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**Universal Spectral Correlation
between Hamiltonians with Disorder II**

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ABSTRACT: We calculate the cross-correlation function between the eigenvalues of two random hermitian matrices taken from separate Gaussian ensembles. In the limit of large matrices the correlation, when suitably smoothed, reproduces a previous result obtained by diagrammatic methods. Unlike previous results, however, this unsmoothed correlation function may be studied at short distances.

Using the results of previous work [1,2,3] on the universality of correlations between eigenvalues in random matrix theory [4,5,6], we study the cross correlation between eigenvalues of two different hermitian matrices drawn from random distributions [7]. The cross correlation is defined by

$$\rho_c(\mu, \nu) = \left\langle \frac{1}{N} \text{Tr} \delta(\mu - \varphi_1) \frac{1}{N} \text{Tr} \delta(\nu - \varphi_2) \right\rangle - \left\langle \frac{1}{N} \text{Tr} \delta(\mu - \varphi_1) \right\rangle \left\langle \frac{1}{N} \text{Tr} \delta(\nu - \varphi_2) \right\rangle, \quad (1)$$

where φ_1 and φ_2 are two N by N hermitean matrices taken from an ensemble to be defined below. The bracket indicates averaging over all the ensemble elements, and the subscript c denotes the connected correlation. For physical application, we may regard these matrices, for example, as Hamiltonians for disordered systems whose eigenvalues correspond to the energy levels of those systems. The correlation between the spectra of such Hamiltonians is of interest in mesoscopic systems [8], disordered metals, and $c=1$ matrix models, for instance.

In a previous paper [7] two of us calculated (1) using a Feynman diagram approach [3]. However, only the “long distance,” i.e., $\mu - \nu$ of order N^0 , correlation was obtained. This is because the diagrammatic method requires that one take the limit $N \rightarrow \infty$ first, and so information about short distance, i.e., $\mu - \nu$ of order N^{-1} , correlations is lost. Taking this limit first has precisely the same effect on correlation functions as the “smoothing” procedure introduced in Ref. 1. In this paper we determine $\rho_c(\mu, \nu)$ over all distance scales using a generalized orthogonal polynomial method.

We consider a general Gaussian ensemble of coupled N by N matrices φ , with probability density

$$P(\varphi_1, \varphi_2) = \frac{1}{Z} e^{-\frac{N}{2} \text{Tr} [m_1^2 \varphi_1^2 + m_2^2 \varphi_2^2 - 2\alpha^2 \varphi_1 \varphi_2]}, \quad (2)$$

where Z is a normalization constant chosen so that $\int d\varphi_1 d\varphi_2 P(\varphi_1, \varphi_2) = 1$. Then $\langle \Theta(\varphi_1, \varphi_2) \rangle = \int d\varphi_1 d\varphi_2 \Theta(\varphi_1, \varphi_2) P(\varphi_1, \varphi_2)$ is the average over all ensemble elements of the observable Θ . In this notation, $d\varphi_a = \prod_{i=1}^N d(\varphi_a)_{ii} \prod_{i>j}^N d\text{Re}(\varphi_a)_{ij} d\text{Im}(\varphi_a)_{ij}$. Here and elsewhere, the factors of N are chosen in our definitions so that in the large N limit the density of eigenvalues has a finite (i.e., of order N^0) width. Note that by making the substitutions $m_1^2 \rightarrow m_1^2 + \alpha^2$, and $m_2^2 \rightarrow m_2^2 + \alpha^2$, and then taking the limit $\alpha^2 \rightarrow \infty$ we force $\varphi_1 - \varphi_2 \rightarrow 0$ and recover the single matrix density considered earlier [8,9,3].

The full cross-correlation, $\rho(\mu, \nu)$, is found by evaluating

$$\rho(\mu, \nu) = \left\langle \frac{1}{N} \text{Tr} \delta(\mu - \varphi_1) \frac{1}{N} \text{Tr} \delta(\nu - \varphi_2) \right\rangle. \quad (3)$$

Since the matrices φ_1 and φ_2 are Hermitian, one begins by integrating out the unitary degrees of freedom in each of the matrix variables [10]. The remaining integration over the N eigenvalues of each matrix may then be performed using the well known method of orthogonal polynomials [6]. These steps are given in more detail in Appendix A. There we find $\rho(\mu, \nu)$, exact for all values of N , to be

$$\rho(\mu, \nu) = \frac{1}{N} K_N(\mu, \nu) w(\mu, \nu) - K_N(\mu, \nu) \bar{K}_N(\mu, \nu) + \frac{1}{N} \sum_{n=0}^{N-1} \phi_n^2(\mu) \frac{1}{N} \sum_{m=0}^{N-1} \psi_m^2(\nu), \quad (4)$$

where

$$w(\mu, \nu) \equiv \sqrt{\frac{Nm_1 m_2}{2\pi}} e^{+\frac{1}{2}[\frac{N}{2}(1-t^2)m_1^2 \mu^2]} e^{+\frac{1}{2}[\frac{N}{2}(1-t^2)m_2^2 \nu^2]} e^{-\frac{N}{2}[m_1^2 \mu^2 + m_2^2 \nu^2 - 2\alpha^2 \mu \nu]} \quad (5)$$

and we have defined the kernels

$$K_N(\mu, \nu) = \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{t^n} \phi_n(\mu) \psi_n(\nu), \quad (6)$$

$$\overline{K}_N(\mu, \nu) = \frac{1}{N} \sum_{n=0}^{N-1} t^n \phi_n(\mu) \psi_n(\nu), \quad (7)$$

in which the constant $t = \frac{\alpha^2}{m_1 m_2}$. t is the dimensionless measure of the strength of the coupling between the two matrices and must be in the range $0 \leq t \leq 1$ in order for (2) to be well defined. We have also defined

$$\begin{aligned} \phi_n(\mu) &\equiv A_n p_n(\mu) e^{-\frac{1}{2}[\frac{N}{2}(1-t^2)m_1^2\mu^2]}, \\ \psi_n(\nu) &\equiv B_n q_n(\nu) e^{-\frac{1}{2}[\frac{N}{2}(1-t^2)m_2^2\nu^2]}. \end{aligned} \quad (8)$$

The p_n 's and q_n 's are orthogonal polynomials defined by the condition

$$\int_{-\infty}^{+\infty} d\mu d\nu p_m(\mu) q_n(\nu) e^{-N[\frac{1}{2}m_1^2\mu^2 + \frac{1}{2}m_2^2\nu^2 - \alpha^2\mu\nu]} = \delta_{mn}, \quad (9)$$

and $A_n \equiv \left[t^n \sqrt{\frac{2\pi}{Nm_1^2}} \right]^{\frac{1}{2}}$, $B_n \equiv \left[t^n \sqrt{\frac{2\pi}{Nm_2^2}} \right]^{\frac{1}{2}}$ so that the functions $\phi_n(\mu)$ and $\psi_m(\nu)$ have the properties

$$\begin{aligned} \int_{-\infty}^{+\infty} d\mu \phi_m(\mu) \phi_n(\mu) &= \delta_{mn}, \\ \int_{-\infty}^{+\infty} d\nu \psi_m(\nu) \psi_n(\nu) &= \delta_{mn}. \end{aligned} \quad (10)$$

It may at first seem odd to define these functions so that p_n and q_n are normalized according to (9) while ϕ_n and ψ_m are according to (10). The normalization of the first set of functions is dictated by the orthogonal polynomial method which results in (4). As can be seen however, we have chosen to construct the ϕ_n 's and ψ_m 's to be individually orthonormal on a single integration with flat measure. They are the convenient functions with which to express the density and correlation functions as they are the functions which take the specific, large N , asymptotic form studied in our previous work.

Equation (4) is the full cross-correlation. The connected part is

$$\rho_c(\mu, \nu) = \frac{1}{N} K_N(\mu, \nu) w(\mu, \nu) - K_N(\mu, \nu) \overline{K}_N(\mu, \nu). \quad (11)$$

In this expression the form of the kernels K and $\overline{K}$, as well as the functions which comprise them, are almost identical to the analogous expressions in the one matrix, two point correlation function but for the appearance of the constant t . If we take the one matrix limit mentioned above then $t \rightarrow 1$, $(1-t^2)m_{1,2}^2 \rightarrow (m_1^2 + m_2^2)$, $w(\mu, \nu) \rightarrow \delta(\mu - \nu)$ and Equation (11) becomes the familiar connected, two-point correlation function for a single Gaussian ensemble.

Finally, though we study only the two-point cross correlation in this paper, all higher order cross-correlations may be similarly defined and calculated [11]. They will be functions of the same two kernels and thus may be studied in the large N limit using the result which follows.

We are interested in finding the large N form of (11) and thus are led to study the kernel $K_N(\mu, \nu)$ in that limit ($\overline{K}_N(\mu, \nu)$ will turn out to follow trivially). If t were unity, as in the single matrix problem, one would be able to multiply (6) by $\mu - \nu$, use the properties of the orthogonal polynomials to collapse the series, and obtain an expression which depends only on the $N-1$ and N th order polynomials, for which the asymptotic form is known [1]. In this two matrix problem, $t \neq 1$ and this procedure fails. Thus, we must calculate the asymptotic form of the entire series itself.

To do this we write the polynomials in an equivalent, integral form which makes the series summable upon exchange of the order of summation and integration. The remaining integrals we evaluate via a saddle point method. The details are given more explicitly in Appendix B. We find that K , to order $1/N$, is given by

$$\begin{aligned} K_N(\mu, \nu) &= \frac{1}{4\pi N} \frac{-e^{Nu}}{\sqrt{\sigma\tau \cos\theta \cos\phi}} \\ &\times \left\{ \frac{\cos[N(h(\theta) + h(\phi)) - (\theta + \phi)] + e^{-u} \cos[N(h(\theta) + h(\phi))]}{[\text{ch } u + \cos(\theta + \phi)]} \right. \\ &\quad \left. - \frac{\cos[N(h(\theta) - h(\phi)) - (\theta - \phi)] - e^{-u} \cos[N(h(\theta) - h(\phi))]}{[\text{ch } u - \cos(\theta - \phi)]} \right\}, \end{aligned} \quad (12)$$

and

$$\overline{K}_N(\mu, \nu) = \{K_N(\mu, \nu)\}_{u \rightarrow -u}, \quad (13)$$

where $h(\theta) \equiv (\frac{1}{2} \sin 2\theta + \theta - \frac{\pi}{2}) + \frac{1}{N}(\frac{1}{2}\theta)$, $u = -\ln t$, and where θ and ϕ are angular variables defined by $\sin \theta \equiv \mu/2\sigma$ and $\sin \phi \equiv \nu/2\tau$, with 2σ and 2τ being the endpoints of the eigenvalue spectra. As μ and ν vary over their allowed ranges, θ and ϕ range from $-\pi/2$ to $+\pi/2$.

Thus we have determined the large N form of K and $\overline{K}$ which we may now substitute into (11) to obtain the large N connected correlation function at all distance scales. This gives a lengthy expression which we will not display explicitly.

Clearly $\rho_c(\mu, \nu)$ will oscillate wildly between μ and ν separated by some finite distance since in general there will be $O(N)$ eigenvalues encountered in such a range. If we follow the "smoothing" procedure of Ref. 1 by integrating $\rho_c(\mu, \nu)$ over intervals $\Delta\mu$ and $\Delta\nu$ small

relative to $O(N^0)$ but large compared to $O(N^{-1})$ we immediately obtain our previous result [3], namely

$$\rho_c(\mu, \nu) = \frac{1}{16N^2\pi^2} \frac{1}{\sigma\tau \cos\theta \cos\phi} \left\{ \frac{1 + \text{ch } u \cos(\theta + \phi)}{[\text{chu} + \cos(\theta + \phi)]^2} + \frac{1 - \text{ch } u \cos(\theta - \phi)}{[\text{chu} - \cos(\theta - \phi)]^2} \right\}. \quad (14)$$

In its unsmoothed form, (12) is valid over all distance scales. Thus we may obtain an entirely new result by examining its short distance behavior. In this regime we take the limit $\mu - \nu$ of order $1/N$, with both variables at a finite distance from the endpoints of their spectra. If we are interested only in short distance correlations with u finite then u regulates any singularity at $\mu = \nu$ and the kernel $K_N(\mu, \nu)$ becomes

$$K_N(\mu, \nu) = \frac{1}{4\pi N} \frac{-e^{Nu}}{a \cos \bar{\theta}} \left\{ \frac{\cos 2Nh(\bar{\theta}) + e^{-u} \cos[2Nh(\bar{\theta}) + \bar{\theta}]}{\text{ch } u + \cos 2\bar{\theta}} - \frac{(1 - e^{-u}) \cos[2N\pi\delta\mu\bar{\rho}(\bar{\mu})]}{\text{ch } u - 1} \right\} \quad (15)$$

where we have defined, $a = \sqrt{\sigma\tau}$, $\frac{\bar{\mu}}{a} \equiv \frac{1}{2}(\frac{\mu}{\sigma} + \frac{\nu}{\tau})$, $\frac{\delta\mu}{a} \equiv \frac{1}{2}(\frac{\mu}{\sigma} - \frac{\nu}{\tau})$, $\delta\theta \equiv \frac{1}{2}(\theta - \phi)$, $\bar{\theta} \equiv \frac{1}{2}(\theta + \phi)$, and $a\bar{\rho}(\bar{\mu}) \equiv \frac{1}{2}(\sigma\rho(\mu) + \tau\rho(\nu))$.

As before, we may obtain $\bar{K}$ by taking $u \rightarrow -u$. The short distance correlation, a somewhat cumbersome expression, is obtained by the substitution of K and $\bar{K}$ into (11).

One may notice that in (15) we have discarded terms which are manifestly small but nonetheless important for small u , i.e., when $t = \alpha^2/m_1m_2$ approaches unity. If we consider the short distance limit while retaining such terms we can make the usual substitutions $m_1^2 \rightarrow m_1^2 + \alpha^2$, $m_2^2 \rightarrow m_2^2 + \alpha^2$, and take the limit $\alpha^2 \rightarrow \infty$ to obtain the familiar, universal result for the one matrix two point correlation function

$$K_N(\mu, \nu) \rightarrow \frac{\sin x}{x} \rho(\bar{\mu}),$$

where $x \equiv 2\pi N\delta\mu\rho(\bar{\mu})$.

In conclusion, we have determined exactly the two matrix cross correlation for Gaussian potential using the method of orthogonal polynomials. Via a saddle point calculation, we have obtained an asymptotic expression for it, in the limit of large matrices, which is valid over all distance scales. We have shown that this result reproduces a previous one for the correlation at “large” distance scales ($\mu - \nu$ of order N^0), and that a new result for “short” distance ($\mu - \nu$ of order N^{-1}) may be obtained.

APPENDIX A

In this appendix we show how the orthogonal polynomial method [6] is used to obtain Equation (4). First let us consider the integral over the two matrix density (2) alone,

$$\frac{1}{Z} \int d\varphi_1 d\varphi_2 e^{-\frac{N}{2} \text{Tr}[m_1^2\varphi_1^2 + m_2^2\varphi_2^2 - 2\alpha^2\varphi_1\varphi_2]}. \quad (A1)$$

For any of the matrices in the domain of integration we may write $\varphi = U^\dagger \Lambda U$, with U being the unitary matrix which diagonalizes φ . If we change integration variables to Λ and U then $d\varphi = d\Lambda \Delta^2(\Lambda) dU$, where $d\Lambda = \prod_{i=1}^N d\lambda_i$, and dU is the measure over unitary matrices. $\Delta^2(\Lambda)$, the Jacobian of the transformation, is the Vandermonde determinant squared and may be written

$$\Delta^2(\Lambda) = \prod_{i < j} (\lambda_i - \lambda_j)^2 = \det[\lambda_i^{j-1}]. \quad (A2)$$

Except for one cross term, all matrices in (A1) appear in single matrix products within traces and thus may be written in diagonal form. So, we write (A1) as

$$\frac{1}{Z} \int d\Lambda d\Lambda' \Delta^2(\Lambda) \Delta^2(\Lambda') e^{-\frac{N}{2} \text{Tr}[m_1^2\Lambda^2 + m_2^2\Lambda'^2]} \int dU e^{-\frac{N\alpha^2}{2} \text{Tr}[\Lambda U \Lambda' U^\dagger]}, \quad (A3)$$

where we have associated Λ with φ_1 and Λ' with φ_2 . The integral over U is known. [10] We substitute its solution to get

$$\frac{1}{Z} \int d\Lambda d\Lambda' \Delta^2(\Lambda) \Delta^2(\Lambda') e^{-\frac{N}{2} \text{Tr}[m_1^2\Lambda^2 + m_2^2\Lambda'^2]} \frac{\det \left[e^{-\frac{N\alpha^2}{2} \lambda_i \lambda'_j} \right]}{\Delta(\Lambda) \Delta(\Lambda')}. \quad (A4)$$

Z may easily be fixed at the end of the calculation so in this and ensuing steps we will ignore most constant factors for the sake of clarity.

Since the integrand is symmetric under the exchange of any two integration variable indices within the same matrix, we replace $\det[e^{-\frac{N}{2}\alpha^2\lambda_{i_1}\lambda_{i_2}}]$ with the product of its diagonal terms and write

$$\frac{1}{Z} \int d\Lambda d\Lambda' \Delta(\Lambda) \Delta(\Lambda') e^{-\frac{N}{2} \sum_{i=1}^N [m_1^2\lambda_i^2 + m_2^2\lambda'_i - 2\alpha^2\lambda_i\lambda'_i]}. \quad (A5)$$

Now consider the product $\Delta(\Lambda)\Delta(\Lambda')$. Adding and subtracting multiples of the columns within each Vandermonde determinant leaves it invariant up to a constant so

$$\begin{aligned} \Delta(\Lambda)\Delta(\Lambda') &\propto \det[p_{i-1}(\lambda_j)] \det[q_{j-1}(\lambda'_k)] \\ &\propto \det \left[\sum_{k=0}^{N-1} p_k(\lambda_i) q_k(\lambda'_j) \right], \end{aligned} \quad (A6)$$

where $p_i(\lambda)$ and $q_j(\lambda')$ are any, nonzero, polynomials of degree i and j respectively. We choose these to be the polynomials defined by the orthonormality condition

$$\int_{-\infty}^{+\infty} d\lambda d\lambda' p_i(\lambda) q_j(\lambda') e^{-N[\frac{1}{2}m_1^2\mu^2 + \frac{1}{2}m_2^2\nu^2 - \alpha^2\lambda\lambda']} = \delta_{ij}. \quad (\text{A7})$$

These polynomials have associated functions which are individually orthonormal on flat measure

$$\begin{aligned} \phi_n(\lambda) &\equiv A_n p_n(\lambda) e^{-\frac{1}{2}[\frac{N}{2}(1-t^2)m_1^2\lambda^2]}, \\ \psi_n(\lambda') &\equiv B_n q_n(\lambda') e^{-\frac{1}{2}[\frac{N}{2}(1-t^2)m_2^2\lambda'^2]}, \end{aligned} \quad (\text{A8})$$

where we have defined $t \equiv \frac{\alpha^2}{m_1 m_2}$, $A_n \equiv \left[t^n \sqrt{\frac{2\pi}{N m_1^2}} \right]^{\frac{1}{2}}$, and $B_n \equiv \left[t^n \sqrt{\frac{2\pi}{N m_2^2}} \right]^{\frac{1}{2}}$, so that the functions $\phi_n(\lambda)$ and $\psi_m(\lambda')$ have the properties

$$\begin{aligned} \int_{-\infty}^{+\infty} d\lambda \phi_i(\lambda) \phi_j(\lambda) &= \delta_{ij}, \\ \int_{-\infty}^{+\infty} d\lambda' \psi_i(\lambda') \psi_j(\lambda') &= \delta_{ij}. \end{aligned} \quad (\text{A9})$$

We recast condition (A7) in terms of ϕ and ψ functions and see that

$$\int d\lambda d\lambda' \frac{1}{t^i} \phi_i(\lambda) \psi_j(\lambda') w(\lambda, \lambda') = \delta_{ij}, \quad (\text{A10})$$

where we have defined

$$w(\lambda, \lambda') \equiv \sqrt{\frac{N m_1 m_2}{2\pi}} e^{\frac{1}{2}[\frac{N}{2}(1-t^2)m_1^2\lambda^2] + \frac{1}{2}[\frac{N}{2}(1-t^2)m_2^2\lambda'^2]} e^{-\frac{N}{2}[m_1^2\lambda^2 + m_2^2\lambda'^2 - 2\alpha^2\lambda\lambda']}. \quad (\text{A11})$$

Using these definitions we can write Equation (A5) as

$$\frac{1}{Z} \int d\Lambda d\Lambda' \det [K_N(\lambda_i, \lambda'_j)] \prod_{k=1}^N w(\lambda_k, \lambda'_k), \quad (\text{A12})$$

where

$$K_N(\lambda, \lambda') \equiv \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{t^n} \phi_n(\lambda) \psi_n(\lambda'). \quad (\text{A13})$$

Now, it is easy to see that the following two identities are true

$$\int d\lambda d\lambda' K_N(\lambda, \lambda') w(\lambda, \lambda') = 1, \quad (\text{A14})$$

$$\int d\lambda d\lambda' K_N(\mu, \lambda') K_N(\lambda, \nu) w(\lambda, \lambda') = \frac{1}{N} K_N(\mu, \nu). \quad (\text{A15})$$

Given (A10) we can also see that $w(\lambda, \lambda')$ has the expansion, in terms of the ϕ and ψ functions,

$$w(\lambda, \lambda') = \sum_{n=0}^{\infty} t^n \phi_n(\lambda) \psi_n(\lambda'), \quad (\text{A16})$$

and so we may readily obtain two additional identities

$$\int d\lambda d\lambda' K_N(\mu, \lambda') K_N(\lambda, \nu) w(\mu, \lambda') w(\lambda, \mu) = \frac{1}{N} \sum_{n=0}^{N-1} \phi_n^2(\mu) \frac{1}{N} \sum_{m=0}^{N-1} \psi_m^2(\nu), \quad (\text{A17})$$

$$\int d\lambda d\lambda' K_N(\lambda, \lambda') w(\mu, \lambda') w(\lambda, \mu) = \bar{K}_N(\mu, \nu), \quad (\text{A18})$$

where we have defined

$$\bar{K}_N(\mu, \nu) \equiv \frac{1}{N} \sum_{n=0}^{N-1} t^n \phi_n(\mu) \psi_n(\nu).$$

Using (A14) and (A15) it can be shown that

$$\int d\lambda_m d\lambda'_m \det [K_N(\lambda_i, \lambda'_j)]_{i,j=1\dots m} w(\lambda_m, \lambda'_m) = \frac{m}{N} \det [K_N(\lambda_i, \lambda'_j)]_{i,j=1\dots m-1}. \quad (\text{A18})$$

We apply (A18) to (A12) N times and find $Z = N!/N^N$.

Now we are well equipped to evaluate Equation (3). In terms of the expressions we have developed here

$$\rho(\mu, \nu) = \frac{N^N}{N!} \int d\Lambda d\Lambda' \frac{1}{N} \sum_{n=1}^N \delta(\lambda_n - \mu) \frac{1}{N} \sum_{m=1}^N \delta(\lambda'_m - \nu) \det [K_N(\lambda_i, \lambda'_j)] \prod_{k=1}^N w(\lambda_k, \lambda'_k). \quad (\text{A19})$$

Applying (A18) $(N-2)$ times and integrating out the delta functions, we reduce this to

$$\begin{aligned} \rho(\mu, \nu) &= \frac{1}{N-1} \int d\lambda d\lambda' K_N(\mu, \nu) K_N(\lambda, \lambda') w(\mu, \nu) w(\lambda, \lambda') \\ &\quad - \frac{1}{N-1} \int d\lambda d\lambda' K_N(\mu, \lambda') K_N(\lambda, \nu) w(\mu, \nu) w(\lambda, \lambda') \\ &\quad + \int d\lambda d\lambda' K_N(\mu, \lambda') K_N(\lambda, \nu) w(\mu, \lambda') w(\lambda, \mu) \\ &\quad - \int d\lambda d\lambda' K_N(\mu, \nu) K_N(\lambda, \lambda') w(\mu, \lambda') w(\lambda, \mu). \end{aligned} \quad (\text{A20})$$

We obtain Equation (4) from the text by applying the four integration identities above.

APPENDIX B

We outline here how the asymptotic form of the kernel $K_N(\mu, \nu)$ is derived. The functions ϕ_n and ψ_n in Eqn.(6) are proportional to the Hermite functions and so may be written

$$\begin{aligned}\phi_n(\mu) &= \left[\sqrt{\frac{N\beta m_1^2}{2\pi}} \frac{1}{2^n n!} \right]^{\frac{1}{2}} e^{-\frac{1}{2}N(\frac{1}{2}\beta m_1^2 \mu^2)} H_n\left(\sqrt{\frac{N\beta m_1^2}{2}} \mu\right), \\ \psi_n(\nu) &= \left[\sqrt{\frac{N\beta m_2^2}{2\pi}} \frac{1}{2^n n!} \right]^{\frac{1}{2}} e^{-\frac{1}{2}N(\frac{1}{2}\beta m_2^2 \nu^2)} H_n\left(\sqrt{\frac{N\beta m_2^2}{2}} \nu\right),\end{aligned}$$

where we have used $\beta \equiv (1 - t^2)$. The Hermite polynomials have the contour integral representation,

$$H_n(x) = \frac{2^n n!}{2\pi i} \oint dz z^{-(n+1)} e^{-\frac{z^2}{4} + xz}, \quad (B1)$$

where the contour of integration is any path enclosing the origin. It may be worth mentioning that, because all classic orthogonal polynomials have some such integral form, the general procedure in this appendix works for the ϕ_n 's and ψ_n 's proportional to any of them.

Using (B1), and inserting $n! = \int_0^\infty ds e^{-s} s^n$, we may write the right hand side of (6) as

$$\begin{aligned}\frac{-1}{(2\pi)^2 N} \left[\frac{N\beta m_1 m_2}{2\pi} \right]^{\frac{1}{2}} \sum_{n=0}^{N-1} e^{-\frac{1}{2}N(\frac{1}{2}\beta m_1^2 \mu^2)} \oint \frac{dw}{w} e^{-\frac{w^2}{4} + \sqrt{\frac{1}{2}N\beta m_1^2} w} \\ \times e^{-\frac{1}{2}N(\frac{1}{2}\beta m_2^2 \nu^2)} \oint \frac{dz}{z} e^{-\frac{z^2}{4} + \sqrt{\frac{1}{2}N\beta m_2^2} \nu z} \int_0^\infty ds e^{-s} \left(\frac{2s}{twz} \right)^n\end{aligned} \quad (B2)$$

We make the variable substitutions, $u \equiv -\ln t$, $\sin \theta \equiv \mu/2\sigma$ and $\sin \phi \equiv \nu/2\tau$, with 2σ and 2τ being the endpoints of the eigenvalue spectra, specifically $\sigma = [\sqrt{\beta m_1^2}]^{-1}$ and $\tau = [\sqrt{\beta m_2^2}]^{-1}$. Note that since t must be in the range $(0, 1)$, u will range over $(0, \infty)$, and also that as μ and ν vary over their allowed ranges, θ and ϕ range from $-\pi/2$ to $+\pi/2$.

Upon substituting, and interchanging the order of integrations and summation we get

$$\begin{aligned}\frac{-1}{(2\pi)^2} \left[\frac{1}{2\pi N\sigma\tau} \right]^{\frac{1}{2}} e^{-N\sin^2 \theta} \oint \frac{dw}{w} e^{-\frac{w^2}{4} + \sqrt{2N}\sin \theta w} \\ \times e^{-N\sin^2 \phi} \oint \frac{dz}{z} e^{-\frac{z^2}{4} + \sqrt{2N}\sin \phi z} \int_0^\infty ds e^{-s} \sum_{n=0}^{N-1} \left(\frac{2se^u}{wz} \right)^n.\end{aligned} \quad (B3)$$

Doing the sum and scaling w , z , and s by $\sqrt{2N}$, $\sqrt{2N}$, and N we obtain

$$\frac{-1}{(2\pi)^2} \left[\frac{N}{2\pi\sigma\tau} \right]^{\frac{1}{2}} e^{-N(\sin^2 \theta + \sin^2 \phi)} \oint dw e^{-Nf(w)} \oint dz e^{-Ng(z)} \int_0^\infty ds e^{-Ns} \left[\frac{(se^u)^N - (wz)^N}{(se^u) - wz} \right] \quad (B4)$$

where we have set

$$\begin{aligned}f(w) &= -\frac{1}{2}w^2 + 2\sin \theta w - \log w, \\ g(z) &= -\frac{1}{2}z^2 + 2\sin \phi z - \log z.\end{aligned} \quad (B5)$$

To this point no approximations have been made so (B4) is exactly equal to $K_N(\mu, \nu)$ for all N .

To determine the leading order behavior of (B3) we now evaluate the three integrals using the method of steepest descent. To leading order

$$\int_0^\infty ds e^{-Ns} \left[\frac{(se^u)^N - (wz)^N}{se^u - wz} \right] = \frac{\sqrt{2\pi}}{\sqrt{N}} \frac{e^{-N} e^{Nu}}{[e^u - wz]} \quad (B6)$$

The integral over z , to leading order, is

$$\oint dz \frac{e^{Ng(z)}}{[e^u - wz]} = \sum_{\eta=\pm} \frac{\sqrt{\pi}}{\sqrt{N} \cos \phi} \frac{e^{Ng(z_\eta)} e^{\eta i \phi}}{[e^u - wz_\eta]} \quad (B7)$$

where $z_\pm = \exp \pm i(\phi - \frac{\pi}{2})$ are the saddle points of the integral. The contribution for the w integral has precisely the same form. Putting this all together, we find

$$K_N(\mu, \nu) = \frac{-1}{4\pi N} \frac{e^{Nu}}{\sqrt{\sigma\tau \cos \theta \cos \phi}} \sum_{\xi, \eta=\pm} \left[\frac{e^{-iN[\xi h(\theta) + \eta h(\phi)]}}{e^u - e^{i[\xi(\theta - \frac{\pi}{2}) + \eta(\phi - \frac{\pi}{2})]}} \right] + O(1/N), \quad (B8)$$

where $h(\theta) \equiv (\frac{1}{2} \sin 2\theta + \theta - \frac{\pi}{2}) + \frac{1}{N}(\frac{1}{2}\theta)$. By summing over ξ and η and formulating the expression in terms of cosines we obtain Equation (12).

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