
&\quad + \frac{1}{2} \oint_{\Sigma_0} d\Sigma \dot{R}_1^{(i)} \left(\rho_0 \psi + \frac{m\tau_0}{3} \phi_{\mu\mu} \right)^{(j)}. \quad (64)
\end{aligned}$$

In a similar way eqs. (39)-(41) lead to

$$\begin{aligned}
\mathcal{E}^{(2)}[\dot{\zeta}_1^{(i)}, \dot{R}_1^{(i)}, \dot{\chi}_{\alpha\beta}^{(i)}, \dot{\zeta}_1^{(j)}, \dot{R}_1^{(j)}, \dot{\chi}_{\alpha\beta}^{(j)}] &\equiv \frac{1}{2} \int_{D_0} d^3x \left[\mathcal{A}_1 \dot{\zeta}_1^{(i)} \dot{\zeta}_1^{(j)} + \frac{\tau_0}{3} \left(\dot{\chi}_{\alpha\beta}^{(i)} \dot{\chi}_{\alpha\beta}^{(j)} + \frac{1}{2} \dot{\chi}_{\mu\mu}^{(i)} \dot{\chi}_{\mu\mu}^{(j)} \right) \right. \\
&\quad \left. - \frac{\rho_0}{2} \left(\dot{\chi}_{\nu\nu}^{(i)} \dot{\zeta}_1^{(j)} + \dot{\chi}_{\nu\nu}^{(j)} \dot{\zeta}_1^{(i)} \right) + \mathcal{C}_3 \left(\mathcal{A}_1 \dot{\zeta}_1 - \frac{1}{2} \rho_0 \dot{\chi}_{\nu\nu} \right)^{(i)} \left(\mathcal{A}_1 \dot{\zeta}_1 - \frac{1}{2} \rho_0 \dot{\chi}_{\mu\mu} \right)^{(j)} \right] \\
&\quad + \frac{\sigma}{2R_0^2} \oint_{\Sigma_0} d\Sigma \dot{R}_1^{(i)} \dot{R}_1^{(j)} [\ell(\ell+1) - 2] \\
&= \omega_j^2 \frac{1}{2} \left\{ \int_{D_0} d^3x \left[\psi^{(j)} \left(\mathcal{A}_1 \dot{\zeta}_1 - \frac{1}{2} \rho_0 \dot{\chi}_{\nu\nu} \right)^{(i)} + m \phi_{\alpha\beta}^{(j)} \left(\delta_{\alpha\beta} \frac{\rho_0}{2} \dot{\zeta}_1 - \frac{\tau_0}{3} \left(\dot{\chi}_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \dot{\chi}_{\mu\mu} \right) \right)^{(i)} \right] \right. \\
&\quad \left. + \oint_{\Sigma_0} d\Sigma \dot{R}_1^{(i)} \left(\rho_0 \psi + \frac{m\tau_0}{3} \phi_{\mu\mu} \right)^{(j)} \right\}. \quad (65)
\end{aligned}$$

We write $\zeta_1^{(j)}(\mathbf{x}, t) = \bar{\zeta}_1^{(j)}(\mathbf{x}) \alpha_j(t)$, $R_1^{(j)}(\theta, \phi, t) = \bar{R}_1^{(j)}(\theta, \phi) \alpha_j(t)$, $\chi_{\alpha\beta}^{(j)}(\mathbf{x}, t) = \bar{\chi}_{\alpha\beta}^{(j)}(\mathbf{x}) \alpha_j(t)$, $\psi^{(j)}(\mathbf{x}, t) = \bar{\psi}^{(j)}(\mathbf{x}) \beta_j(t)$, $\phi_{\alpha\beta}^{(j)}(\mathbf{x}, t) = \bar{\phi}_{\alpha\beta}^{(j)}(\mathbf{x}) \beta_j(t)$, where if $\ell = 1$ or if $\ell > 1$ (electric eigenmodes) and $\sigma = 0$, we have $\alpha_0 = t$ and $\beta_0 = 1$, otherwise $\alpha_j = \sin(\omega_j t + \gamma_j)$ and $\beta_j = \cos(\omega_j t + \gamma_j)$. We are using the index 0 to refer to the zero energy modes (see ref. [4]). From eqs. (64) and (65) the orthogonality relation follows

$$\begin{aligned}
\frac{1}{2} \int_{D_0} d^3x \left\{ \bar{\psi}^{(i)} \left(\mathcal{A}_1 \bar{\zeta}_1 - \frac{\rho_0}{2} \bar{\chi}_{\mu\mu} \right)^{(j)} + m \bar{\phi}_{\alpha\beta}^{(i)} \left[\delta_{\alpha\beta} \frac{\rho_0}{2} \bar{\zeta}_1 - \frac{\tau_0}{3} \left(\bar{\chi}_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \bar{\chi}_{\mu\mu} \right) \right]^{(j)} \right\} \\
+ \frac{1}{2} \oint_{\Sigma_0} d\Sigma \bar{R}_1^{(j)} \left(\rho_0 \bar{\psi} + \frac{m\tau_0}{3} \bar{\phi}_{\mu\mu} \right)^{(i)} = \delta_{ij}. \quad (66)
\end{aligned}$$

We will now show that the present model satisfies the sum rules m_{-1} , m_1 and m_3 for the electric modes and the sum rules m_1 and m_3 for the

magnetic modes. Let $\hat{D} = \sum_{i=1}^A D(\mathbf{x}_i)$ be an excitation operator. We expand $D(\mathbf{x})$ in the basis of the eigenfunctions $\bar{\psi}^{(j)}(\mathbf{x})$, $D(\mathbf{x}) = \sum_j c_j \bar{\psi}^{(j)}(\mathbf{x}) = \sum_j c_j \psi^{(j)}(\mathbf{x}, 0)$, where in eqs. (62) and (63) all the γ_n are taken to be zero. We require $\psi(\mathbf{x}, 0) = D(\mathbf{x})$, and $\phi_{\alpha\beta}(\mathbf{x}, 0) = 0 = \sum_j c_j \bar{\phi}_{\alpha\beta}^{(j)}(\mathbf{x})$. From the orthogonality relation it follows that

$$c_j = \frac{1}{2} \int_{D_0} d^3x D \left(\mathcal{A}_1 \bar{\varsigma}_1 - \frac{\rho_0}{2} \bar{\chi}_{\mu\mu} \right)^{(j)} + \frac{1}{2} \oint_{\Sigma_0} d\Sigma D \rho_0 \bar{R}_1^{(j)}. \quad (67)$$

The coefficient c_j represents the transition amplitude $\langle j | D | 0 \rangle$.

If we now consider eq. (30) and replace $\delta\psi$ and ψ by D we find

$$\begin{aligned} T^{(2)}[D, 0] &\equiv \int_{D_0} d^3x \frac{\rho_0}{2m} (\nabla D) \cdot (\nabla D) \\ &= \frac{1}{2} \left[\int_{D_0} d^3x D \left(\mathcal{A}_1 \dot{\varsigma}_1 - \frac{\rho_0}{2} \dot{\chi}_{\mu\mu} \right) + \oint_{\Sigma_0} d\Sigma D \rho_0 \dot{R}_1 \right]_{t=0}, \end{aligned} \quad (68)$$

where $\dot{\varsigma}_1|_{t=0} = (\sum_j c_j \dot{\varsigma}_1^{(j)})_{t=0}$, $\dot{\chi}_{\alpha\beta}|_{t=0} = (\sum_j c_j \dot{\chi}_{\alpha\beta}^{(j)})_{t=0}$ and $\dot{R}_1|_{t=0} = (\sum_j c_j \dot{R}_1^{(j)})_{t=0}$. Finally using the expansions for D and for the variational fields and the orthogonality relation (66) we obtain the energy weighted sum rule m_1 ,

$$\begin{aligned} T^{(2)}[D, 0] &= \sum_{ij} c_i c_j \frac{1}{2} \left\{ \int_{D_0} d^3x \left[m \bar{\phi}_{\alpha\beta}^{(i)} \left(\delta_{\alpha\beta} \frac{\rho_0}{2} \dot{\varsigma}_1 - \frac{\tau_0}{3} (\dot{\chi}_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \dot{\chi}_{\mu\mu}) \right)^{(j)} \right. \right. \\ &\quad \left. \left. + \bar{\psi}^{(i)} \left(\mathcal{A}_1 \dot{\varsigma}_1 - \frac{\rho_0}{2} \dot{\chi}_{\mu\mu} \right)^{(j)} \right] + \oint_{\Sigma_0} d\Sigma \dot{R}_1^{(j)} \left(\rho_0 \bar{\psi} + \frac{m\tau_0}{3} \bar{\phi}_{\mu\mu} \right)^{(i)} \right\}_{t=0} = c_0^2 + \sum_j \omega_j c_j^2. \end{aligned} \quad (69)$$

We now consider the system perturbed by an external field $D(\mathbf{x})$. The perturbed energy E' is

$$E' = E + \int_{D_0} d^3x \left(\rho_0 + \mathcal{A}_1 \varsigma_1 - \frac{\rho_0}{2} \chi_{\mu\mu} \right) D(\mathbf{x}) + \oint_{\Sigma_0} d\Sigma R_1 \rho_0 D(\mathbf{x}). \quad (70)$$

In order to determine the fields R_1 , ς_1 and $\chi_{\alpha\beta}$ we minimize $E' - \mu' A - TS$, so that for $\mu' = \mu + \mu_1$ we have

$$\begin{aligned} &\delta \left\{ \int_{D_0} d^3x \left(\rho_0 + \mathcal{A}_1 \varsigma_1 - \frac{\rho_0}{2} \chi_{\mu\mu} \right) (D - \mu_1) \right. \\ &\quad \left. + \oint_{\Sigma_0} d\Sigma R_1 \rho_0 (D - \mu_1) + \mathcal{E}^{(2)}[\varsigma_1, R_1, \chi_{\alpha\beta}] \right\} = 0. \end{aligned} \quad (71)$$

Allowing for arbitrary variations of ς_1 we find

$$-\mathcal{A}_1 (D - \mu_1) = (1 + \mathcal{A}_1 \mathcal{C}_3) \left(\mathcal{A}_1 \varsigma_1 - \frac{1}{2} \rho_0 \chi_{\nu\nu} \right), \quad (72)$$

from arbitrary variations of $\chi_{\alpha\beta}$ it follows

$$\delta_{\alpha\beta} \frac{\rho_0}{2} (D - \mu_1) = -\delta_{\alpha\beta} \frac{\rho_0}{2} \left[\varsigma_1 + \mathcal{C}_3 \left(\mathcal{A}_1 \varsigma_1 - \frac{\rho_0}{2} \chi_{\nu\nu} \right) \right] + \frac{\tau_0}{3} \left(\chi_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \chi_{\mu\mu} \right), \quad (73)$$

and finally considering arbitrary variations of R_1 we obtain

$$\left\{ \rho_0 (D - \mu_1) + \frac{\sigma R_1}{R_0^2} [\ell(\ell + 1) - 2] \right\}_{r=R_0} = 0. \quad (74)$$

From eqs. (72) and (73) we find $\chi_{\alpha\beta} = 0$ and so one has

$$D - \mu_1 = -\varsigma_1 (1 + \mathcal{A}_1 \mathcal{C}_3). \quad (75)$$

The Lagrange multiplier μ_1 is only imposed for $\ell = 0$, due to the orthogonality properties of the spherical harmonics. Then it follows from eq. (74) that the external field D should be zero at the boundary for $\ell = 1$ and, if $\sigma = 0$, also for $\ell > 1$. As was mentioned in ref. [4] an operator $D(\mathbf{x})$, which is zero at the surface, does not have the capacity of exciting pure surface modes. These modes occur at $\omega = 0$ for $\ell = 1$ and also for $\ell > 1$ if $\sigma = 0$ (see ref. [10]). We will now consider the following initial conditions $\chi_{\alpha\beta}(\mathbf{x}, 0) = 0$ and $R_1(\theta, \phi, 0)$ and $\varsigma_1(\mathbf{x}, 0)$ given by eqs. (74) and (75). Then, taking into account eqs. (39) and (41) we write, for $t=0$,

$$\begin{aligned} \mathcal{E}^{(2)}[\varsigma_1, R_1, \chi_{\alpha\beta}] &= \frac{1}{2} \int_{D_0} d^3x \mathcal{A}_1 (1 + \mathcal{A}_1 \mathcal{C}_3) \varsigma_1^2 + \frac{1}{2} \sigma \oint_{\Sigma_0} d\Sigma \frac{R_1^2}{R_0^2} [\ell(\ell + 1) - 2] \\ &= -\frac{1}{2} \left[\int_{D_0} d^3x \varsigma_1 \left(\mathcal{A}_1 \dot{\psi} + \frac{m\rho_0}{2} \dot{\phi}_{\mu\mu} \right) + \oint_{\Sigma_0} d\Sigma R_1 \left(\rho_0 \dot{\psi} + \frac{m\tau_0}{3} \dot{\phi}_{\mu\mu} \right) \right]_{t=0}, \end{aligned} \quad (76)$$

where $\varsigma_1 = \varsigma_1(\mathbf{x}, 0) = \sum_j g_j \bar{\varsigma}_1^{(j)}$, $R_1 = R_1(\theta, \phi, 0) = \sum_j g_j \bar{R}_1^{(j)}$, $\chi_{\alpha\beta} = 0 = \chi_{\alpha\beta}(\mathbf{x}, 0) = \sum_j g_j \bar{\chi}_{\alpha\beta}^{(j)}$, $\dot{\psi}(\mathbf{x}, 0) = \sum_j g_j \dot{\psi}^{(j)}(\mathbf{x}, 0)$, and $\dot{\phi}_{\alpha\beta}(\mathbf{x}, 0) = \sum_j g_j \dot{\phi}_{\alpha\beta}^{(j)}(\mathbf{x}, 0)$, and all the γ_n are taken equal to $\pi/2$ in eqs. (62) and (63). The coefficients g_j are

$$g_j = \frac{1}{2} \int_{D_0} d^3x \varsigma_1 \left(\mathcal{A}_1 \bar{\psi} + \frac{m\rho_0}{2} \bar{\phi}_{\alpha\alpha} \right)^{(j)} + \frac{1}{2} \oint_{\Sigma_0} d\Sigma R_1 \left(\rho_0 \bar{\psi} + \frac{m\tau_0}{3} \bar{\phi}_{\mu\mu} \right)^{(j)}. \quad (77)$$

Using the expansions of the variational fields and eqs. (39), (41) and (66) it follows that

$$\begin{aligned} \mathcal{E}^{(2)}[\varsigma_1, R_1, 0] &= \frac{1}{2} \sum_{ij} \omega_i g_i g_j \left\{ \int_{D_0} d^3x \left[\bar{\psi}^{(i)} \left(\mathcal{A}_1 \bar{\varsigma}_1 - \frac{\rho_0}{2} \bar{\chi}_{\mu\mu} \right)^{(j)} \right. \right. \\ &\quad \left. \left. + m \bar{\phi}_{\alpha\beta}^{(i)} \left(\delta_{\alpha\beta} \frac{\rho_0}{2} \bar{\varsigma}_1 - \frac{\tau_0}{3} (\bar{\chi}_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \bar{\chi}_{\mu\mu}) \right)^{(j)} \right] \right. \\ &\quad \left. + \oint_{\Sigma_0} d\Sigma \bar{R}_1^{(j)} \left(\rho_0 \bar{\psi} + \frac{m\tau_0}{3} \bar{\phi}_{\mu\mu} \right)^{(i)} \right\} = \sum_j \omega_j g_j^2. \end{aligned} \quad (78)$$

The coefficients g_n are related with the coefficients c_n , obtained from the expansion of $D(\mathbf{x})$ on the basis of eigenfunctions $\bar{\psi}^{(j)}(\mathbf{x})$ ($D(\mathbf{x}) = \sum_j c_j \bar{\psi}^{(j)}(\mathbf{x})$),

$$\begin{aligned}
-\omega_k g_k &= g_k \frac{1}{2} \left\{ \int_{D_0} d^3x \left[\dot{\psi}^{(k)} \left(\mathcal{A}_1 \bar{\varsigma}_1 - \frac{\rho_0}{2} \bar{\chi}_{\mu\mu} \right)^{(k)} \right. \right. \\
&\quad \left. \left. + m \dot{\phi}_{\alpha\beta}^{(k)} \left(\delta_{\alpha\beta} \frac{\rho_0}{2} \bar{\varsigma}_1 - \frac{\tau_0}{3} (\bar{\chi}_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \bar{\chi}_{\mu\mu}) \right)^{(k)} \right] \right. \\
&\quad \left. + \oint_{\Sigma_0} d\Sigma \bar{R}_1^{(k)} \left(\rho_0 \dot{\psi} + \frac{m\tau_0}{3} \dot{\phi}_{\mu\mu} \right)^{(k)} \right\}_{t=0} \\
&= \frac{1}{2} \left\{ \int_{D_0} d^3x \left[\dot{\psi} \left(\mathcal{A}_1 \bar{\varsigma}_1 - \frac{\rho_0}{2} \bar{\chi}_{\mu\mu} \right)^{(k)} + m \dot{\phi}_{\alpha\beta} \left(\delta_{\alpha\beta} \frac{\rho_0}{2} \bar{\varsigma}_1 - \frac{\tau_0}{3} (\bar{\chi}_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \bar{\chi}_{\mu\mu}) \right)^{(k)} \right] \right. \\
&\quad \left. + \oint_{\Sigma_0} d\Sigma \bar{R}_1^{(k)} \left(\rho_0 \dot{\psi} + \frac{m\tau_0}{3} \dot{\phi}_{\mu\mu} \right) \right\}_{t=0} \\
&= \frac{1}{2} \left\{ \int_{D_0} d^3x \left[\bar{\varsigma}_1^{(k)} \left(\mathcal{A}_1 \dot{\psi} + \frac{m\rho_0}{2} \dot{\phi}_{\mu\mu} \right) - \bar{\chi}_{\alpha\beta}^{(k)} \left(\delta_{\alpha\beta} \frac{\rho_0}{2} \dot{\psi} + \frac{m\tau_0}{3} (\dot{\phi}_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \dot{\phi}_{\mu\mu}) \right) \right] \right. \\
&\quad \left. + \oint_{\Sigma_0} d\Sigma \bar{R}_1^{(k)} \left(\rho_0 \dot{\psi} + \frac{m\tau_0}{3} \dot{\phi}_{\mu\mu} \right) \right\}_{t=0} \\
&= -\frac{1}{2} \left\{ \int_{D_0} d^3x (1 + \mathcal{A}_1 \mathcal{C}_3) \varsigma_1 \left(\mathcal{A}_1 \bar{\varsigma}_1 - \frac{\rho_0}{2} \bar{\chi}_{\mu\mu} \right)^{(k)} + \oint_{\Sigma_0} d\Sigma \bar{R}_1^{(k)} \frac{\sigma R_1}{R_0^2} [\ell(\ell+1) - 2] \right\} = c_k, \quad (79)
\end{aligned}$$

where eqs. (66), (39)-(41), (74) and (75) were taken into account. Using eqs. (78) and (79) we can also write the m_{-1} sum rule as follows

$$\mathcal{E}^{(2)}[\varsigma_1, R_1, 0] = \sum_j \frac{c_j^2}{\omega_j}, \quad (80)$$

where ς_1 and R_1 are related to D through eqs. (74) and (75) and the Lagrange multiplier μ_1 applies only to $\ell = 0$ being evaluated with the help of eqs. (74) and (75) and of the subsidiary condition

$$\int_{D_0} d^3x \left(\mathcal{A}_1 \varsigma_1 - \frac{\rho_0}{2} \chi_{\nu\nu} \right) + \oint_{\Sigma_0} d\Sigma R_1 \rho_0 = 0, \quad (81)$$

which assumes the conservation of the particle number.

We will now be concerned with the sum rule m_3 for the electric modes. We consider the initial conditions $\varsigma_1(\mathbf{x}, 0) = 0 = \sum_j d_j \bar{\varsigma}_1^{(j)}$, $R_1(\theta, \phi, 0) = 0 = \sum_j d_j \bar{R}_1^{(j)}$ and $\chi_{\alpha\beta}(\mathbf{x}, 0) = \partial_\alpha F_\beta + \partial_\beta F_\alpha = \sum_j d_j \bar{\chi}_{\alpha\beta}^{(j)}$ and we put all the coefficients γ_n , in eqs. (62) and (63), equal to $\pi/2$. Then, from eq. (33), if we replace $\delta\chi_{\alpha\beta}$ and $\chi_{\alpha\beta}$, by $\chi_{\alpha\beta}(\mathbf{x}, 0)$, we get

$$\begin{aligned}
&\int_{D_0} d^3x \chi_{\alpha\beta} \left\{ \frac{\tau_0}{3} \left[m \dot{\phi}_{\alpha\beta} - \chi_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} (m \dot{\phi}_{\gamma\gamma} - \chi_{\gamma\gamma}) \right] \right. \\
&\quad \left. + \delta_{\alpha\beta} \frac{\rho_0}{2} \left[\dot{\psi} + \varsigma_1 + \mathcal{C}_3 \left(\mathcal{A}_1 \varsigma_1 - \frac{\rho_0}{2} \chi_{\mu\mu} \right) \right] \right\}_{t=0} = 0, \quad (82)
\end{aligned}$$

where $\dot{\psi}|_{t=0} = \sum_j d_j \dot{\psi}^{(j)}(\mathbf{x}, 0)$, and $\dot{\phi}_{\alpha\beta}|_{t=0} = \sum_j d_j \dot{\phi}_{\alpha\beta}^{(j)}(\mathbf{x}, 0)$. Making use of the previous expansions and of the orthogonality condition it follows that

$$d_k = -\frac{1}{2} \int_{D_0} d^3x \left[\bar{\psi}^{(k)} \frac{\rho_0}{2} \chi_{\mu\mu} + \frac{m\tau_0}{3} \bar{\phi}_{\alpha\beta}^{(k)} \left(\chi_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \chi_{\mu\mu} \right) \right]_{t=0}. \quad (83)$$

From eq. (82) we have

$$\begin{aligned} \mathcal{E}^{(2)}[0, 0, \chi_{\alpha\beta}] &\equiv \frac{1}{2} \int_{D_0} d^3x \chi_{\alpha\beta} \left[\delta_{\alpha\beta} \frac{\rho_0^2 \mathcal{C}_3}{4} \chi_{\mu\mu} + \frac{\tau_0}{3} \left(\chi_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \chi_{\mu\mu} \right) \right]_{t=0} \\ &= \frac{1}{2} \int_{D_0} d^3x \chi_{\alpha\beta} \left[\delta_{\alpha\beta} \frac{\rho_0}{2} \dot{\psi} + \frac{m\tau_0}{3} \left(\dot{\phi}_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \dot{\phi}_{\gamma\gamma} \right) \right]_{t=0} \\ &= -\frac{1}{2} \sum_j d_j d_i \left\{ \int_{D_0} d^3x \left[m \dot{\phi}_{\alpha\beta}^{(j)} \left(\delta_{\alpha\beta} \frac{\rho_0}{2} \bar{\varsigma}_1 - \frac{\tau_0}{3} (\bar{\chi}_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \bar{\chi}_{\gamma\gamma}) \right) \right]^{(i)} \right. \\ &\quad \left. + \dot{\psi}^{(j)} \left(\mathcal{A}_1 \bar{\varsigma}_1 - \frac{\rho_0}{2} \bar{\chi}_{\mu\mu} \right) \right]_{t=0} + \oint_{\Sigma_0} d\Sigma \bar{R}_1^{(i)} \left(\rho_0 \dot{\psi} + \frac{m\tau_0}{3} \dot{\phi}_{\mu\mu} \right)^{(j)} \Big\} = \sum_j \omega_j d_j^2. \end{aligned} \quad (84)$$

If we replace $\mathbf{F}$ by ∇D and express d_j in terms of c_j we obtain the m_3 sum rule. In fact

$$\begin{aligned} d_i &= -\frac{1}{2} \int_{D_0} d^3x D \left[\rho_0 \Delta \bar{\psi} + \frac{2m\tau_0}{3} \left(\frac{1}{2} \Delta \bar{\phi}_{\nu\nu} + \partial_\alpha \partial_\beta \bar{\phi}_{\alpha\beta} \right) \right]^{(i)} \\ &\quad + \frac{1}{2} \oint_{\Sigma_0} d\Sigma D n_\alpha \left[\rho_0 \partial_\alpha \bar{\psi} + \frac{2m\tau_0}{3} \left(\frac{1}{2} \partial_\alpha \bar{\phi}_{\mu\mu} + \partial_\gamma \bar{\phi}_{\alpha\gamma} \right) \right]^{(i)} \\ &\quad - \frac{1}{2} \oint_{\Sigma_0} d\Sigma (\partial_\alpha D) n_\beta \left[\delta_{\alpha\beta} \rho_0 \bar{\psi} + \frac{2m\tau_0}{3} \left(\delta_{\alpha\beta} \frac{1}{2} \bar{\phi}_{\mu\mu} + \bar{\phi}_{\alpha\beta} \right) \right]^{(i)}. \end{aligned} \quad (85)$$

Taking into account the boundary condition (20) and requiring $\partial_r D|_{r=R_0} = 0$, the last integral in the r.h.s. of eq (85) vanishes and, therefore, from eqs. (35) and (36) it follows that $d_i = m\omega_i c_i$, where c_i is given by eq. (67). We then have, from eq. (84), the m_3 sum rule (electric modes) expressed as

$$\frac{1}{m^2} \mathcal{E}^{(2)}[0, 0, \chi_{\alpha\beta}] = \sum_j \omega_j^3 c_j^2, \quad (86)$$

where $\chi_{\alpha\beta} = 2\partial_\alpha \partial_\beta D$. We remark that eq. (84) may also be regarded as the m_1 sum rule corresponding to the transition operator $\frac{1}{2}(\mathbf{p} \cdot \nabla D + (\nabla D) \cdot \mathbf{p})$. Then d_j is the corresponding transition amplitude.

Finally we will be concerned with eq. (31). We consider the initial conditions $\psi(\mathbf{x}, 0) = 0 = \sum_j b_j \bar{\psi}^{(j)}$, $\phi_{\alpha\beta}(\mathbf{x}, 0) = \partial_\alpha F_\beta + \partial_\beta F_\alpha = \sum_j b_j \bar{\phi}_{\alpha\beta}^{(j)}$, where $x_\alpha (\partial_\alpha F_\beta + \partial_\beta F_\alpha)|_{r=R_0} = 0$, and all the γ_n are taken equal to 0 in eqs. (62) and (63). The coefficients b_n are given by

$$b_n = \frac{1}{2} \int_{D_0} d^3x m \phi_{\alpha\beta} \left[\delta_{\alpha\beta} \frac{\rho_0}{2} \bar{\varsigma}_1 - \frac{\tau_0}{3} \left(\bar{\chi}_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \bar{\chi}_{\mu\mu} \right) \right]^{(n)} + \frac{1}{2} \oint d\Sigma \bar{R}_1^{(n)} \frac{m\tau_0}{3} \phi_{\alpha\alpha}. \quad (87)$$

We now insert in eq. (31) the initial conditions replacing $\delta\phi_{\alpha\beta}$ and $\phi_{\alpha\beta}$ by $\phi_{\alpha\beta}(\mathbf{x}, 0) = \partial_\alpha F_\beta + \partial_\beta F_\alpha$ and replacing ψ by 0,

$$\begin{aligned}
T^{(2)}[0, \phi_{\alpha\beta}] &= \frac{1}{2} \left\{ - \int_{D_0} d^3x \phi_{\alpha\beta} \frac{2mG}{15} [\delta_{\alpha\beta} \left(\frac{1}{2} \Delta \phi_{\mu\mu} + \partial_\mu \partial_\nu \phi_{\nu\mu} \right) \right. \\
&\quad \left. + \Delta \phi_{\alpha\beta} + \partial_\beta \partial_\alpha \phi_{\mu\mu} + 2\partial_\alpha \partial_\gamma \phi_{\gamma\beta} + 2\partial_\beta \partial_\gamma \phi_{\gamma\alpha} \right] \\
&+ \oint_{\Sigma_0} d\Sigma \phi_{\alpha\beta} \frac{2mG}{15} \left[\delta_{\alpha\beta} n_\gamma \left(\frac{1}{2} \partial_\gamma \phi_{\mu\mu} + \partial_\mu \phi_{\mu\gamma} \right) + n_\gamma (\partial_\alpha \phi_{\beta\gamma} + \partial_\beta \phi_{\gamma\alpha} + \partial_\gamma \phi_{\alpha\beta}) \right] \Big\}_{t=0} \\
&= \frac{1}{2} \left\{ \int_{D_0} d^3x \phi_{\alpha\beta} \left[\delta_{\alpha\beta} \frac{m\rho_0}{2} \dot{\zeta}_1 - \frac{m\tau_0}{3} \left(\delta_{\alpha\beta} \frac{1}{2} \dot{\chi}_{\nu\nu} + \dot{\chi}_{\alpha\beta} \right) \right] + \oint_{\Sigma_0} d\Sigma \frac{m\tau_0}{3} \phi_{\alpha\alpha} \dot{R}_1 \right\}_{t=0} \\
&= \frac{1}{2} \sum_{ij} b_j b_i \left\{ \int_{D_0} d^3x \left[\bar{\psi}^{(i)} \left(\mathcal{A}_1 \dot{\zeta}_1 - \frac{\rho_0}{2} \dot{\chi}_{\mu\mu} \right)^{(j)} \right. \right. \\
&\quad \left. \left. + m \bar{\phi}_{\alpha\beta}^{(i)} \left(\delta_{\alpha\beta} \frac{\rho_0}{2} \dot{\zeta}_1 - \frac{\tau_0}{3} (\bar{\chi}_{\alpha\beta} + \delta_{\alpha\beta} \frac{1}{2} \dot{\chi}_{\mu\mu}) \right)^{(j)} \right] \right. \\
&\quad \left. + \oint_{\Sigma_0} d\Sigma \dot{R}_1^{(j)} \left(\rho_0 \bar{\psi} + \frac{m\tau_0}{3} \bar{\phi}_{\mu\mu} \right)^{(i)} \right\}_{t=0} = b_0^2 + \sum_i \omega_i b_i^2. \quad (88)
\end{aligned}$$

where b_0^2 corresponds to zero energy modes, $\dot{\zeta}_1(\mathbf{x}, 0) = \sum_j b_j \dot{\zeta}_1^{(j)}(\mathbf{x}, 0)$, $\dot{R}_1(\theta, \phi, 0) = \sum_j b_j \dot{R}_1^{(j)}(\theta, \phi, 0)$, and $\dot{\chi}_{\alpha\beta}(\mathbf{x}, 0) = \sum_j b_j \dot{\chi}_{\alpha\beta}^{(j)}(\mathbf{x}, 0)$. Eq. (88) is a sum rule which applies both to electric and magnetic modes depending on the choice of $\phi_{\alpha\beta}$. As we will see in the case of magnetic modes eq. (88) leads directly to the m_3 sum rule.

We will now be concerned with the magnetic-type excitations, involving only the convection current. For this kind of excitations we have no scalar quantities. The appropriate multipole transition operator has the form

$$\widehat{\mathbf{F} \cdot \mathbf{p}} = \frac{1}{2} \sum_{i=1}^A (\mathbf{F}(\mathbf{x}_i) \cdot \mathbf{p}_i + \mathbf{p}_i \cdot \mathbf{F}(\mathbf{x}_i)), \quad (89)$$

where $\mathbf{F}(\mathbf{x})$ is now a given local transverse field satisfying $\nabla \cdot \mathbf{F} = 0$. In this case we still consider eqs. (82)-(84) and we note that the coefficients d_k (see eq. (83)) may now be written as

$$d_k = -\frac{1}{2} \int_{D_0} d^3x \frac{m\tau_0}{3} (\partial_\alpha F_\beta + \partial_\beta F_\alpha) \bar{\phi}_{\alpha\beta}^{(k)} = \frac{1}{2} m \left(\int_{D_0} d^3x \mathbf{F} \cdot \mathbf{j}^{(k)} \right)_{t=0}, \quad (90)$$

where we took into account the boundary condition (20). It is then clear that eq. (84) leads directly to the m_1 sum rule for magnetic modes

$$\mathcal{E}^{(2)}[0, 0, \chi_{\alpha\beta}] = \sum_j \omega_j d_j^2. \quad (91)$$

where $\chi_{\alpha\beta} = \partial_\alpha F_\beta + \partial_\beta F_\alpha$. The coefficients d_j represent the transition amplitudes $\langle j | \widehat{\mathbf{F} \cdot \mathbf{p}} | 0 \rangle$.

Finally we will consider the m_3 sum rule for magnetic modes. We start from the initial conditions $\phi_{\alpha\beta}(\mathbf{x}, 0) = \partial_\alpha F_\beta + \partial_\beta F_\alpha = \sum_j b_j \bar{\phi}_{\alpha\beta}^{(j)}$, where $x_\alpha(\partial_\alpha F_\beta + \partial_\beta F_\alpha)|_{r=R_0} = 0$, and all the γ_n are taken equal to 0 in eqs. (62) and (63). From eq. (87) we get

$$\begin{aligned} b_n &= -\frac{1}{2} \int_{D_0} d^3x \frac{m\tau_0}{3} \phi_{\alpha\beta} \bar{\chi}_{\alpha\beta}^{(n)} = \omega_n \frac{1}{2} \int_{D_0} d^3x \frac{m^2\tau_0}{3} \phi_{\alpha\beta} \bar{\phi}_{\alpha\beta}^{(n)} \\ &= -\omega_n m^2 \frac{1}{2} \left(\int_{D_0} d^3x \mathbf{F} \cdot \mathbf{j}^{(n)} \right)_{t=0} = -m\omega_n d_n, \end{aligned} \quad (92)$$

where we took into account the relation $\bar{\chi}_{\alpha\beta}^{(n)} = -m\omega_n \bar{\phi}_{\alpha\beta}^{(n)}$. It then follows from eq. (88) the m_3 sum rule for magnetic modes,

$$\begin{aligned} \frac{1}{m^2} T^{(2)}[0, \phi_{\alpha\beta}] &= \frac{1}{m^2} \left[- \int_{D_0} d^3x \frac{2mG}{15} \phi_{\alpha\beta} (\Delta \phi_{\alpha\beta} + 2\partial_\alpha \partial_\gamma \phi_{\gamma\beta} + 2\partial_\beta \partial_\gamma \phi_{\gamma\alpha}) \right. \\ &\quad \left. + \oint_{\Sigma_0} d\Sigma \frac{2mG}{15} \phi_{\alpha\beta} n_\gamma (\partial_\alpha \phi_{\beta\gamma} + \partial_\beta \phi_{\gamma\alpha} + \partial_\gamma \phi_{\alpha\beta}) \right] = \sum_i \omega_i^3 d_i^2. \end{aligned} \quad (93)$$

From eq. (42) and assuming eqs.(62) and (63) it follows that for magnetic modes $b_0 = 0$ since $\bar{\chi}_{\alpha\beta} = 0$.

4. Numerical results

We consider a potential energy density of the form previously used in refs. [1, 2, 3, 8],

$$u = a_2 \rho^2 + a_{2+1/6} \rho^{2+1/6}, \quad (94)$$

where

$$a_2 = -\frac{3}{8} \times 3075.8 \text{ MeV fm}^3, \quad (95)$$

and

$$a_{2+1/6} = \frac{1}{16} \times 20216.4 \text{ MeV fm}^{3+\frac{1}{6}}. \quad (96)$$

We additionally consider a temperature dependent surface tension. Since the surface of the nucleus becomes more diffuse when the temperature increases, we assume that the surface tension σ decreases with temperature according to $\sigma(T) = \sigma(0)(1 - xT^2)$ (see ref. [18]), where $\sigma(0) = 1.0 \text{ MeV fm}^{-2}$ and $x = 6 \times 10^{-3} \text{ MeV}^{-2}$.

We evaluate quantities involving integrals of the Fermi distribution function in momentum space, such as ρ_0 , τ_0 and s_0 , using an expansion in powers of T and taking into account the terms up to T^4 . In the present paper we are mainly interested in extending the results of refs. [9, 10] to a finite temperature including the effect of the surface tension. We arbitrarily consider a

nucleus with $A=208$. In table 1 we give, for different values of the temperature, the equilibrium values of ρ_0 , τ_0 , ς_0 , μ , $E^{(0)}/A$, of the surface energy and of the entropy for the system with $A=208$.

Since the equilibrium state fulfills the stability condition that the free energy is a minimum, it follows that the frequencies of the eigenmodes are real.

For the electric modes we consider the operators

$$D(\mathbf{x}) = r^2 \text{ for } \ell = 0, \quad D(\mathbf{x}) = (r - r^3/R_0^2)Y_{10} \text{ for } \ell = 1, \\ \text{and } D(\mathbf{x}) = r^\ell Y_{\ell 0} \text{ for } \ell > 1. \quad (97)$$

The results presented in table 2 were obtained using this set of parameters and give us the energies (in MeV) and the percentages of the sum rule m_1 corresponding to different electric eigenmodes for a nucleus with $A=208$. For comparison we present in table 3 the results obtained using the model of ref. [4] and the same force parameters.

For electric modes the flow field may be decomposed in vector spherical harmonics,

$$\mathbf{j}(\mathbf{x}) = j_+(r)\mathbf{Y}_{\ell,\ell+1,0} + j_-(r)\mathbf{Y}_{\ell,\ell-1,0}. \quad (98)$$

We also consider the functions $j_{div}(r)$ and $j_{curl}(r)$ defined by the equations,

$$\nabla \cdot \mathbf{j}(\mathbf{x}) = j_{div}(r)Y_{\ell 0}, \quad (99)$$

$$\nabla \times \mathbf{j}(\mathbf{x}) = ij_{curl}(r)\mathbf{Y}_{\ell\ell 0}. \quad (100)$$

The function j_{div} is related to the transition density $\delta\rho$ ($\dot{\rho} = -\nabla \cdot \mathbf{j}$) and the function j_{curl} is also interesting because it indicates how important are the transverse components in the current.

For $\ell = 0$ and $T=0$ we observe that 95.15% of the m_1 sum is concentrated at 15.87 MeV, while in the simpler model of ref. [4] we obtain 92.69% of the m_1 sum located at 14.63 MeV. At a finite temperature we see that new eigenmodes appear, 0_1^+ , 0_4^+ , 0_8^+ ... The new vibrational mode 0_1^+ at $T=5$ MeV absorbs 23.13% of m_1 . The eigenmode 0_3^+ at $T=0$ MeV exhausts 2.25% of m_1 and at $T=5$ MeV it exhausts 6.26% of m_1 . So it is clear that the strength becomes more distributed for $\ell = 0$ when we increase the temperature. In figs. 1 and 2 we plot the functions j_+ (continuous lines) and $-j_{div}$ (dotted lines) for different temperatures for the eigenmodes 0_1^+ and 0_2^+ . The curves are normalized according to eq. (66). We remark that, for the new eigenmodes, the amplitude of the current, as fixed by the normalization condition (66), decreases when the temperature decreases and for small values of the temperature it is proportional to the temperature. Although in fig. 1 we are not considering small values of T , we can clearly see that the amplitude of

the current increases with the temperature. Naturally the particle number is conserved. Therefore eq. (81) is verified for every eigenmode as a consequence of the continuity equation (eqs. (35) and (36)). For $\ell \neq 0$ eq. (81) follows automatically from the orthogonality property of the spherical harmonics. For $\ell = 0$, $\delta\rho$ has to be zero at least at one point. In figs. 1 and 2 we always have j_{div} positive. It should be taken into account that there is always a displacement of the surface represented by R_1 (radial displacement of the nuclear boundary) which is not represented in the figures 1-8.

For $\ell^\pi = 1^-$, we note that large fractions of m_1 are, in the present model, concentrated in the modes 1_4^- , 1_7^- and 1_9^- . The states 1_2^- , 1_5^- , and 1_8^- correspond to mathematical solutions of the eigenvalue problem where $\phi_{\alpha\beta} = \phi_{\alpha\beta}^{(2)}(c_T^{(2)})$ and therefore, since $\partial_\alpha \phi_{\alpha\beta}^{(2)} = 0$, the transition density and the transition current for these modes are zero. This is the reason why these eigenmodes do not exhaust any fraction of the sum rule m_1 , as may be seen in table 2. The norm of these states, given by eq. (66), is equal to zero and therefore they do not correspond to physical states. In fig. 3 we present the flow field of the eigenmode 1_9^- which corresponds to the giant squeezing dipole.

For $\ell^\pi = 2^+$, we note that most of the m_1 sum is concentrated in the state 2_2^+ , which corresponds to the giant quadrupole resonance, and in the low-lying mode 2_1^+ . In figs. 4 and 5 we give the current for the eigenmodes 2_1^+ and 2_2^+ . We see that the low-lying mode 2_1^+ is a surface mode, since j_{div} is rather close to zero in the interior of the nucleus, and we also note that, from the behaviour of j_{curl} , it is clear that it has strong transverse components.

For $\ell^\pi = 3^-$, the m_1 sum is basically distributed through several modes, but still the two states exhausting a larger fraction of m_1 , at $T=0$ MeV, are 3_1^- (fig. 6) which is the low-lying mode, and exhausts 34.10% of m_1 at $T=0$ MeV, and 3_3^- (fig. 7) which exhausts 43.44% of the m_1 sum at $T=0$ MeV and corresponds to the giant octupole resonance (GOR). It is however very interesting to note that at $T=5$ MeV this picture has changed considerably. While we still have the low-lying state 3_1^- , which at $T=5$ MeV exhausts 40.95% of the m_1 sum, the strength corresponding to the GOR is now quite spread since the mode 3_3^- exhausts only 27.93% of the m_1 sum at $T=5$ MeV.

For $\ell^\pi = 4^+$ and $\ell^\pi = 5^-$ we see that the strength is distributed in a larger number of states being the low-lying mode the one which exhausts a larger fraction of the m_1 sum rule. We note that at a finite temperature the new modes 4_8^+ and 5_8^- appear. In particular at $T=5$ MeV the mode 5_8^- exhausts 20.93% of the m_1 sum. In fig. 8 we present the flow field for the low-lying 4_1^+ eigenmode. It is clear, from figs. 4, 6 and 8, that the transverse components for the low-lying modes are important, although they decrease when the temperature increases, and that these modes are surface modes, since the transition density (dotted curve) is small inside the nucleus. In table 4

we present, for different eigenmodes, the constants $a_1^{(T)}$ and $a_2^{(T)}$ appearing in eq. (58) and which are normalized according to the eq. (66). The reason why we present table 4 is that it clearly indicates that the transverse components are very important for the low-lying modes since for these modes $a_1^{(T)}$ and $a_2^{(T)}$ (see eq. (58)) assume much larger values than for the other higher eigenmodes. It follows therefore that for low-lying modes the contribution of the transverse components $\phi_{\alpha\beta}^{(1)}(c_1^{(T)})$ and $\phi_{\alpha\beta}^{(2)}(c_2^{(T)})$ is very important. Thus we find strong transverse components in the transition currents corresponding to the low-lying modes. In particular we note that the flow fields for the modes 3_1^- and 4_1^+ are similar to the corresponding flow fields presented in figures 6 and 8 of ref. [13].

We see that when the temperature increases there is a spreading of the strength corresponding to the giant resonances and that large fractions of the m_1 sum are concentrated at the energies of the low-lying modes 2_1^+ , 3_1^- , 4_1^+ , 5_1^- . The states 3_1^- , 4_1^+ and 5_1^- have energies which are approximately proportional to $A^{-1/2}$ as is also the case in the simpler model of ref. [4], but now have lower energies and strong transverse components in the current. However for $\ell^\pi = 2^+$ we were not able to find an analogous state, since the state 2_1^+ , as well as the other states, have an energy which is approximately proportional to $A^{-1/3}$. In order to make clear the dependence on A of the energies of the eigenmodes we plot in figs. 9 ($T=1$ MeV) and 10 ($T=5$ MeV) the energies of the eigenmodes 0_1^+ , 0_2^+ , 1_3^- , 1_4^- , 2_1^+ , 2_2^+ , 3_2^- , 4_2^+ and 5_2^- multiplied by $A^{1/3}$ and the energies of the eigenmodes 3_1^- , 4_1^+ and 5_1^- multiplied by $A^{1/2}$. Since the lines plotted in figs. 9 and 10 are approximately parallel to the "A" axis, we may say that the eigenmodes 3_1^- , 4_1^+ and 5_1^- have an energy which is approximately proportional to " $A^{-1/2}$ " while the other states have an energy which is approximately proportional to " $A^{-1/3}$ ".

An interesting feature of this model is that new states appear when we consider a finite temperature. For low temperatures, as the temperature decreases, these states exhaust a smaller fraction of the energy weighted sum rule. Examples of the new states at finite temperature are 0_1^+ , 0_4^+ , 0_8^+ , 1_6^- , 1_{11}^- , 2_6^+ , 3_7^- , 4_8^+ and 5_8^- . In fig. 11 we plot for these states the fractions of the m_1 sum as a function of T^2 . It is clear from fig. 11 that, at low temperatures, the fractions of the m_1 sum corresponding to these modes, are proportional to T^2 . As a consequence the amplitude of the transition density and of the transition current, normalized by eq. (66) are proportional to the temperature T. The energies of these states are stable when we consider the limit $T \rightarrow 0$ as may also be seen in table 5 where we give the values of the energies and the percentages of the m_1 sum for the temperature $T=10^{-2}$ MeV.

For $\ell = 0$ the transverse components play no role and therefore the effect of the rising of a new dispersion relation at a finite temperature is more

pronounced and we observe that the relative increase of the number of vibrational modes is larger. This is clear because for $\ell = 0$ the number of dispersion relations increases from 2 to 3 while for $\ell > 0$ the number of dispersion relations increase from 4 to 5 and so the relative change is smaller.

We remark that the results in figs. 2, 3, 6, 8 for $T=0$ MeV are close to the curves presented in ref. [13]. The model in ref. [13] had the same force parameters, related to the constants a_2 and $a_{2+1/6}$, did not include surface tension and did not allow for the surface displacement through the variable R_1 .

We will now discuss the results obtained for magnetic modes. For magnetic type excitations we consider a multipole transition operator given by eq. (89) where $\mathbf{F} = (r - \frac{7r^3}{5R^2})\mathbf{Y}_{110}$ for $\ell^\pi = 1^+$ and $\mathbf{F} = r^\ell \mathbf{Y}_{\ell\ell 0}$ for $\ell > 1$. In table 6 we present, for different values of the temperature, the energies and the fractions of the m_1 sum rule corresponding to different eigenmodes. It is clear from table 6 that with respect to the magnetic eigenmodes there are no significative changes when the temperature increases from 0 MeV to 5 MeV. As in refs. [2, 3], we see that there is a small increase of the energies of the magnetic modes when the temperature increases and that the percentage of the m_1 sum rule exhausted by the different eigenmodes does not depend on the temperature. When $\ell^\pi = 1^+$ there are some eigensolutions where $\phi_{\alpha\beta} = \phi_{\alpha\beta}^{(2)}$. These modes have no physical meaning since the current is zero because $\partial_\alpha \phi_{\alpha\beta}^{(2)} = 0$ and their norm is zero. Examples of such modes are 1_2^+ , 1_3^+ , 1_5^+ , 1_6^+ , 1_8^+ and 1_9^+ . We note that comparing with other calculations [2, 3] we now find a larger number of eigenmodes owing to the fact that we have a richer parametrization. The mode 1_1^+ , which occurs at zero energy, corresponds to the rigid rotation. Finally we note that, in order to improve the results for the magnetic modes, velocity dependent forces should be considered. In this note we are basically interested in investigating electric isoscalar modes and therefore we have only considered a density dependent interaction. Still we have presented the results for the magnetic modes because these modes also appear as solutions of the present model.

5. Conclusions

The present model predicts an appreciable redistribution of transition strength with temperature. This effect agrees, in tendency, with recent experimental findings [19, 20], but is admittedly much less dramatic. This is not surprising because of the artificial discretization of the excitation spectrum due to the constraints imposed on the trial functions, may tend to hide the redistribution of strength. At $T=3$ MeV the redistribution of the strength in the calculations, compared with $T=0$, is sizeable but not very impressive,

while in the experiments the redistribution is much larger. For example, the giant dipole resonance has at $T=2$ MeV a width which, for spherical nuclei, is about twice as much than the $T=0$ width [21]. RPA calculations also predict very weak dependence on temperature in contradiction with experiments [22]. It is natural to expect that the collision terms, which are not taken into account in the present calculation and in RPA contribute significantly to the actual width.

One of the interesting features of the model we have investigated is that it provides, at a finite temperature, an additional dispersion relation which leads to a larger fragmentation of the strength. The effect of the appearance of new vibrational modes is also present in RPA microscopic calculations due to the fact that two particle and two hole transitions are now accessible, owing to the smooth profile of the Fermi surface. This is not the case at zero temperature, where f_0 is a step function in momentum space. Therefore, at a finite temperature T , the configuration space in a microscopic calculation is much larger. The fact that some of these states only exist at finite temperature means that they originate at the Fermi surface, being in this case crucial the profile of the Fermi surface with a finite width. The information concerning the profile in momentum space of the equilibrium distribution function f_0 is given by its moments ρ_0 , τ_0 and G which are used in the dynamical calculations.

The changes of the distribution function describe particle-hole transitions.

Nuclear fluid-dynamical models may be applied to finite systems without much effort. Other fluid-dynamical models [1, 2, 3, 4, 8, 12, 13] have been previously considered for finite systems. However, the present one, due to its parametrization, provides a more complete description of the nuclear dynamics.

With the present parametrization we are able to disentangle the low-lying modes from the giant resonances. We find that the low-lying modes have strong transverse components and have also a surface character since the transition density is close to zero in the interior of the nucleus as may be seen looking at the dotted curves in figs. 4, 6 and 8. However, the transition density corresponding to the giant resonances ($\ell = 2, 3$) is not at all negligible in the interior of the nucleus as may be seen from the dotted curves in figs. 5 and 7.

It has been shown that the present model fulfills the sum rules m_1 , m_{-1} and m_3 for electric modes and also the sum rules m_1 and m_3 for magnetic modes.

When we compare the results of the tables 2 and 3, we conclude that the use of five variational fields R_1 , ψ , ς_1 , $\phi_{\alpha\beta}$ and $\chi_{\alpha\beta}$, instead of only R_1 , ψ and

ς_1 , leads to a much larger number of vibrational states and eventually also to a larger distribution of the strength. In fact when we only consider the three scalar fields we have very collective states because we are imposing this collectivity through the consideration of a very restrictive dynamical space. If we allow for the five fields R_1 , ψ , ς_1 , $\phi_{\alpha\beta}$ and $\chi_{\alpha\beta}$, we are relaxing our constraints and so in principle the strength can be more distributed. In the formalisms of refs. [1, 4] one has a smaller number of states and one finds that considering an operator of the form $r^\ell Y_{\ell 0}$, more than 99% of the m_1 sum is concentrated in a single state which is associated with the corresponding giant resonance. So the collectivity is somehow imposed due to the restrictive parametrization. We see in our results that for $\ell \geq 4$ the strength is rather distributed, which is not the case for $\ell \leq 3$ when we can clearly speak of giant resonances. We can also see that, with increasing temperature, the strength becomes more distributed. This is clear in table 2 when we compare the results at $T=0$ MeV and at $T=5$ MeV. In particular the fact that at $T=5$ MeV the m_1 sum for $\ell = 3$ is rather spread seems to indicate that at such a high temperature the giant octupole resonance seems not to exist any more. We also observe, as is clear from fig. 11 that the fractions of the m_1 sum corresponding to the states which appear at finite temperature, tend to zero in the limit $T \rightarrow 0$ and are proportional to T^2 .

We note that a fluid-dynamical formalism presents itself as a natural way of introducing temperature in the calculation of vibrational modes in an atomic nucleus. Since above 4 MeV the shell effects are washed out and the nucleus is more like a Fermi gas, the use of nuclear fluid-dynamical models appears to be also quite appealing at finite temperatures.

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TABLE CAPTIONS

Table 1- Values of some of the ground state quantities for different values of T : the equilibrium density ρ_0 , the kinetic energy density τ_0 , $\varsigma_0 = \mu - U_0$, the chemical potential μ , $E^{(0)}/A$, the surface energy and the entropy.

Table 2- Energies (in MeV) and fractions of the energy weighted sum rule (considering the excitation operators given by Eq. (97)) are given for the electric modes listed in the first column and calculated according to the present model. A nucleus with $A=208$ was considered.

Table 3- For comparison with ref. [4], energies (in MeV) and fractions of the energy weighted sum rule (considering the excitation operators given by Eq. (97)) are given for the electric modes listed in the first column and calculated according to the model of ref. [4]. A nucleus with $A=208$ was considered.

Table 4- For different eigenmodes we give the values of the quantities $a_1^{(T)}$ and $a_2^{(T)}$ appearing in eq. (58) in order to see the importance of the transverse components for the different modes. The constants $a_1^{(T)}$ and $a_2^{(T)}$ are normalized according to eq. (66).

Table 5- For comparison with Table 2, energies (in MeV) and fractions of the energy weighted sum rule (considering the excitation operators given by Eq. (97)) are given for the "new" electric modes, which appear at a finite temperature. A temperature $T = 10^{-2}$ MeV was considered.

Table 6- For the magnetic modes listed in the first column, energies (in MeV) and percentages of the m_1 sum are given for the temperatures $T=0$ MeV, 3 MeV and 5 MeV. The excitation operators given by eq. (89) are considered where $\mathbf{F} = (r - \frac{7r^3}{5R^2})\mathbf{Y}_{110}$ for $\ell = 1$ and $\mathbf{F} = r^\ell \mathbf{Y}_{\ell 0}$ for $\ell > 1$. A nucleus with $A=208$ is considered.

FIGURE CAPTIONS

Figure 1- The radial functions of $\mathbf{j}$ for the eigenmode 0_1^+ and a nucleus with $A=208$. R stands for the radius of the nucleus. The full curves refer to the function $j_+(r)$ and the dotted curves refer to $-j_{div}(r)$. The thin curves are the results for the temperature $T=1$ MeV ($\hbar\omega = 14.00$ MeV), the medium curves are the results for the temperature $T=3$ MeV ($\hbar\omega = 13.62$ MeV) and the thick curves are the results for the temperature $T=5$ MeV ($\hbar\omega = 12.90$ MeV). The curves are normalized according to eq. (66).

Figure 2- The radial functions of $\mathbf{j}$ for the eigenmode 0_2^+ and a nucleus with $A=208$. R stands for the radius of the nucleus. The full curves refer to the function $j_+(r)$ and the dotted curves refer to $-j_{div}(r)$. The thin curves are the results for the temperature $T=0$ MeV ($\hbar\omega = 15.87$ MeV), the medium curves are the results for the temperature $T=3$ MeV ($\hbar\omega = 15.65$ MeV) and the thick curves are the results for the temperature $T=5$ MeV ($\hbar\omega = 15.25$ MeV). The curves are normalized according to eq. (66).

Figure 3- The radial functions of $\mathbf{j}$ for the eigenmode 1_9^- and a nucleus with $A=208$. R stands for the radius of the nucleus. The full curves refer to the function $j_+(r)$, the dashed lines refer to $j_-(r)$, the dotted curves refer to $-j_{div}(r)$ and the dashed dotted curves refer to $j_{curl}(r)$. The thin curves are the results for the temperature $T=0$ MeV ($\hbar\omega = 28.31$ MeV), the medium curves are the results for the temperature $T=3$ MeV ($\hbar\omega = 28.38$ MeV) and the thick curves are the results for the temperature $T=5$ MeV ($\hbar\omega = 28.41$ MeV). The curves are normalized according to eq. (66).

Figure 4- The radial functions of $\mathbf{j}$ for the eigenmode 2_1^+ . The thin curves are the results for the temperature $T=0$ MeV ($\hbar\omega = 3.47$ MeV), the medium curves are the results for the temperature $T=3$ MeV ($\hbar\omega = 3.64$ MeV) and the thick curves are the results for the temperature $T=5$ MeV ($\hbar\omega = 3.81$ MeV).

Figure 5- The radial functions of $\mathbf{j}$ for the eigenmode 2_2^+ . The thin curves are the results for the temperature $T=0$ MeV ($\hbar\omega = 11.70$ MeV), the medium curves are the results for the temperature $T=3$ MeV ($\hbar\omega = 11.99$ MeV) and the thick curves are the results for the temperature $T=5$ MeV ($\hbar\omega = 12.33$ MeV).

Figure 6- The radial functions of j for the eigenmode 3_1^- . The thin curves are the results for the temperature $T=0$ MeV ($\hbar\omega = 2.92$ MeV), the medium curves are the results for the temperature $T=3$ MeV ($\hbar\omega = 2.96$ MeV) and the thick curves are the results for the temperature $T=5$ MeV ($\hbar\omega = 2.93$ MeV).

Figure 7- The radial functions of j for the eigenmode 3_3^- . The thin curves are the results for the temperature $T=0$ MeV ($\hbar\omega = 18.53$ MeV), the medium curves are the results for the temperature $T=3$ MeV ($\hbar\omega = 18.72$ MeV) and the thick curves are the results for the temperature $T=5$ MeV ($\hbar\omega = 18.73$ MeV).

Figure 8- The radial functions of j for the eigenmode 4_1^+ . The thin curves are the results for the temperature $T=0$ MeV ($\hbar\omega = 4.51$ MeV), the medium curves are the results for the temperature $T=3$ MeV ($\hbar\omega = 4.58$ MeV) and the thick curves are the results for the temperature $T=5$ MeV ($\hbar\omega = 4.56$ MeV).

Figure 9- For the states 0_1^+ (full thin line), 1_3^- (full medium line), 2_1^+ (full thick line), 0_2^+ (dotted thin line), 1_4^- (dotted medium line), 2_2^+ (dotted thick line), 3_2^- (dashed-dotted thin line), 4_2^+ (dashed-dotted medium line) and 5_2^+ (dashed-dotted thick line) we have plotted the energies of the states times $A^{1/3}$ as a function of A and for the states 3_1^- (dashed thin line), 4_1^+ (dashed medium line) and 5_1^- (dashed thick line) we have plotted the energies of the states times $A^{1/2}$ as a function of A for the temperature $T=1$ MeV.

Figure 10- Same as Fig. 9 for the temperature $T=5$ MeV.

Figure 11- We plot, for the new states at a finite temperature, the fractions of the m_1 sum rule as a function of T^2 for temperatures up to 10^{-2} MeV. The curves refer to the fractions of m_1 of the following states: 0_1^+ (multiplied by 10, full thin line), 0_4^+ (dashed thin line), 0_8^+ (multiplied by 10^3 , dotted thin line), 1_6^- (multiplied by 10^2 , dashed-dotted thin line), 1_{11}^- (multiplied by 10^3 , full medium line), 2_6^+ (multiplied by 10^5 , dashed medium line), 3_7^- (dotted medium line), 4_8^+ (multiplied by 10^2 , dashed-dotted medium line), 5_8^- (multiplied by 10^2 , full thick line).

T [MeV]	0	1	2	3	4	5
ρ_0 [fm ⁻³]	0.1479	0.1476	0.1465	0.1447	0.1422	0.1389
τ_0 [MeV fm ⁻³]	3.104	3.102	3.096	3.086	3.068	3.041
ς_0 [MeV]	34.97	34.89	34.66	34.25	33.67	32.90
μ [MeV]	-11.76	-11.85	-12.10	-12.52	-13.11	-13.87
$E^{(0)}/A$ [MeV]	-10.79	-10.74	-10.58	-10.31	-9.926	-9.424
$4\pi R_0^2\sigma$ [MeV]	607.0	604.3	596.2	582.6	563.3	538.1
S	0	29.37	58.88	88.66	118.8	149.6

Table 1

ℓ^π	T=0 MeV		T=1 MeV		T=3 MeV		T=5 MeV	
	$\hbar\omega_i$	m_1 [%]	$\hbar\omega_i$	m_1 [%]	$\hbar\omega_i$	m_1 [%]	$\hbar\omega_i$	m_1 [%]
0_1^+	—	—	14.00	1.223	13.62	9.752	12.90	23.13
0_2^+	15.87	95.15	15.85	93.78	15.65	84.05	15.25	68.07
0_3^+	18.95	2.256	18.93	2.396	18.75	3.566	18.16	6.261
0_4^+	—	—	27.75	2.248×10^{-1}	26.62	4.789×10^{-1}	25.13	6.550×10^{-1}
0_5^+	28.14	2.838×10^{-1}	28.38	9.275×10^{-2}	28.46	4.940×10^{-2}	27.87	5.977×10^{-2}
0_6^+	36.83	3.428×10^{-1}	36.79	3.328×10^{-1}	36.40	2.801×10^{-1}	35.02	2.496×10^{-1}
0_7^+	41.29	1.458	41.31	1.427	40.69	1.347×10^{-2}	38.90	1.060×10^{-3}
0_8^+	—	—	41.99	1.857×10^{-2}	41.45	1.334	41.49	1.154
1_1^-	0	0	0	0	0	0	0	0
1_2^-	6.458	0	6.467	0	6.535	0	6.642	0
1_3^-	9.245	2.456×10^{-1}	9.248	2.663×10^{-1}	9.248	4.728×10^{-1}	9.151	1.149
1_4^-	15.48	26.21	15.47	26.53	15.34	29.21	14.93	34.44
1_5^-	17.01	0	17.03	0	17.21	0	17.50	0
1_6^-	—	—	20.03	4.059×10^{-1}	19.47	2.322	18.64	2.844
1_7^-	23.20	10.22	23.20	9.791	23.10	7.473	22.51	5.563
1_8^-	26.67	0	26.71	0	27.00	0	27.44	0
1_9^-	28.31	53.12	28.32	52.84	28.38	50.70	28.41	46.88
2_1^+	3.473	30.90	3.496	31.43	3.641	35.23	3.810	40.73
2_2^+	11.70	64.19	11.73	63.63	11.99	59.55	12.33	53.00
2_3^+	17.45	2.165	17.43	2.196	17.28	2.525	16.80	3.680
2_4^+	20.54	1.094	20.55	1.114	20.53	1.209	20.40	1.213
2_5^+	21.12	1.002	21.15	9.750×10^{-1}	21.36	8.183×10^{-1}	21.70	6.560×10^{-1}
2_6^+	—	—	25.65	5.393×10^{-4}	24.75	9.638×10^{-5}	23.49	6.407×10^{-3}
2_7^+	27.30	5.717×10^{-2}	27.33	5.717×10^{-2}	27.33	6.421×10^{-2}	26.76	9.119×10^{-2}
3_1^-	2.923	34.10	2.930	34.50	2.960	37.24	2.931	40.95
3_2^-	8.428	2.879×10^{-1}	8.465	2.283×10^{-1}	8.724	6.333×10^{-3}	9.084	2.035×10^{-1}
3_3^-	18.53	43.44	18.55	42.85	18.72	38.08	18.73	27.93
3_4^-	22.80	10.88	22.79	11.15	22.68	13.58	22.37	20.00
3_5^-	25.15	3.847	25.18	3.794	25.43	3.428	25.81	2.916
3_6^-	26.87	5.178	26.89	5.195	26.99	5.300	27.02	5.185
3_7^-	—	—	30.88	2.322×10^{-1}	29.65	1.958×10^{-1}	28.03	3.874×10^{-1}
4_1^+	4.505	33.85	4.518	34.31	4.581	37.45	4.557	41.78
4_2^+	12.26	2.046	12.31	1.894	12.62	1.008	13.07	2.428×10^{-1}
4_3^+	23.36	22.39	23.36	21.90	23.35	18.16	22.96	11.97
4_4^+	27.64	8.859	27.66	9.186	27.79	11.82	27.89	16.63
4_5^+	29.67	17.10	29.70	16.90	29.90	15.22	30.21	11.96
4_6^+	33.45	8.450	33.48	8.281	33.49	4.588	32.13	2.033×10^{-3}
4_7^+	35.38	4.184	35.27	4.224	34.43	7.182	34.10	10.27
4_8^+	—	—	36.56	1.928×10^{-1}	36.21	1.434	35.46	3.949
5_1^-	6.208	33.18	6.227	33.65	6.319	36.88	6.293	41.41
5_2^-	15.82	2.880	15.87	2.709	16.23	1.667	16.74	6.312×10^{-1}
5_3^-	27.24	13.11	27.24	12.81	27.13	10.64	26.54	7.286
5_4^-	31.63	2.676	31.67	2.780	31.93	3.661	32.29	5.602
5_5^-	34.76	22.45	34.78	22.24	34.92	20.41	34.92	14.16
5_6^-	38.96	3.778×10^{-6}	38.90	1.568×10^{-2}	38.27	7.887×10^{-1}	36.55	4.140
5_7^-	40.75	20.08	40.77	19.93	40.70	5.370	39.72	1.855×10^{-1}
5_8^-	—	—	41.73	2.267×10^{-1}	41.18	15.01	41.38	20.93

Table 2

ℓ_i^π	T=0 MeV		T=1 MeV		T=3 MeV		T=5 MeV	
	$\hbar\omega_i$	m_1 [%]	$\hbar\omega_i$	m_1 [%]	$\hbar\omega_i$	m_1 [%]	$\hbar\omega_i$	m_1 [%]
0_1^+	14.63	92.69	14.57	92.69	14.06	92.69	12.99	92.71
0_2^+	29.43	5.560	29.31	5.559	28.29	5.555	26.14	5.543
0_3^+	44.20	1.090	44.01	1.090	42.48	1.088	39.26	1.086
1_1^-	0	0	0	0	0	0	0	0
1_2^-	21.09	85.85	21.00	85.85	20.27	85.85	18.74	85.85
1_3^-	36.26	9.826	36.11	9.826	34.85	9.826	32.21	9.826
1_4^-	51.18	2.475	50.96	2.475	49.19	2.475	45.46	2.475
2_1^+	2.557	99.98	2.549	99.98	2.487	99.98	2.356	99.98
2_2^+	27.17	8.076×10^{-3}	27.06	8.116×10^{-3}	26.12	8.460×10^{-3}	24.15	9.347×10^{-3}
2_3^+	42.77	1.327×10^{-3}	42.58	1.334×10^{-3}	41.11	1.391×10^{-3}	38.00	1.538×10^{-3}
3_1^-	4.903	99.95	4.888	99.95	4.766	99.95	4.513	99.94
3_2^-	33.05	3.423×10^{-2}	32.91	3.440×10^{-2}	31.77	3.584×10^{-2}	29.38	3.954×10^{-2}
3_3^-	49.06	7.143×10^{-3}	48.86	7.177×10^{-3}	47.16	7.483×10^{-3}	43.60	8.270×10^{-3}
4_1^+	7.521	99.88	7.498	99.88	7.310	99.88	6.918	99.87
4_2^+	38.79	7.721×10^{-2}	38.62	7.758×10^{-2}	37.29	8.080×10^{-2}	34.48	8.905×10^{-2}
4_3^+	55.21	1.913×10^{-2}	54.97	1.922×10^{-2}	53.07	2.003×10^{-2}	49.06	2.212×10^{-2}
5_1^+	10.39	99.79	10.36	99.79	10.09	99.78	9.548	99.76
5_2^+	44.43	1.344×10^{-1}	44.24	1.351×10^{-1}	42.72	1.406×10^{-1}	39.50	1.548×10^{-1}
5_3^+	61.24	3.798×10^{-2}	60.98	3.816×10^{-2}	58.87	3.976×10^{-2}	54.43	4.388×10^{-2}

Table 3

ℓ^π	T=0 MeV		T=5 MeV	
	$a_1^{(T)}$	$a_2^{(T)}$	$a_1^{(T)}$	$a_2^{(T)}$
2_1^+	4441	87.19	2241	49.77
2_2^+	4.122	0.1681	3.094	0.1511
2_3^+	0.6619	-2.693×10^{-2}	0.6264	-3.103×10^{-2}
2_4^+	0.6177	2.152×10^{-2}	0.7435	5.886×10^{-3}
2_5^+	9.081×10^{-2}	-0.1043	2.167×10^{-2}	-0.1072
2_6^+	—	—	1.212×10^{-2}	5.999×10^{-4}
2_7^+	8.574×10^{-4}	1.591×10^{-3}	4.881×10^{-3}	-1.875×10^{-3}
3_1^-	9.549×10^4	641.4	7.643×10^4	506.0
3_2^-	83.51	1.446	48.18	1.011
3_3^-	0.3018	1.548×10^{-2}	0.2410	1.376×10^{-2}
3_4^-	0.1385	-1.782×10^{-2}	0.1446	-1.771×10^{-2}
3_5^-	2.694×10^{-2}	3.459×10^{-2}	2.212×10^{-2}	3.591×10^{-2}
3_6^-	0.2924	-3.883×10^{-4}	0.3036	-2.956×10^{-4}
3_7^-	2.531×10^{-3}	-1.273×10^{-3}	6.585×10^{-2}	-7.014×10^{-4}
4_1^+	5.581×10^4	190.7	4.362×10^4	147.1
4_2^+	21.70	0.2866	13.27	0.2176
4_3^+	4.055×10^{-2}	4.097×10^{-3}	2.602×10^{-2}	3.394×10^{-3}
4_4^+	6.842×10^{-2}	-1.508×10^{-2}	9.681×10^{-2}	-1.360×10^{-2}
4_5^+	4.996×10^{-2}	1.088×10^{-2}	5.087×10^{-2}	1.318×10^{-2}
4_6^+	0.1432	1.012×10^{-4}	4.936×10^{-2}	1.241×10^{-3}
4_7^+	1.856×10^{-2}	-1.172×10^{-3}	0.1371	-2.521×10^{-4}
5_1^-	4.368×10^4	78.47	3.441×10^4	60.97
5_2^-	9.998	9.887×10^{-2}	6.306	7.866×10^{-2}
5_3^-	7.274×10^{-3}	-1.688×10^{-3}	2.067×10^{-2}	-1.515×10^{-3}
5_4^-	4.045×10^{-2}	-9.756×10^{-3}	5.760×10^{-2}	-9.500×10^{-3}
5_5^-	4.999×10^{-2}	3.338×10^{-3}	4.707×10^{-2}	4.042×10^{-3}
5_6^-	5.765×10^{-2}	8.940×10^{-4}	4.963×10^{-2}	1.831×10^{-3}
5_7^-	5.911×10^{-2}	-1.017×10^{-3}	3.878×10^{-2}	6.914×10^{-4}

Table 4

ℓ^π	T=10 ⁻² MeV	
	$\hbar\omega_i$	m_1 [%]
0 ₁ ⁺	14.06	1.246×10 ⁻⁴
0 ₄ ⁺	28.11	3.395×10 ⁻³
0 ₈ ⁺	42.17	1.387×10 ⁻⁶
1 ₆ ⁻	20.11	4.320×10 ⁻⁵
1 ₁₁ ⁻	34.57	6.379×10 ⁻⁶
2 ₆ ⁺	25.79	7.882×10 ⁻⁸
3 ₇ ⁻	31.27	5.282×10 ⁻³
4 ₈ ⁺	36.61	1.953×10 ⁻⁵
5 ₈ ⁻	41.86	1.638×10 ⁻⁵

Table 5

ℓ^π	T=0 MeV	T=1 MeV	T=3 MeV	T=5 MeV	m_1 [%]
	$\hbar\omega_i$	$\hbar\omega_i$	$\hbar\omega_i$	$\hbar\omega_i$	
1_1^+	0	0	0	0	0
1_2^+	11.95	11.96	12.09	12.29	0
1_3^+	21.89	21.92	22.15	22.52	0
1_4^+	29.24	29.28	29.59	30.08	42.14
1_5^+	31.41	31.45	31.79	32.31	0
1_6^+	40.79	40.85	41.28	41.96	0
1_7^+	46.14	46.21	46.70	47.47	16.92
2_1^-	8.332	8.344	8.433	8.571	79.96
2_2^-	16.14	16.16	16.34	16.60	8.450
2_3^-	26.01	26.05	26.33	26.76	5.755×10^{-1}
2_4^-	35.59	35.64	36.02	36.61	8.321×10^{-3}
2_5^-	36.20	36.25	36.63	37.24	3.956
2_6^-	45.07	45.14	45.61	46.36	5.818×10^{-2}
2_7^-	53.33	53.41	53.98	54.86	1.759
3_1^+	16.40	16.43	16.60	16.87	62.65
3_2^+	20.23	20.26	20.48	20.81	18.42
3_3^+	29.97	30.02	30.34	30.83	5.797×10^{-1}
3_4^+	39.69	39.75	40.17	40.83	4.897×10^{-2}
3_5^+	42.79	42.85	43.30	44.01	5.974
3_6^+	49.23	49.30	49.82	50.64	6.403×10^{-2}
3_7^+	58.65	58.73	59.36	60.33	5.579×10^{-3}
3_8^+	60.26	60.35	60.99	61.99	2.924
4_1^-	22.13	22.16	22.39	22.76	25.85
4_2^-	24.88	24.92	25.19	25.60	49.67
4_3^-	33.82	33.87	34.23	34.79	6.749×10^{-1}
4_4^-	43.66	43.73	44.19	44.92	6.606×10^{-2}
4_5^-	49.19	49.27	49.79	50.60	7.194
4_6^-	53.29	53.37	53.94	54.82	8.543×10^{-2}
4_7^-	62.78	62.87	63.54	64.58	1.403×10^{-2}
4_8^-	67.00	67.09	67.81	68.92	3.662
5_1^+	26.31	26.35	26.63	27.06	5.800
5_2^+	30.44	30.49	30.81	31.32	65.18
5_3^+	37.58	37.63	38.03	38.66	9.121×10^{-1}
5_4^+	47.55	47.62	48.12	48.91	7.781×10^{-2}
5_5^+	55.45	55.53	56.12	57.04	7.897
5_6^+	57.29	57.37	57.98	58.93	1.934×10^{-1}
5_7^+	66.84	66.93	67.64	68.75	1.789×10^{-2}

Table 6

Figure 1

Figure 2

Figure 3

$$\ell_i^\pi = 1_g^-, \quad A=208$$

Figure 4

Figure 5

Figure 6

Figure 7

$$\ell_i^\pi = 3_3^-, \quad A=208$$

Figure 8

Figure 9

Figure 10

Figure 11