Derivative Expansion in Quantum Theory of Gravitation

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ABSTRACT

The cosmological term prevents perturbation based on derivative expansion in Einstein gravity. We consider quantum theory of gravitation invariant under volume-preserving diffeomorphism and Weyl transformation, which is suitable for derivative expansion.

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1. INTRODUCTION

Universality in quantum field theory leads to natural expectation that low-energy physics can be described by effective Lagrangian.\(^1\) In fact, the standard model of electroweak and strong interactions may be viewed as effective field theory with cutoff stemming from new physics beyond it such as grand unification. Within perturbation theory, its renormalizable part provides dominant contribution to amplitudes computed in the model.

Low-energy description of the other fundamental force — graviton on the flat background — seems slightly different due to essential non-polynomiality of gravitational field theory. It is to be compared with chiral Lagrangian in hadron physics, which is also non-polynomial. Chiral perturbation theory is based on derivative expansion.\(^2\)

In the case of Einstein gravity without the cosmological term, derivative expansion in powers of \(p/M\) yields effective field theory of graviton,\(^3\) where \(p\) denotes a characteristic momentum and \(M\) is the Planck mass. However, the cosmological term cannot be suppressed when a coupled system of gravity and field theory with spontaneously broken symmetry is considered. In order to obtain the flat background on the true vacuum, we generically need a non-zero value of the cosmological constant on the symmetric vacuum.

The cosmological term brings in non-derivative interaction of gravitons, which prevents perturbation based on derivative expansion in quantum theory of Einstein gravity. The order of derivative expansion in four dimensions behaves as

\[
2L + \sum_n (D_n - 2),
\]

\((1)\)
where $L$ denotes the number of loops and $D_n$ is the number of derivatives in each vertex labeled by $n$ in a Feynman diagram. This indicates troublesome nature of non-derivative interaction $D_n = 0$ for perturbation based on derivative expansion.

In this paper, we consider quantum theory of gravitation which is invariant under volume-preserving diffeomorphism and Weyl transformation. Weyl invariance excludes the apparent cosmological term from the effective Lagrangian, which makes perturbation based on derivative expansion possible in the theory of gravitation. Hence this prescription gives a low-energy description of fundamental forces when it is combined with a model of other interactions of elementary particles.

2. LAGRANGIAN

In order to formulate a theory of massless spin-two particle in terms of a symmetric tensor field of degree two, it is necessary to impose gauge symmetries which eliminate spin-one and spin-zero components in it. The candidate symmetries are diffeomorphism and Weyl transformation. The ordinary choice is the full diffeomorphism, which results in Einstein gravity. On the other hand, one can adopt volume-preserving diffeomorphism and Weyl transformation, which is the choice made in this paper.

Let us consider the form of effective Lagrangian for the metric field $g_{\mu\nu}$ invariant under volume-preserving diffeomorphism and Weyl transformation. For simplicity, we take a scalar $\phi$ as an example of matter field.

Weyl symmetry

$$\delta_W g_{\mu\nu} = \Lambda g_{\mu\nu}, \quad \delta_W \phi = 0$$

(2)
implies that the metric field comes in the Lagrangian solely in the combination

\[ \bar{g}_{\mu\nu} = g^{-\frac{1}{4}} g_{\mu\nu}, \]  

where \( g = - \det g_{\mu\nu} \).

Invariance under volume-preserving diffeomorphism

\[ \delta_V \bar{g}_{\mu\nu} = -\varepsilon^\rho \partial_\rho \bar{g}_{\mu\nu} - \bar{g}_{\mu\rho} \partial_\nu \varepsilon^\rho - \bar{g}_{\nu\rho} \partial_\mu \varepsilon^\rho, \quad \delta_V \phi = -\varepsilon^\mu \partial_\mu \phi; \quad \partial_\mu \varepsilon^\mu = 0 \]

can be achieved[^4] in complete analogy to the case of full diffeomorphism invariance. The Lagrangian bears the same form as that in Einstein gravity except for the field \( g_{\mu\nu} \) in the latter replaced by \( \bar{g}_{\mu\nu} \) in the former:

\[ \mathcal{L} = V(\phi) + K(\phi) \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + U(\phi) \bar{R} + \cdots; \quad U(0) = \frac{1}{\kappa^2}, \]

where \( \bar{g}^{\mu\nu} \) denotes the inverse of \( \bar{g}_{\mu\nu} \), \( \bar{R} \) is the scalar curvature corresponding to \( \bar{g}_{\mu\nu} \), and the ellipsis stands for higher-derivative terms. Note that the volume density (in particular, the apparent cosmological term) is absent from the expression (5) since \( \det \bar{g}_{\mu\nu} = -1 \). Hence this theory can serve as a basis for perturbative treatment based on derivative expansion.

3. Classical Theory

[^4]: By means of a scalar field \( \varphi \) with a Weyl transformation law \( \delta_{\varphi} \varphi = -\Lambda \varphi \), one can construct an action \( I[\varphi g_{\mu\nu}] \) which classically has both diffeomorphism and Weyl invariances, where \( I[\varphi g_{\mu\nu}] \) denotes a diffeomorphism-invariant action. A gauge choice \( \varphi = 1 \) in \( I[\varphi g_{\mu\nu}] \) leads to Einstein gravity \( I[g_{\mu\nu}] \), whereas another choice \( \varphi = g^{-\frac{1}{4}} \) results in the present action with residual symmetries of volume-preserving diffeomorphism and Weyl transformation.
In this section, we investigate classical contents of the theory (5). Let $S[\bar{g}_{\mu\nu}, \phi]$ be the corresponding action, whose equations of motion are obtained as

$$\frac{\delta S}{\delta \bar{g}_{\mu\nu}} = g^{-\frac{1}{2}} \left( \frac{\delta S}{\delta \bar{g}_{\mu\nu}} - \frac{1}{4} \bar{g}^{\mu\nu} \bar{g}_{\rho\sigma} \frac{\delta S}{\delta \bar{g}_{\rho\sigma}} \right) = 0, \quad \frac{\delta S}{\delta \phi} = 0$$

by means of the definition (3).

On the other hand, we get an equation

$$\bar{D}_\mu \frac{\delta S}{\delta \bar{g}_{\mu\nu}} = 0$$

with the aid of the second equation in (6) and the Noether identity due to diffeomorphism invariance of an action $S[\bar{g}_{\mu\nu}, \phi]$, where $\bar{D}_\mu$ denotes the covariant derivative corresponding to $\bar{g}_{\mu\nu}$. Hence the first equation in (6) yields

$$\bar{D}_\mu \left( \frac{\delta S}{\delta \bar{g}_{\mu\nu}} - \frac{1}{4} \bar{g}^{\mu\nu} \bar{g}_{\rho\sigma} \frac{\delta S}{\delta \bar{g}_{\rho\sigma}} \right) = - \frac{1}{4} \bar{g}^{\mu\nu} \partial_\mu (\bar{g}_{\rho\sigma} \frac{\delta S}{\delta \bar{g}_{\rho\sigma}}) = 0,$$

which indicates that

$$\bar{g}_{\rho\sigma} \frac{\delta S}{\delta \bar{g}_{\rho\sigma}} = 4\lambda$$

is a constant independent of spacetime.

Therefore we conclude that the equations of motion are given by

$$\frac{\delta S}{\delta \bar{g}_{\mu\nu}} - \lambda \bar{g}^{\mu\nu} = 0, \quad \frac{\delta S}{\delta \phi} = 0,$$

which are none other than those in Einstein gravity with a partial gauge-fixing $g = 1$. This shows that the theory (5) is classically equivalent to Einstein gravity.
with the cosmological constant as an integration constant determined by an initial condition.\footnote{We may use the Weyl transformation (2) to impose the unimodular condition $g = 1$ in the theory (5). Then the theory is reduced to the unimodular gravity,\cite{5} which is known to have the features mentioned above.\cite{5}}

A demand of the flat background $\langle g_{\mu\nu} \rangle = \eta_{\mu\nu}$ (or the unbroken translational invariance) automatically makes the cosmological constant to be zero on the vacuum. This enables us to make perturbation based on derivative expansion in quantum theory of (5).

4. QUANTIZATION

Quantization of the theory proceeds via covariant gauge-fixing. Let us consider the harmonic gauge $\partial_\mu \tilde{g}^{\mu\nu} = 0$, where we define $\tilde{g}^{\mu\nu} = \sqrt{g} g^{\mu\nu}$.

The BRS transformation $\delta$ for the gauge symmetries (2) and (4) is given by

$$ \delta \tilde{g}^{\mu\nu} = \kappa C \tilde{g}^{\mu\nu} - c^\rho \partial_\rho \tilde{g}^{\mu\nu} + \tilde{g}^{\rho\sigma} \partial_\rho c^{\mu\nu} + \tilde{g}^{\mu\rho} \partial_\rho c^{\nu}, \quad \delta \phi = - e^\mu \partial_\mu \phi; \quad e^\mu = \kappa \partial_\mu e^{\nu}; \quad e^{\nu} = - e^{\mu}.$$  

$$ \delta C = - e^\mu \partial_\mu C, \quad \delta e^{\mu} = - \kappa \partial_\rho e^{\rho \mu} \partial_\sigma e^{\sigma \nu} + i e^{\mu \rho \sigma} \partial_\rho d_\sigma, \quad \delta d_\mu = \partial_\mu f, \quad (11)$$

where ghosts for ghosts $d_\mu$ and $f$ have been introduced and $e^{\mu \rho \sigma}$ denotes the Levi-Civita symbol. We further introduce additional fields

$$ \delta \tilde{C}_\mu = i B_\mu, \quad \delta \tilde{d}_\mu = \tilde{e}^{\mu}, \quad \delta e = id, \quad \delta \tilde{f} = i \tilde{d} \quad (12)$$

to adopt gauge conditions

$$ \kappa^{-1} \partial_\mu \tilde{g}^{\mu \nu} = 0, \quad e^{\mu \rho \sigma} \tilde{g}^{\rho \sigma} \partial_\tau e^{\theta \sigma} = 0, \quad \partial_\mu \tilde{e}^{\mu} = 0, \quad \tilde{g}^{\mu \nu} \partial_\mu d_\nu = 0, \quad (13)$$
which are implemented by a gauge-fixing term

\[ \mathcal{L}_B = -i\delta(\partial_\mu \bar{\mathcal{C}}_\nu \kappa^{-1} \partial_\nu \tilde{g}^{\mu\nu} + \partial_\rho \epsilon_{\mu\nu\rho\sigma} \tilde{g}^{\nu\tau} \partial_\tau \epsilon^{\rho\sigma} + c\partial_\rho \tilde{g}^{\mu\nu} + f\tilde{g}^{\mu\nu} \partial_\mu \partial_\nu). \]  

(14)

With a gauge-fixed Lagrangian \( \mathcal{L}_T = \mathcal{L} + \mathcal{L}_B \) in hand, we can make a perturbative quantization of our theory on the flat background.

Let us adopt \( h^{\mu\nu} \) as our basic variable, where we define

\[ \tilde{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}. \]  

(15)

The quadratic part in the field \( h^{\mu\nu} \) of the gauge-fixed Lagrangian \( \mathcal{L}_T \) is given by

\[ \mathcal{L}_0 = \frac{1}{4} (\partial_\mu h_{\nu\rho} \partial^\mu h^{\nu\rho} - 2\partial_\mu h^{\mu\nu} \partial^\rho h_{\rho\nu} - \frac{3}{8} \partial_\mu h \partial^\mu h + \partial_\mu h^{\mu\nu} \partial_\nu h) + B_\mu \partial_\nu h^{\mu\nu}, \]  

(16)

where \( h = \eta_{\mu\nu} h^{\mu\nu} \). Hence the corresponding propagator of \( h^{\mu\nu} \) is obtained in momentum space as follows:

\[ \langle h^{\mu\nu} h^{\rho\sigma} \rangle_0 = \frac{i}{k^2} (\partial^{\mu\nu} \partial^{\rho\sigma} + \partial^{\rho\sigma} \partial^{\mu\nu} - 6\partial^{\mu\nu} \partial^{\rho\sigma}); \quad \partial^{\mu\nu} = \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}, \]  

(17)

which describes desired propagation of massless spin-two particle on the flat background.

A general theorem established by Kugo and Uehara \cite{7} in relativistic quantum field theory states that massless spin-two particle with non-vanishing coupling in the infrared limit \( p_\mu \to 0 \) is uniquely the graviton. Therefore the present theory provides quantum theory of gravitation though it is not based on Einstein gravity. This may be expected from the classical analysis in section 3 that it is essentially equivalent to Einstein gravity with a partial gauge-fixing.
5. Conclusion

We have made a perturbative construction of quantum gravitational theory on the flat background. It is based on derivative expansion by means of effective Lagrangian invariant under volume-preserving diffeomorphism and Weyl transformation.

The construction confirms the view that quantum theory of gravitation is no less than the standard model of electroweak and strong interactions as effective field theory. In fact, the former may be more accurate than the latter under the circumstances that the effective cutoff in the former is expected to be larger than that in the latter.

Thus one can include the graviton among the participants in the Standard Model of elementary particles.

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REFERENCES

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