NUCLEAR CHARGE DENSITY DISTRIBUTIONS
FROM ELASTIC ELECTRON SCATTERING DATA

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Abstract. The model independent procedure of extracting charge density distributions from elastic electron scattering data is investigated. The charge density distributions are expanded on an orthonormal basis and the parameters of the expansions are fixed by the comparison with the experimental data. Two bases with different analytical properties (Fourier-Bessel and Hermite) are used. This allows us to disentangle the uncertainties coming from the choice of the expansion basis from those intrinsic to the extraction procedure. We design a set of tests to select the number of the expansion coefficients adequate for a proper description of the data. The procedure is applied to elastic data measured on $^{12}$C, $^{40}$Ca and $^{208}$Pb nuclei.
1. Introduction

The advantages of using electrons in the investigation of the nuclear structure are mainly related to the fact that the electron-nucleus interaction is relatively weak. For this reason multiple scattering effects are usually neglected and the scattering process is described in terms of perturbation theory.

Since the reaction mechanism in perturbation theory is well under control the connection between the cross section and quantities such as charge distributions, transition densities, response functions etc., is well understood. This is the reason why the comparison between theory and experiment is often done not at the cross section level, but using these quantities.

The techniques used to extract these quantities from the cross section data are, in principle, grouped under two basic approaches. A first approach consists in the solution of the inverse scattering problem, or in other words, in obtaining the potential from the cross section data. In the second approach one solves the direct scattering problem making a guess about the form of the potential and calculating the cross section. The parameters fixing the form of the potential are changed until the experimental cross sections are reproduced.

Neither of these procedures is trouble free.

The inverse scattering approach applied to a set of experimental data, which are measured on a discrete grid and have error bars, does not ensure a unique solution because of numerical instabilities [1]. This makes the use of this approach extremely impractical and for this reason it has never been applied in this context.

On the other hand, the direct scattering approach is limited by the need of guessing an analytical expression of the potential, or of the quantity generating it.

In the past good descriptions of the experimental data were obtained with analytical expressions of the density depending from few parameters (e.g the Fermi distribution). In these cases the parameters are related to the characteristics of the shape of the density, for example the half-density radius, the diffuseness, the central depression. The development of the accelerator and detector technology has produced data of better quality and covering an increasing range of momentum transfer. The simple, and intuitive, analytical expressions mentioned above are not able to provide satisfactory descriptions of these new high quality data. This has been achieved, within the direct scattering approach, with the introduction of the so-called model independent methods.

These methods are based upon the assumption that the function describing the density can be expanded on a complete set of orthonormal functions. The coefficients of the expansion are changed in order to reproduce the experimental cross section. Since the basic assumption is very general (it is hard to believe that functions describing density distributions or quantities related to them like potentials have such odd analytical properties which prohibit them to be expanded on a basis) from the mathematical point of view the success of these methods is guaranteed. The mathematical certainty of the success of these methods should however pay the price of two shortcomings: first the parameters of the expansion are not expected to have a particular physical meaning, and second, in practical applications the expansion must be truncated to a finite number of terms and this generates uncertainties in the determination of the distribution.

The uncertainties related to the extraction procedure should be added to those related to the experimental errors. Our claim is that, given the high accuracy of the modern electron scattering experiments, these theoretical uncertainties could be of the same order of magnitude, or even bigger, than those coming from the experimental errors.

In this paper we present the results of our study on the determination of the charge density distributions of doubly-magic nuclei from elastic scattering processes. Specifically, we have been interested in investigating the source and the magnitude of the theoretical uncertainties linked to the model independent extraction techniques.

To obtain information independent from the choice of the expansion basis, we have used two orthonormal bases with quite different analytical properties. These are the Fourier-Bessel (FB) basis, widely used in the literature to analyze both elastic and inelastic experimental data [2, 3, 4], and a basis of Hermite functions which, as far as we know, we are using for the first time in this context.

In the next section of the paper we describe the properties of the two bases and the mechanisms which allow them to reproduce the cross section. For simplicity this study is done in Plane Wave Born Approximation (PWBA).
In section 3 we use the two bases to reproduce pseudo data generated by model charge density distributions as well as really measured cross sections. This analysis is done solving the Dirac equation for an electron scattered by a Coulomb potential in the ultra-relativistic limit. We present a comparison between the performances of the two bases in reproducing the data and we discuss the common features and limitations. We discuss the propagation of the experimental errors from the cross section to the density. In this section we analyze the uncertainties related to the truncation of the expansion basis and we define a procedure to fix the number of expansion coefficients.

Finally, in the last section we summarize the work and we draw our conclusions.

2. Model independent analysis.

Our study of the determination of the charge distributions from elastic scattering processes is done solving directly the scattering problem. The potential is generated by a trial charge distribution whose parameters are chosen to reproduce the experimental cross section.

In the model independent procedure the trial density distribution is expanded on an orthonormal basis

\[ \rho(r) = \sum_{n=1}^{\infty} A_n P_n(r). \]

(2.1)

and the expansion coefficients \( A_n \) are fixed to reproduce the cross section.

This is the straightforward application of the model independent method. In reality, this method is more general, and can be applied to any function which is determining the density in a unique way. For example, in the generalized Helm model of ref. [5] the density is generated as a convolution of two terms, one of these (the so-called Helm density) is expanded on a basis of Legendre polynomials.

In Plane Wave Born Approximation (PWBA) the link between the charge density distribution and the cross section is straightforward. In this work we shall limit our considerations to even-even nuclei. In this case, defining the form factor as

\[ F(q) = 4\pi \int_0^{\infty} dr r^2 j_0(qr) \rho(r), \]

(2.2)

where \( j_0 \) is the spherical Bessel function of zero order and \( q \) the absolute value of the three momentum transfer, the cross section can be written as:

\[ \frac{d\sigma(q)}{d\Omega} = \sigma_M f_{\text{rec}} |F(q)|^2. \]

(2.3)

In eq. 2.3 we have indicated with \( \sigma_M \) the Mott cross section and with \( f_{\text{rec}} \) the recoil term:

\[ f_{\text{rec}} = (1 + \frac{2\epsilon}{M_A} \sin^2 \frac{\theta}{2})^{-1} \]

where \( \epsilon \) is the electron energy, \( M_A \) is the mass of the nucleus and \( \theta \) is the scattering angle.

In PWBA the inverse scattering problem can be easily solved. It is possible to extract the form factor from the cross section and then, with an inversion of the Fourier transform, to get the charge density distribution

\[ \rho(r) = \frac{1}{2\pi^2} \int_0^{\infty} j_0(qr) F(q) q^2 \, dq. \]

(2.4)

The intrinsic limitations of the direct and inverse approaches in the determination of the charge density distribution are well shown by the above equations.

Eq. 2.1 shows that the direct approach requires an infinite sum of the expansion terms. From the pragmatical point of view this sum has to be truncated and this produces an error in the determination of the charge density distribution.

On the other hand, it is shown by eq. 2.4 that, in the solution of the inverse scattering problem, a precise determination of the charge density distribution is connected to the knowledge of the cross section.
up to infinite momentum transfer. The measurements are clearly performed in a limited range of $q$ and this produces an uncertainty in the determination of the charge density distribution.

The relation between these intrinsic limitations of the two procedures can be better seen by specifying the expansion basis of the density.

In our study we made the hypothesis that the charge distribution is a continuous and even function of the distance $r$ from the center of the nucleus. This means that an analytical continuation of the density in the unphysical region of the negative distances would provide $\rho(-r) = \rho(r)$.

2.1 The Fourier Bessel expansion

The Fourier Bessel (FB) basis has been widely used to analyze both elastic and inelastic data [4], and here we simply sketch the basic ideas underlying the expansion procedure referring to ref. [2,3] for further details.

The basic idea of the FB expansion is related to the possibility of expanding a single valued function defined in a finite domain. For this purpose one uses the orthogonality relation between spherical Bessel functions in a finite domain:

$$\int_0^{R_c} drr^2 j_l(qn r) j_l(qm r) = \frac{R_c^2}{2} [j_{l+1}(qn R_c)]^2 \delta_{n,m},$$

where the $q_n$ are defined such as

$$j_l(qn R_c) = 0.$$

In our case the use of the FB basis implies that the charge density $\rho(r)$ should be zero for values of $r$ larger than $R_c$:

$$\rho(r) = \begin{cases} \sum_{n=1}^{\infty} a_n j_0(qn r) & \text{for } r \leq R_c \\ 0 & \text{for } r > R_c \end{cases},$$

where, from eq. 2.6,

$$q_n = \frac{n\pi}{R_c}.$$

In principle it is possible to obtain the $a_n$ coefficients measuring directly the cross section at the $q_n$ momentum transfer. If the form factor 2.2 is known in $q_n$, the coefficients $a_n$ can be obtained inserting of the relations 2.5 and 2.7 in the definition 2.2 of the form factor:

$$a_n = \frac{F(q_n)}{2\pi R_c^2 [j_1(qn R_c)]^2}.$$

In general the cross section is measured at $q$ values different from $q_n$. Using the expansion 2.7 of the charge density we find for the form factor the relation:

$$F(q) = \frac{4\pi R_c^2}{q} \sum_n a_n \frac{(-1)^n}{(qR_c)^2 - (n\pi)^2} \sin(qR_c).$$

Before applying the FB expansion of the density to realistic cases we illustrate in a simple example the mechanism which allows this expansion to reproduce the form factors. In order to keep the calculations as simple as possible we have performed this study in PWBA. The form factor that the FB techniques should reproduce has been generated by a density of analytical form. Specifically we have used densities given by symmetrized Fermi distributions [6]

$$\rho(r) = \alpha \frac{\cosh(R/d)}{\cosh(R/d) + \cosh(r/d)}.$$
For this density an analytical expression of the form factor can be easily obtained \[7\]:

\[
F(q) = -\frac{4\pi^2\alpha d \cosh(R/d)}{q} \sinh(R/d) \left[ \frac{R\cos(qR)}{\sinh(\pi qd)} - \frac{\pi d\sin(qR)\cosh(\pi qd)}{\sinh^2(\pi qd)} \right].
\] (2.12)

The above expression is composed of oscillating terms damped by exponentials. This is better seen taking the limit for \(qd \gg 1\). We obtain:

\[
F(q) \simeq -\frac{4\pi^2\alpha d}{q} \left[ R\cos(qR) - \pi d\sin(qR) \right] e^{-\pi qd},
\] (2.13)

where the oscillating and damping terms are evident.

Only two of the above parameters \(\alpha, R\) and \(d\) are independent since the normalization condition

\[4\pi \int dr^2 \rho(r) = Z\]

relates them.

In the panels a) and c) of fig.1, together with the form factor given by eq.(2.12) (thick full lines), we present the results obtained using respectively 4 and 6 FB coefficients calculated from eq. (2.9). The thin lines represent the results obtained with 4 and 6 FB expansion coefficients and the contribution of the last coefficient of the expansion is shown by the dashed lines.

The contribution to the form factor of the \(n\)-th expansion term is localized around the \(q_n\) value of eq. (2.8a). Let us consider for example the sixth term. Its major contribution is peaked around \(q = 2.5 \text{ fm}^{-1}\), and it oscillates on the full momentum transfer range. For \(q < 2.5 \text{ fm}^{-1}\) the contribution of this term is orders of magnitude smaller than the full form factor, therefore irrelevant for the fit. The presence of this term is however necessary to reproduce the form factor around \(q = 2.5 \text{ fm}^{-1}\). For larger values of \(q\) the sixth term presents an oscillating behavior completely different from that of the true form factor, and the fit above this value is no longer reasonable.

This analysis confirms the naive idea that the higher expansion coefficients of the charge density distribution influence only the high \(q\) part of the form factor. There is therefore a link between the experimental limitation of the data at a maximum value of the momentum transfer and the truncation of the FB expansion. The number of expansion coefficients determined by experiments performed up to \(q_{\text{max}}\) can be obtained inverting eq. 2.8:

\[
n_{\text{max}} = \frac{q_{\text{max}} R_c}{\pi}.
\] (2.8b)

The use of the FB expansion implies a strong hypothesis on the shape of the density forcing it to be zero for \(r > R_c\). This is producing a discontinuity in the analytical shape of the density with consequences on the form factor.

In fig.2 we compare the form factors calculated for the cases of \(^{12}\text{C}\), \(^{40}\text{Ca}\) and \(^{208}\text{Pb}\) nuclei using the exact expression (2.12) with those obtained truncating the density (2.11) at different values of \(R_c\). The calculations have been done with the densities of eq. (2.11) with \(R = 1.1 A^{1/3} \text{ fm}\) and \(d = 0.626 \text{ fm}\).

The thick lines are representing the squares of the form factors calculated with eq. (2.12). The thin lines show the square of the form factor calculated with the truncated density distribution \((F_c(q))\). An estimate of effect of the truncation on the form factor is given by the expression:

\[
\Delta F(q) \equiv F(q) - F_c(q) = \frac{4\pi}{q} \int_{R_c}^{\infty} dr sin(qr) \rho(r)
\]

\[
\simeq \frac{8\pi}{q} d^2 \alpha \cosh(R/d) \left\{ \frac{e^{-\frac{n_c}{\pi}}}{(1 + (qd)^2)} \left[ (qd)\cos(qR_c)\left(\frac{R_c}{d}(1 + (qd)^2) + 2\right) + sin(qR_c)\left(\frac{R_c}{d}(1 + (qd)^2) + 1 - (qd)^2\right) \right] \right\},
\] (2.15)
which results in all the cases here considered in excellent agreement with the exact result obtained with a numerical integration.

The difference between the two form factors is not spread over the full range of the momentum transfer but it shows up at a certain value of $q$ after which the true form factor decreases while the form factor obtained with the truncated density oscillates around a roughly constant value.

Eq. 2.15 shows that $\Delta F(q)$ becomes smaller the larger is the value of $R_c$.

$R_c$ can be fixed by the condition that for a given $q$ the ratio $\Delta F(q)/F(q)$ (eqs. 2.15 and 2.12) should be smaller than a fixed value. For example, the $R_c$ radii used in fig. 2 are 8 fm for both $^{12}\text{C}$ and $^{40}\text{Ca}$, and 11 fm for $^{208}\text{Pb}$.

The truncation of the density generates an unphysical contribution which, for large values of $q$, dominates on the true value of the form factor. Using the FB expansion one should take care to choose the $R_c$ parameter such that the unphysical contribution remains small in the range of $q$ values investigated.

2.2 The Hermite expansion

In order to disentangle the uncertainties related to the choice of the expansion basis from those intrinsic of the extraction method we have worked with two bases having different analytical properties. In addition to the FB expansion presented above, we have used a basis of Hermite functions [8]:

$$u_n(x) = (\sqrt{\pi} 2^n n!)^{-\frac{1}{4}} e^{-\frac{x^2}{2}} H_n(x), \quad (2.16)$$

where we have indicated with $H_n$ the Hermite polynomial of order $n$.

The orthogonality relation of the $h_n$ functions is given by:

$$\int_{-\infty}^{\infty} u_n(x) u_m(x) \, dx = \delta_{n,m}. \quad (2.17)$$

The charge density distribution is expanded as:

$$\rho(r) = \sum_{n=0}^{\infty} c_n u_n\left(\frac{r}{\beta}\right), \quad (2.18)$$

where $\beta$ is an arbitrary quantity, with the units of a length, fixing the scale of the expansion basis. The Hermite expansion removes the strong hypothesis done by the FB expansion that the density should be zero for $r > R_c$.

The expression of the form factor obtained inserting in eq. (2.2) the Hermite expansion (2.18) is given by:

$$F(q) = \frac{\pi^{3/2}}{q} 2\beta^2 \sum_{n=0}^{\infty} \left[ c_{n+2} \sqrt{n+2} + c_n \sqrt{n+1} \right] (-1)^{n/2} u_{n+1}(q\beta). \quad (2.19)$$

The sum in eq. (2.18, 2.19) is restricted only to the even values of $n$, since we have assumed that $\rho(r)$ is an even function of $r$.

Like in the case of the FB expansion, we have studied the mechanism which enables the Hermite basis to reproduce the form factors. In the panels b) and d) of fig.1 the thick lines are the squared form factors obtained with the symmetric Fermi distribution, eq.(2.11), the thin lines have been obtained summing respectively 4 and 6 terms of the Hermite expansion, and the dashed lines represent the contribution of the last term.

The comparison between the figures relative to the FB calculation with the analogous figure for the Hermite expansion shows a remarkable difference between the two bases. The $n$-th term of the FB basis is oscillating on the full range of $q$ and it presents a pronounced peak in correspondence with the $q$ value where it gives its main contribution to the full form factor. The $n$-th term of the Hermite basis is oscillating only
up to a maximum value of $q$, corresponding to the value where it gives its main contribution to the full form factor, and, after, it decreases without oscillations. Contrary to the FB case the contribution of the n-th term does not show any pronounced peak. It seems that, while in the FB expansion there is a localization of the contribution of each expansion term, in the Hermite case the form factor is built up by the calibrated sum of several terms.

Also in the case of the Hermite expansion the number of expansion coefficients is related to the maximum value of the experimental momentum transfer. Fig. 1 shows that the addition of new terms enables one to reproduce the form factor on a wider range of momentum transfer. Finding a rule linking the number of coefficients with the maximum value of the momentum transfer is not straightforward as in the FB case where the coefficients are directly related to $q_n$. In order to have an empirical estimate of the maximum number of expansion coefficients fixed by a given $q_{max}$ we have imposed that the last inflection point of the form factor produced by the n-th coefficient should lie at higher momentum than $q_{max}$. With this assumption we found the approximate relation:

$$n \sim 0.5(q_{max}^2 \beta^2).$$ (2.20)

3. Specific applications

In the determination of a quantity, such as the nuclear charge distribution, from a measured quantity, the cross section, experimental and theoretical uncertainties are both contributing. In the specific case we are discussing, the experimental uncertainties on the charge density are related to the errors on the data and to the fact that the cross section is measured on a limited range of momentum transfer values. The theoretical uncertainties are linked to the use of a limited number of expansion coefficients which produces also a dependence upon the choice of the expansion basis.

The two bases described in the previous section have been used to investigate the theoretical uncertainties of the charge distribution.

So far we have studied the properties of the two bases in the framework of the PWBA, but a realistic description of the elastic electron scattering cross section requires the full solution of the Dirac equation. The Dirac equation for the elastic scattering of electrons from a potential has been solved in the ultrarelativistic limit, i.e. neglecting the terms containing the electron mass, using standard procedures [9].

The uncertainties of the fit procedures have been studied working on a set of pseudodata generated by known charge densities. We have evaluated the ability of the expansion techniques to reproduce the known densities and we have designed a procedure to minimize the theoretical uncertainties. In a second step we have applied this procedure to analyze the experimental data on $^{12}C$, $^{40}Ca$ and $^{208}Pb$ [10–13].

3.1 Application to pseudodata

The set of pseudodata we have used to study the uncertainties of the fit procedures has been generated by shell model densities produced by a Woods-Saxon potential:

$$V_{WS}(r) = \frac{-V_0}{1 + e^{(r-R)/a}} + \left(\frac{h}{m_\pi c}\right)^2 \frac{d}{dr} \left(\frac{-V_{ls}}{1 + e^{(r-R)/a}}\right) \vec{l} \cdot \vec{\sigma} + V_{Coul} \tag{3.1}$$

where we have indicated with $m_\pi$ the pion mass. In the following we shall call these charge distributions Woods-Saxon densities.

The values of the parameters used in our calculations are given in Tab.1. With the electrostatic potentials generated by these charge distributions we have calculated the elastic cross sections at the same angles where the experimental points of ref. [10–13] have been measured. We have associated to each point the percentage error of the analogous experimental point. More information about the pseudodata are given in Tab.2.

The fit procedure of the pseudodata consisted in modifying the N expansion coefficients of the charge distribution until the minimum $\chi^2$ was found.
We should remark that not all the N coefficients are independent. The normalization of the charge density, eq. (2.14), imposes a condition on the parameters. For the FB expansion, we have imposed the further condition that the first derivative of the density should be zero at $R_c$. In the following, if not explicitly mentioned, we shall refer to the N expansion coefficients keeping in mind that for the FB expansion only N-2 coefficients are independent and for the Hermite expansion N-1. In the fitting procedure only the independent coefficients are changed.

We show in figs. 3 and 4 the sensitivity of the fit to the scale parameters $R_c$ and $\beta$. In these figures the full lines represent the Woods-Saxon density used to generate the pseudodata.

We have already discussed in sect. 2 the method used to fix $R_c$, and for $^{12}C$ with data up to $q_{\text{max}} \sim 3.0$ fm$^{-1}$ we found values of the cut-off radius larger than 8 fm.

The value of $\beta$ is chosen imposing two conditions on the n-th term of the Hermite expansion: the first one is that the last inflection point in its contribution to the form factor is beyond $q_{\text{max}}$, see sect. 2.2. The second condition is asking that the last inflection point of its contribution to the charge distribution should be far enough from the nuclear radius $R$ in a position $R_c$ where the density is negligible.

After imposing these two conditions we found: $\beta = \sqrt{R_c/q_{\text{max}}}$. For $R_c$ we choose the values of cut off radius used in the FB case, therefore we obtained $\beta \sim 1.5, 1.7$ fm.

We compare in fig. 3 two fits of the $^{12}C$ pseudodata done with a FB basis using two values of $R_c$. In this figure the Woods-Saxon density used to generated the pseudodata is shown by the full lines.

In the upper panel of fig.3 we show the fit done with $R_c = 8.0$ fm. With 8 FB coefficients the charge distribution is already well described as well as the cross section pseudodata, (we obtain a $\chi^2$ of about 1 per degree of freedom). Increasing the number of coefficients improves the agreement with the true density. The situation remains stable until we used 14 coefficients. In this case, while the description of the cross section remains very good (still we get $\chi^2 \sim 1$ per degree of freedom), the density obtained shows unreasonable oscillations around the true density.

The results of the lower panel of fig.3, where we used $R_c = 10$ fm, show that 8 FB coefficients are not enough to obtain a reasonable description of the density while the density obtained with 14 coefficients is still adequate.

The general features of the fit procedure are moved to higher order of coefficients when the value of $R_c$ is increased. This is consistent with the empirical formula $n_{\text{max}} = q_{\text{max}}R_c/\pi$. A small value of $R_c$ allows one to describe the data with a lower number of coefficients, on the other hand, also the instabilities on the density show up for a lower number of coefficients.

We reached the same conclusions also in the case of the Hermite basis, shown in fig. 4. In the upper panel of this figure we present the results of the fits done with $\beta = 1.5$ fm, and we observe that a fit with 6 coefficient already reproduces very well the true density. In the lower panel we show the results of the fits done with $\beta = 1.7$ fm. In this case 6 coefficients are not enough to reproduce reasonably well the density.

The comparison of fig. 3 and 4 seems to indicate that the expansion on the Hermite basis is more stable than the FB expansion. This means that, with respect to the FB basis, it is possible to achieve a reasonable description of the data, and of the true density, with a smaller number of terms, and that the unphysical oscillations become evident only for a larger number of terms. This is clear if one compares the result of the fit with 16 terms of fig. 3 with the analogous fit in fig. 4.

The two figures we have presented contain all the relevant features of the model independent procedure for extracting charge distributions from cross section data.

There is a minimum number of coefficients necessary to get a reasonable fit to the data. For a set of data, this minimum number depends on the choice of the expansion basis and on the value of the scale parameter.

There is also an upper limit of the number of coefficients which allows one to produce reasonable densities distributions. Adding coefficients to the expansion maintains constant the quality of the fit of the cross section but it produces oscillations of the density. This fact is easily understood looking at the mechanism which allows the FB and the Hermite expansions to reproduce the cross section (see fig. 1). High order terms are effective at high values of the momentum transfer. Therefore all the terms contributing in a momentum region not covered by the data are undetermined. They do not effect the fit of the cross section but their presence is strongly felt by the charge distribution.

The minimum number of coefficients is easily found when adequate values of the $\chi^2$ and of the confidence
levels are reached. It is more difficult to find the upper limit of the number of expansion coefficients. This
problem is rather well known in the literature [3] and it has been usually solved guessing the behavior of the
cross section, or of the form factor, in the momentum transfer region beyond the maximum value measured.
The continuation of the cross section in the unmeasured region fixes the values of the high-order coefficients
preventing them from producing unwanted oscillation of the density distribution.

In spite of the more or less reasonable hypotheses done about the behavior of the cross section in an
unmeasured region, it is clear that this procedure adds further uncertainties to the charge distribution.

We believe that the charge density distributions should be extracted considering only the measured
data. The truncation of the basis is intrinsically related to the knowledge of the data on a limited range of
momentum transfer values, and this is part of the experimental uncertainty.

The maximum number of coefficients determined by a set of data extended up to \( q_{\text{max}} \) is given in PWBA
by the eqs. (2.8b) and (2.20). These rules can be used as an indication, but in the realistic description of
the scattering process they are not precise.

We have tackled the problem of cutting the expansion together with the problem of propagating the
experimental errors of the cross section on the charge distribution. The number of expansion terms and the
error on the charge distribution are both fixed by the data.

The estimate of the error on the density distribution is usually obtained with the error propagation
rules [3]. These rules assume a quadratic behavior of the covariance matrix in the region of the values of the
parameters producing the minimum \( \chi^2 \).

We have calculated the error propagation with a different procedure. From the original cross section
calculated with the Woods-Saxon densities, we have generated sets of pseudodata. Each set has been
constructed by random values normally distributed around the exact cross section values with standard
deviation equal to the associated error.

Every fit produces a new charge distribution. Putting together all the charge distributions we obtain a
band of lines which automatically take into account the uncertainty due to the errors of the cross sections.

Our procedure is equivalent to the standard procedure if the quadratic behavior of the covariance matrix
around the point of minimum of the fit is a good approximation.

As example, we show in figs. 5 and 6 some of the density distributions bands obtained performing a set
of 100 fits of the randomly generated pseudodata of \(^{40}\text{Ca}\). We have applied this procedure for both FB and
Hermite expansion.

Instead of drawing every density with a continuous line, we have indicated each line with a fixed number
of dots randomly distributed in \( r \). The darkening of one region gives an indication of the fact that many
charge distributions are overlapping.

In these figures the full lines are representing the Woods-Saxon densities used to generate the cross
section data.

We should point out that not all the lines which can be drawn within the density band are reproducing
a set of pseudodata. The density points are strongly correlated among them. For example, one of the
constraints on these points is the fact that the charge distribution should satisfy the normalization condition
(2.14).

Every fit produces also a new set of best fit parameters. Superimposed to the figures of the distribution
bands, for each expansion coefficient, we show the absolute value of the ratio between the standard deviation
of the n-th expansion coefficient and its average value. When this ratio is above 1 the value of the expansion
coefficient is compatible with zero. This means that the data are not sensitive to this term of the expansion.

In each figure we have inserted the value of the average \( \chi^2 \) per number of degrees of freedom and the
value of the average confidence level.

In fig. 5 we observe that the fits done with 9, 10 and 11 FB terms have similar characteristics: the same
value of the averaged \( \chi^2 \), similar values of the averaged confidence levels and density bands concentrated
around the valued of the true distribution. The lower panel on the right shows that the simple addition of
another coefficient produces an unstable situation. The density band is quite spread around the true value
of the density and 4 out of 12 coefficients are compatible with zero. This instability of the fit it is not shown
by the values of the averaged \( \chi^2 \) and confidence level.

This fact is not specific of the chosen expansion basis because also in fig. 6, where the Hermite basis
has been used, we move suddenly from a stable situation to a situation where 6 coefficients are compatible

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with zero. We remark in fig. 6 that the presence of coefficients compatible with zero does not mean that the density band is oscillating around the true density distribution. Both the fits with 10 and 11 terms have one term compatible with zero, but they shows stable density bands. We notice that the fit with 11 parameters produces an error band narrower than the fit with 10. The fit with 13 terms starts to show some instability, but it is the fit with 14 terms which produces a strongly spread density band. In this last case 10 out of 14 terms are undetermined. Note that, like in the FB case, the values of the average $\chi^2$ and confidence level are not sensitive to the instability of the fit.

A comparison between fig. 5 and 6 emphasizes the differences between the two bases. In fig. 5 we observe that the increase of the number of coefficients of the FB expansion is affecting only the last coefficients. For example in the fit with 12 expansion terms, the last four coefficients are undetermined, while the first eight have roughly the same values presented in the other panels. The situation is reversed in the Hermite case where the addition of new coefficients strongly modifies also the other ones, as we note on fig. 6. This fact indicates that the Hermite coefficients are more strongly correlated than the FB ones. This correlation is clearly indicated by the covariance matrices obtained in the fits [14] and is connected to the different characteristic of the two bases we have already discussed in the description of fig. 1. Each term of the FB basis produces contributions rather well localized in momentum space, while the contribution of each term of the Hermite basis is spread over a wider momentum range.

The instability of the charge distributions is not connected to the fact that high order terms are not constrained by the data, this is simply an accident related to the FB expansion basis. There are general properties of the charge distributions not restricted by the data, independently from the chosen basis.

The tests performed for the $^{12}$C and $^{208}$Pb show a clear jump in the values of the $\chi^2$ and of the confidence level from a fit with a too small number of parameters to a fit with an adequate number of parameters. This allows one to fix a minimum number of expansion terms.

Unfortunately, as we have discussed for the $^{40}$Ca case, there is not such a clear way to fix the maximum number of coefficients. The criterion of excluding all the fits containing at least one undetermined coefficient, i.e. compatible with zero, is not adequate. As we have seen in the above discussion, there are situations where some of the expansion coefficients are compatible with zero but the distribution band is concentrated around the Woods-Saxon distribution.

We can summarize our aim saying that we are looking, in the space of the coefficients, for a vector which minimizes the $\chi^2$ and, at the same time, is stable against statistical variations of the cross section values.

The stability of a set of expansion coefficients with respect to the variations of the cross section can be measured by the quantity:

$$S(N_c) = \frac{\sum_{n=1}^{N} <a_n>^2 - \sum_{n=1}^{N} <a_n^2>}{\sum_{n=1}^{N} <a_n>^2},$$

(3.2)

where $N$ is the number of expansion coefficients, and

$$<a_n> = \frac{1}{M} \sum_{\mu=1}^{M} a_n^{(\mu)}$$

(3.3)

is the value of the coefficient $a_n$ averaged on $M$ samples, and

$$<a_n^2> = \frac{1}{M} \sum_{\mu=1}^{M} |a_n^{(\mu)}|^2.$$  

(3.4)

A small value of the quantity $S(N)$ indicates that the vector constructed with the average values of the coefficients is rather well defined. This means that the errors on the coefficients are quite small and this produces a narrow distribution band. Of course large values of $S(N)$ are connected with broad distribution bands.

Together with the quantity defined in eq. (3.2), which gives information about the stability of the fit with respect to statistical variations, we should also include an indicator of the quality of the fit. For this reason we define:

$$G(N) = \frac{<cl(N)>}{S(N)},$$

(3.5)
where \( <c_l(N)> \) is the average confidence level. The quantity defined in eq. (3.5) is not linked to the accuracy in the definition of the single expansion coefficient. For particular reasons one of the coefficients of the expansion can be compatible with zero without producing instabilities in the charge distribution. The value of \( G \) gives information about the global features of the fit and the stability of the distribution density.

In fig. 7 we show the values of \( G \), calculated for the cases we have studied in figs. 5-6, as a function of the number of expansion coefficients \( N \). The values of \( G \) have been normalized in order to have all the maxima equal to one.

The FB fits show sharp maxima, different for each nucleus considered. For the Hermite case the situation is more complicated. The maxima are not so sharp and there is a set of parameters with analogous characteristics. This confirms the fact that the Hermite expansion is more stable against variations of the number of coefficients than the FB expansion.

The maxima of \( G \) correspond to the ideal number of expansion coefficients. This number optimizes the quality of the fit of the cross section data and the stability of the result against the statistical fluctuations. In the next subsection we shall discuss the results we have obtained applying this procedure to the experimental data of ref. [10–13].

Before closing we would like to remark that the procedure we have designed provides the best distribution band compatible with the data, statistical errors included. However this does not ensure that the density generating the cross section certainly lies in the distribution band obtained applying the procedure.

This point is quite well illustrated in fig.8. The full lines represent a density generated with 15 FB coefficients. We have used this density to produce \(^{12}\text{C}\) pseudodata. Since the shape of the density is quite odd, we verified that the cross section was not showing anomalous behavior at high \( q \), at least within a reasonable range of values of the momentum transfer. For this reason we have slightly increased the momentum transfer covered with respect to the \(^{12}\text{C}\) pseudodata used for the previous analysis. With the charge distribution of fig.8 we generated data up a scattering angle of 140 degrees corresponding to a maximum momentum transfer of 3.6\( \text{fm}^{-1} \).

We have applied the fit procedure to this set of pseudodata. We found an optimal value of 10 FB coefficients and this produces the distribution band shown in the upper panel of fig. 8. The original charge distribution in not included in this band. On the other hand it is shown in the central panel of fig. 8 that the band generated by a 15 parameters fit also contains the original distribution. The optimal fit done with 10 Hermite functions shown in the bottom panel produces a distribution band practically identical to that obtained with 10 FB coefficients. This gives us confidence of the fact that the procedure we have designed is independent from the choice of the expansion basis and the result is only related to the set of data. This procedure fixes the number of expansion coefficients to obtain a reasonable compromise between the quality of the fit and the uncertainties on the density. This, however, does not ensure that the real density is contained in the found band.

### 3.2 Application to experimental data

The methodology presented in the previous section has been applied to the study of the experimental data of \(^{12}\text{C}\), \(^{40}\text{Ca}\) and \(^{208}\text{Pb}\) given in refs. [10–13].

We have assumed that the quoted experimental errors are only statistical and we have generated sets of 100 angular distributions with the same method used for the pseudodata. We considered about the fact that we are sampling on data which are already a statistical sample by multiplying each error by the factor \( \sqrt{2} \). These data have been fitted with both the FB and Hermite basis changing the number of expansion coefficients.

The results of these calculations are summarized in tab. 3 where we present the average values of the \( \chi^2 \) and of the confidence level for each set of fits. We have performed the test of the \( G \) factor defined in eq. (3.5) and we found, for each expansion basis, the optimal number of expansion term.

We present in figs. 9–11 the distribution bands produced by the optimal number of coefficients for both expansion bases and for the three nuclei considered. In these figures the full lines are the charge distributions obtained with the FB coefficients published in ref. [15]. For \(^{12}\text{C}\) and \(^{40}\text{Ca}\) these densities present small differences with respect to our density bands.
In fig. 11 together with our distribution bands we present two charge distributions both obtained with FB parametrizations quoted in ref. [15]. The full line is produced by the 11 coefficients parametrization and it overlaps completely with the distribution bands. The dashed line is instead generated with the 17 coefficients parametrization.

We have performed a consistency test between the distribution bands obtained with the FB and Hermite expansions and we found that, for each nucleus considered, the differences between the FB and the Hermite densities bands are compatible with zero. Looking in detail, however, we found that they are not always equally spaced around the zero, contrary to the result we have obtained with the pseudodata. This, together with the relatively low values of the confidence levels shown in tab. 3, can be an indication of the fact that the set of data we have used are not a good statistical sample.

4. Conclusions

The work we have presented in this paper has been addressed to the study of the theoretical uncertainties in the procedure of extracting the charge distributions from elastic electron scattering cross sections. We have worked within the direct scattering approach to investigate the uncertainties of the so-called model independent techniques consisting in expanding the charge distributions on a orthonormal basis and finding the coefficients of the expansion to obtain the best fit to the data.

In order to obtain information independent from the choice of the expansion basis we have used two bases with different analytical properties. The FB basis is rather localized in \( q \) space, as it is shown in fig.1, and it is spread in the coordinate space. The Hermite basis, used for the first time in this context, has analogous characteristics in both \( q \) and \( r \) space, the Fourier transform of a Hermite function is still a Hermite function. Fig.1 shows that the contribution of each term of the Hermite expansion is not localized around a specific value of \( q \).

We found few advantages in using the Hermite basis with respect to the FB ones. The Hermite expansion seems to be more stable and usually it requires a smaller number of expansion terms to obtain fits of the same quality (see fig. 3 and 4). In addition, while with FB one should impose that the distribution should be zero after a certain radius, the Hermite expansion is not requiring any hypothesis on the shape of the charge density.

In any case the aim of our work was not the proposal of a new expansion basis to be used in model independent method, but rather the investigation of the uncertainties related to the method in itself. We found that in both the expansion bases it is not possible to increase at will the number of expansion terms. There is an upper limit when the uncertainty band of the charge distribution starts to become very large and shows big oscillations, in spite of the fact that the quality of the fit to the cross section is not worsening. The increase of the distribution uncertainty band appears when the full set of expansion terms is not any more constrained by the data.

This problem is not related to the choice of the basis but it is intrinsic to the extraction method, due to the necessary discretization and limitation in \( q \) of the experimental data and to the presence of statistical error. These are also the reasons why the solution of the inverse problem is not unique.

It is clearly possible to set up methods to stabilize the result of the fit. For example, we have designed a procedure to determine the number of coefficients giving the best compromise between the quality of the fit and the stability of the set of parameters with respect to statistical fluctuations of the data. This procedure consists in repeating the fit with various numbers of expansion terms. Each set of fits is analyzed with the indicator defined in eq. (3.5) whose maximum provides the optimal number of coefficients.

This is only a useful pragmatic recipe but it does not guarantee that the unknown quantity is within the obtained density band. This is strikingly shown in fig. 8 where the cross section produced by a oscillating density distribution has been fitted. The application of the prescription to both FB and Hermite bases allows us to find an optimal value of the number of expansion terms, but the obtained density bands are not containing the true density distribution.

In the model independent method the truncation error is unavoidable. It is even not possible to give a realistic estimation of it, because it is related to the procedure used to stabilize the density distribution band.
The model independent method does not provide stable solution of the inverse problem, in the sense that small variations of the cross section produce large variations of the density. It is necessary to perform an opportune truncation of the basis to stabilize the solution. The criteria used to perform the truncation are based upon pragmatical considerations. For these reasons the distributions extracted with these methods must be considered with caution when comparing to nuclear models.

It is a pleasure to acknowledge the useful discussions with S.Caracciolo, A.M.Lallena, G.Lolos, G.Mancarella, R.Perrino, P.Rotelli, R.Schiavilla and I.Sick.
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    B.Frois, J.Bellicard, J.M.Cavedon, M.Huet, P.Leconte, A.Nakada, P.X.Ho and I.Sick,


Table 1. Parameters of the Woods-Saxon potential (eq. 3.1) producing the densities used to generate the pseudodata.

<table>
<thead>
<tr>
<th></th>
<th>( V_o ) (MeV)</th>
<th>( V_{LS} ) (MeV)</th>
<th>( R ) (fm)</th>
<th>( a ) (fm)</th>
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</thead>
<tbody>
<tr>
<td>(^{12}\text{C})</td>
<td>55.0</td>
<td>3.2</td>
<td>2.86</td>
<td>0.57</td>
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<tr>
<td>(^{40}\text{Ca})</td>
<td>57.0</td>
<td>11.1</td>
<td>4.1</td>
<td>0.53</td>
</tr>
<tr>
<td>(^{208}\text{Pb})</td>
<td>60.4</td>
<td>6.75</td>
<td>7.2</td>
<td>0.59</td>
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</table>

Table 2. Information about the pseudodata. For each nucleus considered we present: the number of pseudodata generated (\(ndata\)), the energy of the electrons (\(\epsilon\)), the maximum scattering angle (\(\theta_{max}\)) and the corresponding maximum momentum transfer (\(q_{max}\)).

<table>
<thead>
<tr>
<th></th>
<th>(ndata)</th>
<th>(\epsilon) (MeV)</th>
<th>(\theta_{max}) (deg.)</th>
<th>(q_{max}) (fm(^{-1}))</th>
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</thead>
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<tr>
<td>(^{12}\text{C})</td>
<td>99</td>
<td>400</td>
<td>132</td>
<td>3.5</td>
</tr>
<tr>
<td>(^{40}\text{Ca})</td>
<td>148</td>
<td>502</td>
<td>110</td>
<td>3.9</td>
</tr>
<tr>
<td>(^{208}\text{Pb})</td>
<td>148</td>
<td>502</td>
<td>87</td>
<td>3.2</td>
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</table>
Table 3. Averaged $\chi^2$ and confidence level for the fits to the experimental data obtained with various number of expansion coefficients.

<table>
<thead>
<tr>
<th></th>
<th>$^{12}$C FB</th>
<th>$^{12}$C Hermite</th>
<th>$^{40}$Ca FB</th>
<th>$^{40}$Ca Hermite</th>
<th>$^{208}$Pb FB</th>
<th>$^{208}$Pb Hermite</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>$\chi^2$</td>
<td>c.l.</td>
<td>$\chi^2$</td>
<td>c.l.</td>
<td>$\chi^2$</td>
<td>c.l.</td>
</tr>
<tr>
<td>7</td>
<td>1.08</td>
<td>0.35</td>
<td>1.17</td>
<td>0.17</td>
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<td></td>
</tr>
<tr>
<td>8</td>
<td>1.93</td>
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<td>9</td>
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<td>0.19</td>
<td>1.07</td>
<td>0.39</td>
<td>1.12</td>
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<td>1.13</td>
<td>0.28</td>
<td>1.10</td>
<td>0.33</td>
<td>1.11</td>
<td>0.15</td>
</tr>
<tr>
<td>12</td>
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<td></td>
<td></td>
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<td>0.22</td>
</tr>
<tr>
<td>14</td>
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<td></td>
<td></td>
<td></td>
<td>1.13</td>
<td>0.23</td>
</tr>
</tbody>
</table>

$\chi^2$ and confidence level for the fits to the experimental data obtained with various number of expansion coefficients.
Figure captions

Fig.1 Form factor squared shown as a function of the momentum transfer. The thick lines represent the form factor produced by the test density distribution of eq. (2.11). The panels a and c show the form factor obtained with 4 and 6 FB expansions terms respectively. The panels b and d show the analogous fit performed with the Hermite expansion basis. The full thin lines show the form factors generated summing the N terms of the expansion and the dashed line show the contribution of the n-th term. The form factors are all normalized such as \( F(0) = 1.0 \).

Fig.2 Comparison between exact form factors, eq. (2.12), (thick lines) and form factors calculated with a truncated density, eq. (2.15), (thin lines).

Fig.3 Charge densities obtained from a FB fit of the \(^{12}\text{C}\) pseudodata done with various number of expansion terms. The results of the upper panel have been obtained with \( R_c = 8 \) fm, those of the lower panel with \( R_c = 10 \) fm. The full lines represent the Woods-Saxon charge distribution used to generate the pseudodata.

Fig.4 The same as fig. 3 for the Hermite expansion basis. Also in this case we presents results obtained with two different values of the scale parameters: \( \beta = 1.5 \) fm (upper panel) and \( \beta = 1.7 \) fm (lower panel).

Fig.5 The bottom and left scale refer to the distribution bands produced by the 100 FB fits to \(^{40}\text{Ca}\) pseudodata represented by the randomly scattered dots \( (R_c = 8.0 \) fm). The full lines show the Woods-Saxon charge distribution generating the pseudodata. The upper scale shows the number of expansion coefficients used in the fit and the right scale refers to the absolute value of the ratio between the error and the average value of the expansion coefficient. These ratios are shown by the circles. The average values of the reduced \( \chi^2 \) and of the confidence level are also given.

Fig.6 The same as fig. 5 for the case of the Hermite expansion \( (\beta = 1.5 \) fm).

Fig.7 The quantity \( G \) defined in eq. (3.5) as a function of the expansion coefficients for the three nuclei under investigation. The full lines are related to \(^{12}\text{C}\), the dotted lines to \(^{40}\text{Ca}\) and the dashed lines to \(^{208}\text{Pb}\). Each line has been normalized in order to have the maximum equal to 1.

Fig.8 Fit to the pseudodata generated by the charge distribution shown by the full lines. The distribution band of the upper panel has been obtained with 10 FB coefficients and the density band of the middle panel with 15 FB coefficients. The lower panel shows a density band obtained with 10 Hermite coefficients.

Fig.9 Charge density bands obtained from the fit to the experimental data of ref.[10] using the FB basis (10 coefficients) and the Hermite basis (10 coefficients). The full lines represent the density obtained by the FB parametrization given in ref.[15].

Fig.10 Same as fig. 9 for \(^{40}\text{Ca}\). The experimental data are from refs. [11] and we used 10 coefficients for both FB and Hermite basis. Like in fig. 9 the full lines are giving the FB density of ref.[15].

Fig.11 Same as the previous two figures for \(^{208}\text{Pb}\) where we used the data of refs. [12,13]. The number of coefficients used in the fits are 12 and 11 for the FB and the Hermite basis respectively. Full and dashed lines are showing the distributions given by the FB parametrization in ref. [15]. The full lines have been obtained with the 11 FB coefficients parametrization, the dashed lines with the 17 FB coefficients parametrization.