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Forcing of a Distribution Type

by

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Integrable Equations with a Forcing of a Distribution Type

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Abstract

We consider the nonlinear Schrodinger equation in the variable $q(x, t)$ with the forcing $iu(t)\delta(x) + iu_1(t)\delta'(x)$. We assume that $q(x, 0)$, $u(t)$, $u_1(t)$ are given and that these functions as well as their first two derivatives belong to $L_1 \cap L_2(\mathbb{R}^+)$. We show that the solution of this problem can be reduced to solving two Riemann-Hilbert (RH) problems in the complex k -plane with jumps on $Im(k^2) = 0$. Each RH problem is equivalent to a linear integral equation which has a unique global solution. These linear integral equations are uniquely defined in terms of certain functions (scattering data) $b(k)$, $c(k)$, $d(k)$ and $f(k)$. The functions $b(k)$ and $d(k)$ can be effectively computed in terms of $q(x, 0)$. However, although the analytic properties of $c(k)$ and $f(k)$ are completely determined, the relationship between $c(k)$, $f(k)$, $q(x, 0)$, $u(t)$, and $u_1(t)$ is highly nonlinear. In spite of this difficulty we can give an effective description of the asymptotic behavior of $q(x, t)$ for large t . In particular we show that as $t \rightarrow \infty$, solitons are generated moving away from the origin. It is important to emphasize that the analysis of this problem, in addition to techniques of exact integrability, requires the essential use of general PDE techniques.

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1 Introduction

For integrable equations such as the nonlinear Schrodinger (NLS) and the Korteweg-deVries (KdV) equations, a method exists for solving the Cauchy problem on the infinite line for decaying initial data. This method reduces the solution of the Cauchy problem to the solution of a linear integral equation. An asymptotic analysis of this integral equation shows that for large time and for $\frac{x}{t} = O(1)$, any initial data will decompose into a number of solitons. This type of asymptotic information, which is both mathematically and physically very important cannot be obtained by the usual PDE techniques. Recently a method has been developed [1], for solving the initial-boundary value problem on the half-infinite line for decaying initial and boundary data. This method also reduces the solution of the initial-boundary value problem to the solution of a linear integral equation whose large time asymptotic investigation shows how the boundary data can generate solitons. This method has already been applied to the NLS [2], to the sine-Gordon [3], and to the KdV [4].

It was shown in [5] that the methodology for studying initial-boundary value problems can be used to study forced problems when the forcing is of a certain distributional type. Such problems arise in several physical applications. Consider for example, the problem of an inviscid flow over a locally non-flat topography (see Figure 1.1) when the flow velocity is near the critical, long wave speed.

Figure 1.1

It can be shown (see [6]-[10]) that in the usual inviscid, weakly nonlinear, shallow water, long wave approximation, the first order term of the outer solution satisfies a forced KdV, where the forcing is of a distribution type. Actually there exist three distinguished cases depending on the specific topography. These cases are characterized by,

$$2\chi_\tau + 2u\chi_\xi - 3\chi\chi_\xi - \frac{1}{3}\chi_{\xi\xi\xi} = \begin{cases} B_0\delta(\xi), & m = 2 \\ A_0\delta'(\xi), & m = \frac{3}{2} \\ -A_1\delta''(\xi), & m = 1 \end{cases},$$

where $\zeta(x, t, \varepsilon) = \chi(\xi, \tau) + O(\varepsilon^{\frac{1}{2}})$, $\tau = \varepsilon^{\frac{3}{2}}t$, $\xi = \varepsilon^{\frac{1}{2}}x$. The three cases above correspond to: (i) the flow over a step (either a rise, $B_0 > 0$, or a drop, $B_0 < 0$); (ii) the flow over a bump (either of elevation, $A_0 > 0$, or of depression $A_0 < 0$); (iii) the flow over a bump of zero net value matrix displacement but with a nonvanishing first wavelet.

In this paper we show how the new methodology developed for initial-boundary value problems, can be used to linearize forced integrable equations, where the forcing is of a distribution type. For simplicity we consider the forced NLS equation. However, similar considerations apply to other integrable equations. In particular our results for the half-line problem of the KdV and sine-Gordon equations, and the procedure used here, can be used for the linearization of the forced KdV and of the forced sine-Gordon equations.

We will consider the forced NLS equation

$$iq_t + q_{xx} - 2\lambda|q|^2q = iu(t)\delta(x) + iu_1(t)\delta'(x); \lambda = \pm 1, \quad x \in (-\infty, \infty), \quad t \in [0, \infty), \quad (1.1)$$

where $q(x, 0)$, and $u(t)$, $u_1(t)$ are given functions decaying for large x and t respectively. In equation (1.1), $\delta(x)$ and $\delta'(x)$ denote the Dirac delta function and its derivative. The cases of $\lambda = 1$ and $\lambda = -1$ are usually referred to as the defocusing and focusing cases respectively. Let $q_0(x) \doteq q(x, 0)$. We will assume that

$$q_0(x) \in H_2(\mathbb{R}), \quad xq_0(x), \quad x^2q_0(x) \in L_2(\mathbb{R}), \quad v(t) \in C_2(\mathbb{R}^+), \quad v(t) \in L_1 \cap L_2(\mathbb{R}^+), \quad (1.2)$$

$$v'(t), \quad tv(t), \quad tv'(t), \quad tv''(t) \in L_1(\mathbb{R}^+),$$

where $v(t)$ is either $u(t)$ or $u_1(t)$, H_2 denotes that a function and its first two derivatives belong to L_2 , and C_2 denotes that a function is twice differentiable.

Equation (1.1) implies that both q and q_x are discontinuous at $x = 0$, and that they satisfy the jump conditions

$$[q_x]_{x=0} = iu(t), \quad [q]_{x=0} = iu_1(t), \quad (1.3)$$

where we use the notation $[f(x)]_{x=0} = f(+0) - f(-0)$.

Let

$$q(x, t) = q_l(x, t) + q_r(x, t); \quad q_l(x, t) = q(x, t)H(-x), \quad q_r(x, t) = q(x, t)H(x), \quad (1.4)$$

where $H(x)$ denotes the Heaviside function, i.e. $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x < 0$. Then the forced equation (1.1) is equivalent to the following two coupled initial-boundary value problems for the homogeneous NLS equation

$$iq_t + q_{xx} - 2\lambda|q|^2q = 0; \quad (1.5)$$

$$q_l(x, t) : x \in (-\infty, 0], \quad t \in [0, \infty); \quad q_l(x, 0) = q(x, 0)H(-x), \quad (1.6)$$

$$q_r(x, t) : x \in [0, \infty), \quad t \in [0, \infty); \quad q_r(x, 0) = q(x, 0)H(x). \quad (1.7)$$

The boundary data associated with these two initial-boundary value problems are coupled via

$$\frac{\partial}{\partial x} q_r(0, t) - \frac{\partial}{\partial x} q_l(0, t) = iu(t), \quad q_r(0, t) - q_l(0, t) = iu_1(t). \quad (1.8)$$

We recall that the NLS equation (1.5) is the compatibility condition of the Lax pair

$$\psi_x = (-ik\sigma_3 + Q)\psi, \quad \sigma_3 = \text{diag}(1, -1), \quad Q(x, t) = \begin{pmatrix} 0 & q(x, t) \\ \lambda \bar{q}(x, t) & 0 \end{pmatrix}, \quad (1.9)$$

and

$$\psi_t + 2ik^2[\sigma_3, \psi] = \hat{Q}\psi, \quad \hat{Q}(x, t, k) = -i\lambda|q|^2\sigma_3 + 2kQ - iQ_x\sigma_3. \quad (1.10)$$

We now summarize the main results.

Theorem 1.1 Let the 2×2 matrix $\hat{Z}_r(x, t, k)$ be the solution of the following RH problem (right RH problem).

- (i) $\hat{Z}_r$ is a sectionally meromorphic function in the complex k -plane cut along $\text{Im}(k^2) = 0$.
- (ii) $\hat{Z}_r$ has unit determinant and satisfies the jump conditions

$$\hat{Z}_r^-(x, t, k) = \hat{Z}_r^+(x, t, k)e^{-i(kx+2k^2t)\sigma_3}G_r(k)e^{i(kx+2k^2t)\sigma_3}, \quad k^2 \in \mathbb{R}; \quad \hat{Z} \rightarrow I \text{ as } k \rightarrow \infty, \quad (1.11)$$

where the jumps $G_r(k)$, depicted in Figure 1.2, are defined in terms of $b(k)$ and $c(k)$. The function $b(k)$ is defined by

$$b(k) \doteq \frac{s_1(k)}{s_2(k)}, \quad k \in \mathbb{R}. \quad (1.12)$$

The x, t dependence enters only through $\theta(x, t) = 2i(kx + 2k^2t)$.

$x > 0$

$$\begin{pmatrix} 1 & 0 \\ c(k)e^\theta & 1 \end{pmatrix} \begin{array}{c|c} II & I \\ \hline - & + \\ \hline III & IV \end{array} \begin{pmatrix} 1 & -b(k)e^{-\theta} \\ \lambda \bar{b(k)}e^\theta & 1 - \lambda|b(k)|^2 \end{pmatrix} \begin{pmatrix} 1 & -\lambda \bar{c(k)}e^{-\theta} \\ 0 & 1 \end{pmatrix}$$

Figure 1.2

(iii) The functions $s_1(k)$ and $s_2(k)$ are uniquely defined in terms of $q_r(x, 0)$ by the equations $s_1(k) \doteq \psi_1(0, k)$, $s_2(k) \doteq \psi_2(0, k)$, where $(\psi_1, \psi_2)^T$ is the solution of (1.9) with $q(x, t)$ replaced by $q_r(x, 0)$, satisfying the boundary condition $\lim_{x \rightarrow \infty} [(\psi_1, \psi_2)^T \exp(-ikx)] = 0$. Both $s_1(k)$ and $s_2(k)$ have analytic extensions in the upper half of the complex k -plane which will be denoted by $\mathbb{C}^+$. Although $c(k)$ depends in a nonlinear way on $q_0(x)$, $u(t)$ and $u_1(t)$, its analytic structure is completely determined by the conditions that $\lim_{k \rightarrow \infty} c(k) = 0$ and that $c(k)$ is meromorphic in the second quadrant of the complex k -plane (which will be denoted by II).

(iv) $\hat{Z}_r(x, t, k)$ can have poles only at the poles of $c(k)$, $k \in II$, which do not coincide with the zeros of $s_2(k)$, and at the complex conjugates of these points. If k_0 is such a pole, then

$$\text{res}_{k_0} \hat{Z}_r \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \text{res}_{k_0} c e^{\theta(k_0)} \hat{Z}_r \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\text{res}_{\bar{k}_0} \hat{Z}_r \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda \overline{\text{res}_{k_0} c} e^{-\theta(\bar{k}_0)} \hat{Z}_r \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let the 2×2 matrix $\hat{Z}_l(x, t, k)$ be the solution of the following RH problem (left RH problem).

(i) $\hat{Z}_l$ is a sectionally meromorphic function in the complex k -plane cut along $Im(k^2) = 0$.

(ii) $\hat{Z}_l$ has unit determinant and satisfies jump conditions similar to (1.11) with the matrix $G_r(k)$ replaced by $G_l(k)$, depicted in Figure 1.3 and defined in terms of $d(k)$ and $f(k)$. The function $d(k)$ is defined by

$$d(k) = \frac{l_1(k)}{\overline{l_2(k)}} \quad k \in \mathbb{R}. \quad (1.13)$$

$$\boxed{x < 0} \quad \begin{pmatrix} 1 & -\lambda \overline{f(\bar{k})} e^{-\theta} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -d(k) e^{-\theta} \\ \lambda \overline{d(k)} e^{\theta} & 1 - \lambda |d(k)|^2 \end{pmatrix} \begin{array}{c|c} II & I \\ \hline - & + \\ \hline + & - \\ \hline III & IV \end{array} \begin{pmatrix} 1 - \lambda |d(k) - \lambda \overline{f(\bar{k})}|^2 & (d(k) - \lambda \overline{f(\bar{k})}) e^{-\theta} \\ -\lambda (\overline{d(k)} - \lambda f(k)) e^{\theta} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ f(k) e^{\theta} & 1 \end{pmatrix}$$

Figure 1.3

(iii) The functions $l_1(k)$ and $l_2(k)$ are uniquely defined in terms of the $q_l(x, 0)$ by the equation $l_1(k) \doteq \varphi_1(0, k)$, $l_2(k) \doteq \varphi_2(0, k)$ where $(\varphi_1, \varphi_2)^T$ is the solution of equation (1.9) with $q(x, t)$ replaced by $q_l(x, 0)$, satisfying the boundary condition $\lim_{x \rightarrow \infty} [(\varphi_1, \varphi_2)^T \exp(-ikx)] = (0, 1)^T$. Both $l_1(k)$ and $l_2(k)$ have analytic continuations in the lower half of the complex k -plane which will be denoted by $\mathbb{C}^-$. The function $f(k)$, which again depends in a nonlinear way on $q_0(x)$, $u(t)$ and $u_1(t)$, is specified by the condition that $\lim_{k \rightarrow \infty} f(k) = 0$ and that $f(k)$ is meromorphic in the fourth quadrant of the complex k -plane (which will be denoted by IV).

(iv) $\hat{Z}_l(x, t, k)$ can have poles only at those poles of $f(k)$, $k \in IV$, which do not coincide with the zeros of $l_2(k)$, and at the complex conjugates of these points. If k_0 is a such pole, then

$$\text{res}_{k_0} \hat{Z}_l \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \text{res}_{k_0} f e^{\theta(k_0)} \hat{Z}_l \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\text{res}_{\bar{k}_0} \hat{Z}_l \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \overline{\lambda \text{res}_{k_0} f} e^{-\theta(\bar{k}_0)} \hat{Z}_l \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For the defocusing case we assume that $\hat{Z}_r$ and $\hat{Z}_l$ have no poles. For the focusing case we assume that they may have only a finite number of poles, that they do not occur on the imaginary axis, that the poles of $\hat{Z}_r$ do not occur on $\mathbb{R}^-$, and that the poles of $\hat{Z}_l$ do not occur on $\mathbb{R}^+$. Then, under the assumption (1.2), both of the above RH problems are uniquely solvable in terms of linear integral equations. Having obtained $\hat{Z}_r$ and $\hat{Z}_l$, $q(x, t)$ is determined from

$$q_r(x, t) = 2i \lim_{k \rightarrow \infty} (\hat{Z}_r(x, t, k))_{12}, \quad q_l(x, t) = 2i \lim_{k \rightarrow \infty} (\hat{Z}_l(x, t, k))_{12}, \quad k \in I, \quad (1.14)$$

where Z_{12} denotes the 12 components of the matrix Z .

Unfortunately, given $u(t)$ and $u_1(t)$ we have no simple method of constructing $c(k)$ and $f(k)$. However, if $c(k)$ with $k \in II$, and $f(k)$ with $k \in IV$, are any suitably decaying meromorphic functions, then the solution of the two RH problems of Figure 1.2 and 1.3 will generate a solution of (1.1) for some functions $u(t)$ and $u_1(t)$. Furthermore, this solution will satisfy the correct initial data $q(x, 0)$.

Under certain symmetry conditions it is possible to obtain the solution of equation (1.1) by solving only one RH problem. This is the case for the forcing $iu(t)\delta(x)$ and even initial conditions ($q(x, 0) = q(-x, 0)$), as well as with the forcing $iu_1(t)\delta'(x)$ and odd initial conditions.

As a consequence of the fact that the general solution $q(x, t)$ of equation (1.1) can be obtained by solving the RH problems of Figure 1.2 and 1.3, we can give an effective description of the long time behavior.

Theorem 1.2 Let $\lambda = -1$ (focusing case) and suppose that the solution $q(x, t)$ of the problem (1.1) is characterized by the RH data $b(k)$, $c(k)$, $d(k)$, and $f(k)$.

Assume that both the right and the left-RH problems have a finite number of poles denoted by $k_j^+ \in II$, $j = 1, \dots, N_+$ and $k_j^- \in IV$, $j = 1, \dots, N_-$ respectively. Also denote $c_1, \dots, c_{N_+}$ and $f_1, \dots, f_{N_-}$ the corresponding residues of the functions $c(k)$ and $f(k)$. Then, the asymptotic behavior of $q(x, t)$ as $t \rightarrow +\infty$ is given by

(i) $t \rightarrow +\infty, x > 0, -\frac{x}{4t} = \xi_j^+ + O(\frac{1}{t}), j = 1, \dots, N_+$ (right solitons):

$$q_r(x, t) = -2\eta_j^+ \frac{\exp\{-2i\xi_j^+ x - 4i[(\xi_j^+)^2 - (\eta_j^+)^2]t - i\varphi_j^+\}}{\cosh\{2\eta_j^+(x + 4\xi_j^+ t) - \Delta_j^+\}} + O(t^{-1/2}) \quad (1.15)$$

where

$$\eta_j^+ = \text{Im} k_j^+, \quad \xi_j^+ = \text{Re} k_j^+,$$

$$\begin{aligned} \varphi_j^+ &= -\frac{\pi}{2} + \arg c_j + \sum_{l=1, l \neq j}^{N_+} [\text{sign}(\xi_l^+ - \xi_j^+) - 1] \arg \frac{k_j^+ - k_l^+}{k_j^+ - \bar{k}_l^+} + \frac{1}{\pi} \int_{-\infty}^{-x/4t} \frac{\log[1 + |b(\mu) + \overline{c(\mu)}|^2]}{(\mu - \xi_j^+)^2 + (\eta_j^+)^2} (\mu - \xi_j^+) d\mu, \\ \Delta_j^+ &= -\log 2\eta_j^+ + \log |c_j^+| + \sum_{l=1, l \neq j}^{N_+} [\text{sign}(\xi_l^+ - \xi_j^+) - 1] \log \left| \frac{k_j^+ - k_l^+}{k_j^+ - \bar{k}_l^+} \right| - \frac{\eta_j^+}{\pi} \int_{-\infty}^{-x/4t} \frac{\log[1 + |b(\mu) + \overline{c(\mu)}|^2]}{(\mu - \xi_j^+)^2 + (\eta_j^+)^2} d\mu. \end{aligned}$$

(ii) $t \rightarrow +\infty, x < 0, -x/4t = \xi_j^- + O(1/t), j = 1, \dots, N_-$ (left solitons):

$$q_l(x, t) = -2\eta_j^- \frac{\exp\{-2i\xi_j^- x - 4i[(\xi_j^-)^2 - (\eta_j^-)^2]t - i\varphi_j^-\}}{\cosh\{2\eta_j^-(x + 4\xi_j^- t) - \Delta_j^-\}} + O(t^{-1/2}) \quad (1.16)$$

where

$$\eta_j^- = \text{Im} k_j^-, \quad \xi_j^- = \text{Re} k_j^-,$$

$$\begin{aligned} \varphi_j^- &= \frac{\pi}{2} + \arg f_j + \sum_{l=1, l \neq j}^{N_-} [\text{sign}(\xi_j^- - \xi_l^-) - 1] \arg \frac{k_j^- - k_l^-}{k_j^- - \bar{k}_l^-} - \frac{1}{\pi} \int_{-x/4t}^{+\infty} \frac{\log[1 + |d(\mu) + \overline{f(\mu)}|^2]}{(\mu - \xi_j^-)^2 + (\eta_j^-)^2} (\mu - \xi_j^-) d\mu, \\ \Delta_j^- &= -\log 2|\eta_j^-| + \log |f_j| + \sum_{l=1, l \neq j}^{N_-} [\text{sign}(\xi_j^- - \xi_l^-) - 1] \log \left| \frac{k_j^- - k_l^-}{k_j^- - \bar{k}_l^-} \right| + \frac{\eta_j^-}{\pi} \int_{-x/4t}^{+\infty} \frac{\log[1 + |d(\mu) + \overline{f(\mu)}|^2]}{(\mu - \xi_j^-)^2 + (\eta_j^-)^2} d\mu. \end{aligned}$$

(iii) $t \rightarrow +\infty, x > 0, -x/4t \neq \xi_j^+, j = 1, \dots, N_+$ (right dispersive part):

$$q_r(x, t) = \alpha_+(-\frac{x}{4t}) t^{-1/2} \exp \left\{ \frac{ix^2}{4t} + 2i\alpha_+^2(-\frac{x}{4t}) \log t + i\varphi_+(-\frac{x}{4t}) \right\} + o(t^{-1/2}), \quad (1.17)$$

where

$$\begin{aligned} \alpha_+(k) &> 0, \quad \alpha_+^2(k) = \frac{1}{4\pi} \log[1 + |b(k) + \overline{c(k)}|^2], \quad k < 0 \\ \varphi_+(k) &= 2\alpha_+^2(k) \log 2 + \frac{3\pi}{4} + \arg(b(k) + \overline{c(k)}) + \arg \Gamma(-2i\alpha_+^2(k)) \\ &+ 4 \int_{-\infty}^k \log |\mu - k| d\alpha_+^2(\mu) + 2 \sum_{j=1}^{N_+} \arg(k_j^+ - k) \text{sign}(\xi_j^+ - k), \quad k < 0, \quad k \neq \xi_j^+, \end{aligned}$$

and Γ denotes the gamma function.

(iv) $t \rightarrow +\infty$, $x < 0$, $-x/4t \neq \xi_j^-$, $j = 1, \dots, N_-$ (left dispersive part):

$$q_l(x, t) = \alpha_-(-\frac{x}{4t})t^{-1/2} \exp \left\{ \frac{ix^2}{4t} + 2i\alpha_-^2(-\frac{x}{4t}) \log t + i\varphi_-(-\frac{x}{4t}) \right\} + o(t^{-1/2}) \quad (1.18)$$

where

$$\alpha_-(k) > 0, \quad \alpha_-^2(k) = \frac{1}{4\pi} \log[1 + |d(k) + \overline{f(k)}|^2], \quad k > 0,$$

$$\varphi_-(k) = 2\alpha_-^2(k) \log 2 - \frac{\pi}{4} + \arg(d(k) + \overline{f(k)}) + \arg \Gamma(-2i\alpha_-^2(k))$$

$$-4 \int_k^{+\infty} \log |\mu - k| d\alpha_-^2(\mu) - 2 \sum_{j=1}^{N_-} \arg(k_j^- - k) \operatorname{sign}(\xi_j^- - k), \quad k > 0, \quad k \neq \xi_j^-.$$

A detailed and rigorous investigation of the results for the initial-boundary value problem with $x > 0$ is given in [2]. Some of these results, as well as the modifications needed for the initial-boundary value problem with $x < 0$ are summarized in §2. Further discussion can be found in §3.

2 The RH-Problems

Since the forcing term in equation (1.1) has its support at $x = 0$, it is natural to split the solution q on two parts q_r and q_l defined on the right and left half-lines in x . Although the boundary conditions for q_r and q_l are unknown at $x = 0$, they are linked by the jump conditions (1.8). Suppose we have found q_r and q_l satisfying the homogeneous NLS equation for $x > 0$ and $x < 0$ respectively, and such that $q(x, t)$ and $q_x(x, t)$ have finite limits as $x \rightarrow \pm 0$, and the jump conditions (1.8) are satisfied. Then the piecewise smooth function

$$q(x, t) = \begin{cases} q_l(x, t), & x < 0, \\ q_r(x, t), & x > 0. \end{cases} \quad (2.1)$$

is a solution of the NLS equation with the forcing term (1.1).

In order to prove this statement let us apply the NLS operator on $q(x, t)$, considering the latter as a distribution. The resulting function has support in x consisting of only the point $x = 0$. According to the theory of distributions such function is a sum of a delta-function and its x -derivatives [11]. Since $q(x, t)$ has only the jump (1.8) at $x = 0$ and no other singularities, and since the NLS operator involves the second derivative ∂_{xx} , one conclude that the sum of a delta-function and its x -derivatives consists of only the two terms: $iu(t)\delta(x) + iu_1(x, t)\delta'(x)$.

2.1 The Right-RH Problem

The derivation of the right RH problem can be found in [2]. It involves the following considerations:

(a) The investigation of the x -part of the Lax pair, i.e. equation (1.9), with $q(x, t) = q_r(x, t)$. The 2×2 matrices $\varphi(x, t, k)$ and $\psi(x, t, k)$ are defined as the solutions of equation (1.9) specified

by the boundary conditions $\varphi(0, t, k) = I$, and $\lim_{x \rightarrow \infty} \psi(x, t, k) \exp(ikx\sigma_3) = I$, respectively. The matrices $\Phi(x, t, k)$ and $\Psi(x, t, k)$ are defined by the equations $\Phi(x, t, k) = \varphi(x, t, k) \exp(ikx\sigma_3)$ and $\Psi(x, t, k) = \psi(x, t, k) \exp(ikx\sigma_3)$. These functions are analytic in the complex k -plane cut along $\text{Im}(k) = 0$, they tend to the unit matrix as $k \rightarrow \infty$, and they have unit determinant. Actually $\Psi = (\Psi^-, \Psi^+)$, and the matrix $\Phi = (\Phi^+, \Phi^-)$ is an entire function with respect to k . The eigenfunctions Φ and Ψ satisfy certain symmetry conditions involving complex conjugation. The 2×2 matrix χ_r is defined by

$$\chi_r^-(x, t, k) = (\Psi^-(x, t, k), \lambda \Phi^-(x, t, k) / \overline{\psi_2^+(0, t, k)}), \quad \chi_r^+(x, t, k) = (\Phi^+(x, t, k) / \psi_2^+(0, t, k), \Psi^+(x, t, k)), \quad (2.2)$$

for $k \in \mathbb{C}^-$, and $k \in \mathbb{C}^+$ respectively. The matrix χ_r has unit determinant and satisfies a certain jump condition

$$\chi_r^-(x, t, k) = \chi_r^+(x, t, k) F_r(x, t, k), \quad k \in \mathbb{R}, \quad (2.3)$$

where $F(x, t, k)$ is a 2×2 matrix involving $\psi_1^+(0, t, k)$ and $\psi_2^+(0, t, k)$.

(b) The t -part of the Lax pair is used to analyze $\psi(0, t, k)$. This function satisfies equation (1.10) evaluated at $x = 0$, where $c(t) = 2ik^2\sigma_3$. The 2×2 matrices $\hat{\Phi}(t, k)$ and $\hat{\Psi}(t, k)$ are defined by the requirement that they satisfy the same equation as $\psi(0, t, k)$, and that $\hat{\Phi}(0, k) = I$, $\lim_{t \rightarrow \infty} \hat{\Psi}(t, k) = I$. These functions are analytic in the complex k -plane cut along $\text{Im}(k^2) = 0$, they tend to the unit matrix as $k \rightarrow \infty$, and they have unit determinant. Actually $\hat{\Psi} = (\hat{\Psi}^-, \hat{\Psi}^+)$, and the matrix $\hat{\Phi} = (\hat{\Phi}^+, \hat{\Phi}^-)$ is an entire function with respect to k . The eigenfunctions $\hat{\Phi}$ and $\hat{\Psi}$ satisfy the same symmetry condition as those satisfied by Φ and Ψ . The matrix $\hat{Y}_r(t, k)$ is defined by

$$\left(\frac{\hat{\Phi}^+(t, k)}{s_2(k)}, \psi^+(0, t, k) \right), \left(\frac{\hat{\Psi}^-(t, k)}{\rho(k)}, \psi^+(0, t, k) \right), \left(\psi^-(0, t, k), \frac{\hat{\Psi}^+(t, k)}{\rho(\bar{k})} \right), \left(\psi^-(0, t, k), \frac{\hat{\Phi}^-(t, k)}{s_2(\bar{k})} \right), \quad (2.4)$$

for $k \in I, \dots, IV$, where the scalars $s_2(k)$ and $\rho(k)$ are defined by

$$s_2(k) \doteq \psi_2^+(0, 0, k), k \in \mathbb{C}^+; \rho(k) \doteq \hat{\Psi}_1^-(0, k) s_2(k) - \hat{\Psi}_2^-(0, k) s_1(k), k \in II; s_1(k) \doteq \psi_1^+(0, 0, k), k \in \mathbb{C}^+. \quad (2.5)$$

The matrix $\hat{Y}_r$ has unit determinant and satisfies a certain jump condition

$$\hat{Y}_r^-(t, k) = \hat{Y}_r^+(t, k) e^{-2ik^2 t \sigma_3} G_r(k) e^{2ik^2 t \sigma_3}, \quad k^2 \in \mathbb{R}, \quad (2.6)$$

where the 2×2 matrix $G(k)$ involves the scalars $b(k)$ and $c(k) = \hat{\Psi}_2^-(0, k) / s_2^+(k) \rho(k)$.

(c) The formulation of a RH problem whose solution gives the solution of the initial-boundary value problem of the NLS equation. Let the 2×2 matrix $\hat{Z}_r(x, t, k)$ be defined by

$$\chi_r^+(x, t, k) \begin{pmatrix} 1 & 0 \\ \frac{(\hat{Y}_r)_{21}(t, k)}{(\hat{Y}_r)_{22}(t, k)} e^{2ikx} & 1 \end{pmatrix}, \quad k \in \mathbb{C}^+; \quad \chi_r^-(x, t, k) \begin{pmatrix} 1 & \frac{(\hat{Y}_r)_{12}(t, k)}{(\hat{Y}_r)_{11}(t, k)} e^{-2ikx} \\ 0 & 1 \end{pmatrix}, \quad k \in \mathbb{C}^-. \quad (2.8)$$

Then the meromorphic function $\hat{Z}_r(x, t, k)$, which has unit determinant, satisfies the RH problem (1.11). The jumps of this RH problem are given in Figure 1.2. The possible poles of $\hat{Z}_r(x, t, k)$ can occur only at the zeros of $\rho(k)$ for $k \in II$, and at the complex conjugates of these zeros. We note that the zeros of $\rho(k)$ are those poles of $c(k)$ which do not coincide with the zeros of $s_2(k)$.

2.2 The Left-RH Problem

This RH problem is constructed in a similar way as the right-RH problem. The only difference comes from the x -problem, since now $x < 0$. The obvious symmetry $x \rightarrow -x$, $k \rightarrow -k$, changes the domains of analyticity of the Ψ and Φ functions. In this case χ_l is defined by

$$\chi_l^-(x, t, k) = (\Phi^-(x, t, k)/\psi_2^-(0, t, k), \Psi^-(x, t, k)), \quad \chi_l^+(x, t, k) = (\Psi^+(x, t, k), \Phi^+(x, t, k)/\overline{\psi_2^-(0, t, k)}). \quad (2.9)$$

The symmetry $k \rightarrow -k$ changes the quadrants $I \Leftrightarrow III$, $II \Leftrightarrow IV$, so that the matrix $\hat{Y}_l(t, k)$ is defined by

$$\left(\psi^+(0, t, k), \frac{\hat{\Psi}^+(t, k)}{\tilde{\rho}(\bar{k})} \right), \left(\psi^+(0, t, k), \frac{\hat{\Phi}^-(t, k)}{l_2(\bar{k})} \right), \left(\frac{\hat{\Phi}^+(t, k)}{l_2(k)}, \psi^-(0, t, k) \right), \left(\frac{\hat{\Psi}^-(t, k)}{\tilde{\rho}(k)}, \psi^-(0, t, k) \right), \quad (2.10)$$

for $k \in I, \dots, IV$ respectively, where

$$l_1(k) = \psi_1^-(0, 0, k), \quad l_2(k) = \psi_2^-(0, 0, k), \quad k \in \mathbb{C}^-; \quad \tilde{\rho}(k) = \hat{\Psi}_1(0, k)l_2(k) - \hat{\Psi}_2(0, k)l_1(k), \quad k \in IV. \quad (2.11)$$

The matrix $\hat{Y}_r$ satisfies a jump condition determined by $G_l(k)$ which is defined in terms of $d(k) = l_1(k)/\overline{l_2(\bar{k})}$, $k \in \mathbb{R}^-$ and $f(k) = \hat{\Psi}_2^-(0, k)/\tilde{\rho}(k)l_2(k)$, $k \in IV$. The 2×2 matrix $\hat{Z}_l(x, t, k)$ is defined by

$$\chi_l^+(x, t, k) \begin{pmatrix} 1 & \frac{(\hat{Y}_l)_{12}(t, k)}{(\hat{Y}_l)_{11}(t, k)} e^{-2ikx} \\ 0 & 1 \end{pmatrix}, \quad k \in \mathbb{C}^+; \quad \chi_l^-(x, t, k) \begin{pmatrix} 1 & 0 \\ \frac{(\hat{Y}_l)_{21}(t, k)}{(\hat{Y}_l)_{22}(t, k)} e^{2ikx} & 1 \end{pmatrix}, \quad k \in \mathbb{C}^-. \quad (2.12)$$

This matrix satisfies the jump condition of Figure 1.3.

2.3 Special Cases

Let us consider the particular cases of the NLS equation with separate delta and delta prime forcing:

$$q_t + q_{xx} - 2\lambda|q|^2q = iu(t)\delta(x), \quad (2.13)$$

$$q_t + q_{xx} - 2\lambda|q|^2q = iu_1(t)\delta'(x). \quad (2.14)$$

The case (2.13) corresponds to

$$[q]_{x=0} = 0 \quad (2.15)$$

while the case (2.14) corresponds

$$[q_x]_{x=0} = 0 \quad (2.16)$$

Unfortunately it is in general difficult to find the conditions on the scattering data associated with the constraints (2.15) and (2.16). However, there exist two particular cases which admit an effective description of the scattering data for the above conditions. These are the cases when $q(x, 0)$ is either even or odd. In these cases $q(x, t)$ is either even or odd, and then the following well know symmetries are valid.

$$q(x, t) = q(-x, t) \Leftrightarrow \Psi(x, t, k) = \sigma_3 \Psi(-x, t, -k) \sigma_3,$$

$$q(x, t) = -q(-x, t) \Leftrightarrow \Psi(x, t, k) = \Psi(-x, t, -k).$$

These reductions imply

$$\sigma_3 \hat{Z}_l(x, t, k) \sigma_3 = \hat{Z}_r(-x, t, -k), \quad (2.17)$$

and

$$\hat{Z}_l(x, t, k) = \hat{Z}_r(-x, t, -k), \quad (2.18)$$

for even and odd potentials respectively. Equations (2.17) and (2.18) in turn imply the following conditions for the scattering data:

$$q(x, t) = q(-x, t) \Leftrightarrow c(k) = -f(-k), \quad b(k) = -d(-k), \quad (2.19)$$

$$q(x, t) = -q(-x, t) \Leftrightarrow c(k) = f(-k), \quad b(k) = d(-k). \quad (2.20)$$

3 Discussion

The linearization scheme developed in this paper can be summarized as follows: Given $q(x, 0)$ construct $b(k)$ and $d(k)$ (see equations (1.12) and (1.13)). Then if $c(k)$ for $k \in II$ and $f(k)$ for $k \in IV$ are any suitably decaying meromorphic functions, the solution of the RH problems of Figure 1.2 and 1.3 generate the solution $q(x, t)$ corresponding to initial data $q(x, 0)$ and some forcing functions $u(t)$ and $u_1(t)$. The main limitation of our result is that for given $q(x, 0)$, $u(t)$, $u_1(t)$ we cannot construct $c(k)$ and $f(k)$ by solving a linear problem. Nevertheless, we claim that for any $q(x, 0)$, $u(t)$, $u_1(t)$ satisfying the assumption (1.2), $c(k)$ and $f(k)$ exist, i.e. the RH problems of Figures 1.2 and 1.3 solve (1.1) for **general** initial and forcing data.

The RH problems of Figure 1.2 and 1.3 are quite natural. To appreciate this we first recall the RH problem that corresponds to the NLS with $x \in (-\infty, \infty)$. This RH problem is depicted in Figure 3.1.

$$\begin{array}{c} + \\ \hline - \end{array} \left(\begin{array}{cc} 1 & -b(k)e^{-\theta} \\ \lambda \overline{b(k)}e^{\theta} & 1 - \lambda |b(k)|^2 \end{array} \right).$$

Figure 3.1

Comparing the RH problems of Figure 1.2 and Figure 3.1 we see that the jumps for $k \in \mathbb{R}^+$ are identical. The jump for $k \in i\mathbb{R}^+$ can not have a nonzero entry in the 12 position since $e^{-\theta}$ is unbounded for $k \in i\mathbb{R}^+$. The jump for $k \in i\mathbb{R}^-$ follows by symmetry considerations. Finally the jump for $k \in \mathbb{R}^-$ follows from the cyclic condition that the product of the jump matrices equals unity (this is a reflection of continuity at $k = 0$). Indeed this jump can be written as

$$\begin{pmatrix} 1 & -\lambda \bar{c} e^{-\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -b e^{-\theta} \\ \lambda \bar{b} e^{\theta} & 1 - \lambda |b|^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ c e^{\theta} & 1 \end{pmatrix}.$$

The fact that $c(k)$ has analytic continuation for $k \in II$ can also be easily understood. At $t = 0$, the RH problem of Figure 1.2 must be reduced to the one that defines $q_r(x, 0)$. At $t = 0$, the term e^{θ} has analytic continuation in II . Thus the jumps along the imaginary axis can be mapped to a jump on the negative real axis. In this way, at $t = 0$ one finds the RH problem of Figure 3.1 with θ replaced by $2ikx$. This RH problem corresponds precisely to $q_r(x, 0)$. Similar considerations apply to the RH problem of Figure 1.3.

The fact that $c(k)$ and $f(k)$ are related in a nonlinear way with $q(x, 0)$, $u(t)$, $u_1(t)$, is a reflection of the fact that $q(0, t)$ and $q_x(0, t)$ depends nonlinearly on $q(x, 0)$, $u(t)$, and $u_1(t)$. To appreciate this we first consider the solution of the linearized problem

$$iq_t + q_{xx} = iu(t)\delta(x) + iu_1(t)\delta'(x), \quad x \in (-\infty, \infty), \quad t \in [0, \infty), \quad (3.1)$$

where $q(x, 0)$, $u(t)$, and $u_1(t)$ are given and decaying for large t and large x . In order to draw comparisons with the nonlinear problem we decompose (3.1) into two half-line problems. Let

$$\hat{q}_r(k, t) = \int_0^\infty dx e^{ikx} q_r(x, t). \quad (3.2)$$

The evolution of the Fourier data $\hat{q}_r(k, t)$ is given by

$$\frac{\partial}{\partial t} \hat{q}_r + ik^2 \hat{q}_r = i\partial_x q_r(0, t) + kq_r(0, t). \quad (3.3)$$

The evolution of $\hat{q}_l(k, t)$ defined by

$$\hat{q}_l(k, t) = \int_{-\infty}^0 dx e^{ikx} q_l(x, t) \quad (3.4)$$

satisfies an equation similar to (3.3) with the subscript r replaced by l . The evolutions of $\hat{q}_r$ and $\hat{q}_l$ involve four unknowns ($q_r(0, t)$, $q_l(0, t)$, $\partial_x q_r(0, t)$, $\partial_x q_l(0, t)$), while we only have two equations for these unknowns (see equation (1.8)). This apparently ominous situation can be bypassed by using the crucial facts that the solution $\hat{q}_r(k, t)$ of equation (3.3) is analytic in the upper half of the k complex plane (this follows from equation (3.2)). Similarly $\hat{q}_l(k, t)$ is analytic in the lower half of the k complex plane. It turns out that these requirements imply

$$\hat{q}_r(k, 0) = - \int_0^\infty e^{ik^2 t} (i\partial_x q_r(0, t) + kq_r(0, t)) \quad (3.5a)$$

and

$$\hat{q}_l(k, 0) = - \int_0^\infty e^{ik^2 t} (i \partial_x q_l(0, t) + k q_l(0, t)) dt. \quad (3.5b)$$

Using the substitutions $k = e^{i\pi/4} \sqrt{\rho}$, $\rho > 0$ and $k = -e^{i\pi/4} \sqrt{\rho}$ in (3.5a) and (3.5b) respectively, we find two more equations for the boundary values of $q(0, t)$ and $\partial_x q(0, t)$.

It is important to notice that if $\partial_x q_r(0, t)$ and $q_r(0, t)$ are arbitrary functions, then the r.h.s. of equation (3.5a) will be analytic for $k \in I \cup III$. However, in order for $\partial_x q_r(0, t)$ and $q_r(0, t)$ to be the boundary values of the solution of equation (3.1), it is necessary and sufficient that the r.h.s. of equation (3.5a) has analytic continuation across the positive imaginary k axis ($\hat{q}_r(k, 0)$ is analytic for $k \in I \cup II$). Similar considerations apply to $\partial_x q_l(0, t)$ and $q_l(0, t)$.

We now discuss the nonlinear problem. Let $(\hat{\psi}_1(t, k), \hat{\psi}_2(t, k))^T$ be the solution of the vector equation

$$\hat{\psi}_t + (2ik^2 \sigma_3 + i\lambda |q(0, t)|^2 \sigma_3 - 2kQ(0, t) + i\partial_x Q_r(0, t) \sigma_3) \hat{\psi} = 0, \quad (3.6)$$

satisfying the boundary condition $\lim_{t \rightarrow \infty} [(\hat{\psi}_1, \hat{\psi}_2) \exp(-2ik^2 t)] = (0, 1)^T$. Let $r(k) \doteq \hat{\psi}_1(0, k)/\hat{\psi}_2(0, k)$. It turns out that for arbitrary decaying functions $q_r(0, t)$ and $\partial_x q_r(0, t)$, $r(k)$ is a meromorphic function for $k \in I \cup III$ and $r(k) \rightarrow 0$ as $k \rightarrow \infty$. However, if $q_r(0, t)$ and $\partial_x q_r(0, t)$ are the boundary values of the NLS defined on $x, t \in [0, \infty)$, then in addition $r(k)$ satisfies

$$r(k) = \frac{s_1(k)}{s_2(k)}, \quad k \in I, \quad (3.7)$$

where s_1 and s_2 are determined from $q_r(x, 0)$ and are analytic for $k \in I \cup II$. Equation (3.7) is the analogue of equation (3.5a). It shows that although the relationship between $q_r(x, 0)$, $q_r(0, t)$, and $\partial_x q_r(0, t)$ is highly nonlinear, its reflection in the k -plane (scattering space) is rather simple: $r(k)$ has analytic continuation across the positive imaginary k axis.

Similarly, let $(\hat{\varphi}_1(t, k), \hat{\varphi}_2(t, k))^T$ be the solution of vector equations similar to (3.6) with the subscript r replaced by l , satisfying the boundary conditions $\lim_{t \rightarrow \infty} [(\hat{\varphi}_1, \hat{\varphi}_2) \exp(2ik^2 t)] = (0, 1)^T$. Let $l(k) \doteq \hat{\varphi}_1(0, k)/\hat{\varphi}_2(0, k)$. For arbitrary decaying functions $q_l(0, t)$ and $\partial_x q_l(0, t)$, $l(k)$ is a meromorphic function for $k \in I \cup III$ and $l(k) \rightarrow 0$ as $k \rightarrow \infty$. However, if $q_l(0, t)$ and $\partial_x q_l(0, t)$ are the boundary values of the NLS defined on $x \in [-\infty, 0)$, $t \in [0, \infty)$, then in addition $l(k)$ satisfies

$$l(k) = \frac{l_1(k)}{l_2(k)}, \quad k \in III, \quad (3.8)$$

where l_1 and l_2 are determined for $q_l(x, 0)$ and are analytic for $k \in III \cup IV$. Equation (3.8) is the analogue of equation (3.5b). Again, the reflection of the nonlinear relationship between $q_l(x, 0)$, $q_l(0, t)$ and $\partial_x q_l(0, t)$, is simply that $l(k)$ has analytic continuation across the negative imaginary k axis.

There exists an invertible correspondence between the "potential" $\{q_r(0, t), \partial_x q_r(0, t)\}$ and the scattering data $r(k)$: Given $q_r(0, t)$ and $\partial_x q_r(0, t)$ equation (3.6) implies $r(k)$. Conversely given a meromorphic function $r(k)$, one can find $q_r(0, t)$ and $\partial_x q_r(0, t)$ by solving a RH problem with the jump $r(k)$ for $k^2 \in \mathbb{R}$. In the following discussion we shall refer to this RH problem as the right inverse problem. This provides an effective way for deriving pairs of functions $q_r(0, t)$ and $\partial_x q_r(0, t)$ compatible with a given $q_r(x, 0)$. Indeed, given $q_r(x, 0)$ one first computes $r(k)$ for $k \in I$ from equation (3.7). Let $r(k)$, $k \in III$ be any suitably decaying meromorphic function. Then the solution

of the above inverse problem yields $q_r(0, t)$ and $\partial_x q_r(0, t)$. Similarly starting with $q_l(x, 0)$ and any suitably meromorphic function $l(k)$, $k \in I$, one finds $q_l(0, t)$ and $\partial_x q_l(0, t)$ by solving a left inverse problem. In this way one finds the solution of (1.1) with some $u(t)$ and $u_1(t)$.

Unfortunately, given $q(x, 0)$, $u(t)$, and $u_1(t)$ we cannot compute $r(k)$ for $k \in III$ and $l(k)$ for $k \in I$. This is a consequence of the fact that we now have two “mixed” problems where we know “half” the scattering data ($r(k)$ for $k \in I$ and $l(k)$ for $k \in III$) and two relations among the two potentials $\{q_r(0, t), \partial_x q_r(0, t)\}$, $\{q_l(0, t), \partial_x q_l(0, t)\}$ (see equation (1.8)).

We point out that in addition to the forced equation (1.1), there exist other problems which can be formulated as two coupled initial-boundary value problems. For example, such a problem arises in connection with optical switches. In this case the important physical question is to understand the interaction of a light beam with an interface separating two media with different dielectrics. For a wide variety of situations the relevant envelope $A(x, z)$ satisfies (see [12]),

$$2in_0 \frac{w}{c} \frac{\partial A}{\partial z} + \frac{\partial^2 A}{\partial x^2} + \frac{w^2}{c^2} \alpha_0 A |A|^2 = V(x, z)A,$$

where $V(x, z) = 0$ for $x < 0$, and $V(x, z) = \frac{w^2}{c^2} [(n_0^2 - n_1^2) - (\alpha_0 - \alpha_1)|A|^2]$ for $x > 0$. In this equation α_0 and α_1 are constants, w is the frequency of light, c is the speed of light, and n_0, n_1 are the two different dielectrics. The physical question reduces to studying two different NLS equations each defined on a half-infinite line, with $A(x, 0)$ a soliton located at $x \rightarrow -\infty$. The boundary conditions at $x = 0$ are coupled by the requirement that A and A_x are continuous.

This problem gives rise to two initial boundary value problems coupled via

$$q_l(0, t) = \alpha q_r(0, t), \quad \partial_x q_l(0, t) = \alpha \partial_x q_r(0, t) \quad (3.9)$$

where α is a given constant. This problem can also be formulated as the solution of the RH problems of Figure 1.2 and 1.3. However, the problem of finding $c(k)$ and $f(k)$ is again nonlinear. This is a consequence of the fact that in order to find $c(k)$ and $f(k)$ we need $q_r(0, t)$ and $\partial_x q_r(0, t)$ which depend in a nonlinear way on $q(x, 0)$: Given $q(x, 0)$ compute $r(k)$, $k \in I$ and $l(k)$, $k \in III$. Then determine $r(k)$, $k \in III$ and $l(k)$, $k \in I$ by the requirement that the solution of the right and the left inverse problems discussed earlier give rise to potentials satisfying (3.9). In the linear limit this requirement reduces to

$$\hat{q}_r(e^{i\pi/4} \sqrt{\rho}, 0) = - \int_0^\infty e^{-\rho t} (i w(t) + e^{i\pi/4} \sqrt{\rho} v(t)), \quad (3.10a)$$

$$\hat{q}_l(-e^{i\pi/4} \sqrt{\rho}, 0) = -\alpha \int_0^\infty e^{-\rho t} (i w(t) - e^{i\pi/4} \sqrt{\rho} v(t)), \quad (3.10b)$$

where $w(t) = \partial_x q_r(0, t)$, $v(t) = q_r(0, t)$, $\hat{q}_r(e^{i\pi/4} \sqrt{\rho}, 0)$ is the Fourier transform of $q(x, 0)$ with $x > 0$ evaluated at $k = e^{i\pi/4} \sqrt{\rho}$ and $\hat{q}_l(-e^{i\pi/4} \sqrt{\rho}, 0)$ is the Fourier transform of $q(x, 0)$ with $x < 0$ evaluated at $k = -e^{i\pi/4} \sqrt{\rho}$. Equations (3.10) can be easily solved for $w(t)$ and $v(t)$.

The asymptotic results described in Theorem 1.2 indicate that as $t \rightarrow \infty$, the solution of this problem gives rise to solitons moving away from the origin. Unfortunately, since we can not express $c(k)$ in terms of $f(k)$, we cannot calculate the number of the solitons moving to the right in terms of the number of these solitons moving to the left. A numerical approach to finding the number of these solitons is under investigation.

It is important to mention that the analysis of equation (1.1) requires an essential use of general PDE techniques. This follows from the fact that in order to study the map between $\{q(0, t), q_x(0, t)\}$ and $r(k)$ or $l(k)$ one needs apriori estimates for $q(0, t)$ and $q_x(0, t)$. The uniqueness and existence of a global solution for the NLS on the quarter plane is established in [13]. This result makes fundamental use of certain equations which are the analogues of the first three conserved quantities. Therefore, this theory uses L_2 estimates. However, the methods of exact integrability are based on L_1 estimates. It is therefore crucial for us to extend the results of [13] from L_2 to L_1 . This poses significant technical difficulties. A discussion of this problem as well as details of the results regarding the initial-boundary value problem of the NLS for $x > 0$ can be found in [2] and [14].

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