Regularisation, the BV method, and the antibracket cohomology

Walter Troost and Antoine Van Proeyen

Instituut voor Theoretische Fysica
Katholieke Universiteit Leuven
Celestijnenlaan 200D
B–3001 Leuven, Belgium

Abstract

We review the Lagrangian Batalin–Vilkovisky method for gauge theories. This includes gauge fixing, quantisation and regularisation. We emphasize the role of cohomology of the antibracket operation. Our main example is $d = 2$ gravity, for which we also discuss the solutions for the cohomology in the space of local integrals. This leads to the most general form for the action, for anomalies and for background charges.

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2 Onderzoeksleider, NFWO, Belgium; E-mail : Walter.tf@fys.kuleuven.ac.be
E-mail : Walter.Troost@fys.kuleuven.ac.be.

3 Onderzoeksleider, NFWO, Belgium; E-mail : Antoine.VanProeyen@fys.kuleuven.ac.be.
1 Introduction.

The Batalin-Vilkovisky method for quantisation of general gauge theories has been introduced in [1]. It is applicable to all known gauge theories. Another advantage of the method is that it gives a comprehensive picture of the gauge fixing procedure. It uses a space of fields and antifields endowed with a symplectic structure defining an antibracket operation. The latter is a Poisson-like structure, and field transformations which are canonical with respect to this antibracket play an important part. Gauge fixing is essentially such a canonical transformation. We will summarize the essential ingredients and tools. In comparison with previous lectures [2, 3] we will put more emphasis here on the role of cohomology. We will use $d = 2$ gravity as our main example, and refer to [3] for chiral $W_3$.

In section 2 we will recall some definitions, introduce antifields, antibrackets and the ‘extended classical action’. In the third section we introduce three different cohomologies and indicate the different roles they play. In section 4 we will review how gauge fixing is obtained by a canonical transformation. Then we will turn to the quantum theory. General principles will be given in section 5, where also the regularisation (based on Pauli-Villars) will be introduced [4, 5]. Anomalies are discussed in section 6, and illustrated with a calculation for $d = 2$ gravity. The latter will be treated also from a purely cohomological point of view in section 7. That section reports work in collaboration with F. Brandt [6]. Besides the most general form for the classical action and for anomalies, we also obtain the most general form for background charges.

This lecture is not self-contained. The reader is urged to have also [2, 3] at hand, as we will refer to specific formulas in these reviews. The relevant equations and sections will be referred to as (I.), resp. (II.). A more complete text is still in preparation [7]. Many examples can also be found in [8].

2 The ingredients

Let us start by introducing the setup for $d = 2$ gravity with scalar matter. The classical fields are $\phi^i = \{X^\mu, g_{\alpha\beta}\}$, where $X^\mu$ are the $D$ scalar fields, and $g_{\alpha\beta}$ is the two dimensional metric. There are two types of gauge symmetries, for which we introduce ghosts $c^a = \{\xi^a, c\}$, namely $\xi^a$ for the general coordinate transformations ($\alpha = +$ or $-$) and $c$ for the local dilatations. Note that we do not introduce gauge fields for these transformations. For the diffeomorphisms they are nevertheless effectively included in the metric $g_{\alpha\beta}$, and in fact one could equally set up this same problem using the zweibeins instead of the metric. For the dilatations however they remain excluded.

We denote all these fields collectively as $\Phi^A = \{\phi^i, c^a\}$. We introduce antifields for all of them, $\Phi^*_{\alpha}$. They have opposite statistics. Then one defines an extended action $S(\Phi, \Phi^*)$, function of all the Fields\(^1\). The construction of this extended action was explained in detail

\(^1\)We need a common denomination for talking about fields and antifields together. For this we will use ‘Fields’ with a capital F. We will use the notation $z^a = \{\Phi^A, \Phi^*_{\alpha}\}$ (with $A = 1, ..., N$, and $\alpha = 1, ..., 2N$).
in [2, 3], and leads to\(^2\)

\[
S = -\frac{1}{2} \sqrt{g} g^{\alpha \beta} \partial_{\alpha} X_{\mu} \cdot \partial_{\beta} X_{\mu} \\
+ X_{\mu}^{*} \xi^{\alpha} \partial_{\alpha} X_{\mu} + g^{* \alpha \beta} \left( \xi^{\gamma} \partial_{\gamma} g_{\alpha \beta} + 2 g_{\gamma (\alpha} \partial_{\beta)} \xi^{\gamma} + c g_{\alpha \beta} \right) \\
- \xi^{*} \xi^{\alpha} \partial_{\alpha} \xi_{\alpha} - c^{*} \xi^{\alpha} \partial_{\alpha} c .
\]

(1)

The first line is the classical action, and the other lines are the BRST transformations of the fields multiplied by their corresponding antifields. In general, more terms may be present, for example expressing relations in an open gauge algebra. Introducing now an antibracket

\[
(F, G) = F \left( \frac{\partial}{\partial \Phi^A} \right) \cdot \frac{\partial}{\partial \Phi^*_A} G - F \left( \frac{\partial}{\partial \Phi^*_A} \right) \cdot \frac{\partial}{\partial \Phi^A} G ,
\]

(2)

It is clear that the extended action satisfies

\[
(S, S) = 0 .
\]

(3)

When one checks the terms in this equation with a specific type of antifields, one will notice that it expresses the gauge invariance of the classical action, as well as the commutator algebra of the gauge transformations and their Jacobi identities. In general, for example with open gauge algebras, it will also include the other defining relations of the gauge theory.

One assigns ghost numbers 0 for the classical fields \( \phi^i \), 1 for the ghosts \( c^a \), and for the antifields a ghost number such that the sum of the ghost number of a field and its corresponding antifield always adds up to \(-1\). As a consequence, by (2), this gives a ghost number assignment to \((F, G)\) which is 1 lower than that of \(FG\).

We invite the reader now to look to the example of \(W_3\) in section II.1 and II.2. This example was treated in full using the BV method in [9].

Let us summarize the essential properties of the extended classical action. It is a function \(S(z)\) of all the Fields \(z\).

**Classical limit :** \(S(\Phi, \Phi^* = 0)\) is the classical action.

**Master equation :** \((S, S) = 0\), which implies that\(^3\) \(R_{\alpha \beta} \equiv \frac{\partial}{\partial z^a} \cdot \frac{\partial}{\partial z^a} S\) has rank \(\leq N\).

**Properness condition :** the rank of \(R\) is equal to \(N\). This property essentially means that we have to include all symmetries using ghosts, and also all zero modes. It will allow us to define the path integral.

It has been proven in several steps [10, 11, 9] that there is a local solution to these conditions for all classical actions (under certain regularity conditions satisfied by all reasonable gauge

\(^2\)We omit \(\int d^{3}x\) for actions and anomaly expressions, here and in the sequel. (Anti-)Symmetrisation of indices is defined by \(f_{(\mu \nu)} = \frac{1}{2}(f_{\mu \nu} + f_{\nu \mu})\), etc.

\(^3\)The argument was repeated in section 1.3.1.
theories). The proof is very long, and will not be repeated here. The main tool is the ‘Koszul–Tate’ differential, which we will define below.

3 Cohomologies

We now introduce several different but related differentials and cohomologies. From the extended action one defines the nilpotent operator

\[ SF(z) \equiv (S, F) \, . \tag{4} \]

The nilpotency follows from the Jacobi identity of antibrackets and the master equation:

\[ S^2 F = (S, (S, F)) = \frac{1}{2}((S, S), F) = 0 \, . \tag{5} \]

The operator \( S \) raises the ghost number by 1, i.e.: \( gh(SF) = gh(F) + 1 \). The cohomology of this operator in the space of local functions of ghost number 0 gives physically meaningful quantities. They are formed by arbitrary gauge invariant functionals, and are defined up to field equations.

We now introduce a second grading: the antifield number (\( afn \)). This is \(-gh\) for the Fields of negative ghost number (antifields in the ‘classical basis’, the one used so far), and 0 for Fields of non-negative ghost number. Check that the extended action (1) has zero ghost number, but the 3 lines have antifield numbers 0, 1 and 2 respectively. The action of \( S \) can be split according to its change in the antifield number:

\[ S = \delta_{KT} + \Omega + D_1 + D_2 + \ldots \, . \tag{6} \]

Observe that the antifield number at most diminishes by one. This part defines the Koszul–Tate (KT) differential (introduced in BV quantisation in [11]). Explicitly, the main properties are:

\[ \delta_{KT}\Phi = 0 \quad \text{vanishes on fields} \]

\[ \delta_{KT}\hat{\phi}_i^* = \frac{S^0 \partial}{\partial \phi_i} \quad \text{field equations} \]

\[ \delta_{KT}\hat{c}_a^* = \hat{\phi}_i^* R_i^a \]

\( (S^0 \) is the action without antifields, and \( R_i^a \) determines its gauge transformations). Now the nilpotency property \( S^2 = 0 \) implies

\[ \delta_{KT}^2 = 0 \quad \text{KT is nilpotent} \]

\[ \Omega\delta_{KT} + \delta_{KT}\Omega = 0 \]

\[ \Omega^2 = -\delta_{KT} D_1 - D_1\delta_{KT} \quad \text{On-shell nilpotent BRST} \tag{8} \]

The equation (6) also defines \( \Omega \), which will be called the BRST operator. Its action on functionals of the fields is given by

\[ \Omega\Phi = S\Phi|_{\phi^*=0} \, . \tag{9} \]

We will come back to the meaning of the last lines of (8) in a moment.
We have now 3 different cohomologies, related to $\mathcal{S}$, $\delta_{KT}$ and $\Omega$. We should also be careful to distinguish cohomology in the space of local functions, or in the space of integrals thereof (this is cohomology mod $d$, denoted $H^q(\mathcal{S} \mid d)$, where $d$ is the space-time differential).

1. $\delta_{KT}$ acts in the space of functions of $\Phi_A^*$ and $\phi^i$, i.e. functions of ghost number negative or zero: it vanishes on the ghosts, and also the image of $\delta_{KT}$ on the antifields or classical fields does not contain the ghosts. As a grading one uses the antifield number, and $H^k(\delta_{KT})$ then denotes the cohomology of $\delta_{KT}$ in local functions of antifield number $k$. Its main property is that

$$H^k(\delta_{KT}) = \delta_0^k \times \text{functions on the stationary surface},$$

where the stationary surface refers to the surface in the space of fields of ghost number 0 (`classical fields' $\phi^i$) where the classical field equations are satisfied\(^4\). Also for integrals over local functions of ghost number zero, or $H^q(\delta_{KT} \mid d)$, such a property holds. But there is not such a general statement for $H^k(\delta_{KT} \mid d)$ with $k > 0$ (in [12] some general results have been given, relating e.g. the $k = 1$ case to rigid symmetries).

2. The last line of (8) implies that $\Omega$ is nilpotent in the cohomology space of $\delta_{KT}$. One can define a cohomology of $\Omega$ in the cohomology space $H^k(\delta_{KT})$. This is the BRST cohomology, which therefore is defined on the stationary surface. So a function $F$ is in the cohomology of $\Omega$, and non trivial, if

$$\Omega F \approx 0 \quad \text{and} \quad F \not\approx \Omega G,$$

where $\approx$ means equality on the stationary surface, defined with the classical field equations, and $F$ satisfies $\delta_{KT} F = 0$. This is denoted by $H^k(\Omega \mid H(\delta_{KT}))$. The grading of $\Omega$ is the ghost number. Therefore $k$ refers here to the ghost number. For $k = 0$ the cohomology of local functions now identifies functions which differ by gauge transformations, and, combined with the result for $\delta_{KT}$, this implies that we are left with the physical observables.

3. $\mathcal{S}$ acts on fields and antifields. Its grading is the ghost number. The cohomology of $\mathcal{S}$ combines the results of the previous two cohomologies:

$$\begin{align*}
\text{for } k < 0 : & \quad H^k(\mathcal{S}) = 0 \\
\text{for } k \geq 0 : & \quad H^k(\mathcal{S}) = H^k(\mathcal{S} \mid H(\delta_{KT})).
\end{align*}$$

Note that this statement for $k \geq 0$ applies also for cohomology mod $d$ (local integrals). It is important that to define the cohomologies with the BRST method we have to use ‘weak equalities’ (even for a ‘closed algebra’), while in the antibracket cohomology the inclusion of the antifields allows us to work with unqualified equalities.

4 Gauge fixing

Let us illustrate the gauge fixing with a simple example. Consider the theory with classical fields $\phi^i = \{X^\mu, h\}$ and classical action

$$S_0 = \partial X^\mu \bar{\partial} X_\mu.$$

\(^4\)To be more exact, in some cases one should define this concept using the concept of functions ‘not proportional to field equations’, see the discussion on ‘evanescent functions’ in section 3.1 of [9].
Trivially, \( h \) does not occur in it, which we can interpret as a gauge invariance \( \delta h = \epsilon \). The extended action is

\[
S = \partial X^\mu \bar{\partial} X_\mu + h^* \epsilon .
\]  

(14)

For propagators, one wishes to invert the matrix of second derivatives of the action w.r.t. fields, but it is singular. Consider however the matrix of second derivatives w.r.t. Fields, \( \partial_{\alpha} \partial_{\beta} S \), and make a slight rearrangement:

\[
\begin{pmatrix}
X^\mu & h & c & X^* & h^* & c^* \\
X^\mu & h^* & c & X^* & h & c^* \\
X^\mu & c & \begin{pmatrix} * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 \end{pmatrix} & \begin{pmatrix} h^* & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
X^\mu & c & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 \end{pmatrix} & \begin{pmatrix} h^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
X^\mu & c & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} h^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{pmatrix}.
\]  

(15)

The stars in the matrix denote the non-zero entries. The rank of the matrix is indeed half its dimension (properness condition). The invertible part is just the left upper corner of the second matrix. (The other entries happen to be zero because of the simplicity of the model). If we reassign the antifield \( h^* \) to be a field, renaming it \( b \), then \( h \) becomes an antifield \( b^* \), and the action in terms of the new fields has no zero-modes. This is what we call gauge-fixed.

In general gauge fixing is a canonical transformation such that the new ‘fields’ have no zero-modes. Later we will choose these fields as integration variables of the path integral.

Also in \( d = 2 \) gravity gauge fixing is accomplished by just such a change of name. We define

\[
g_{\alpha \beta} = \eta_{\alpha \beta} + b^*_{\alpha \beta} ; \quad g^{*\alpha \beta} = -b^{\alpha \beta} .
\]  

(16)

We can now consider two bases:

\[
\begin{array}{c|cccccc}
\text{classical basis} & X^\mu & g_{\alpha \beta} & \xi^\alpha & c & X^* & g^{*\alpha \beta} \\
\text{statistics} & + & + & - & - & - & + \\
\text{ghost number} & 0 & 0 & 1 & 1 & -1 & -2 & -2 \\
gauge fixed basis & X^\mu & b^*_{\alpha \beta} & \xi^\alpha & c & X^* & b^{\alpha \beta} & \xi^* & c^*
\end{array}
\]  

(17)

It is now easy to see that, substituting (16) in (1), the part of the extended action which is independent of antifields in the gauge-fixed basis has no gauge invariances any more, i.e. it is properly called gauge fixed. Note that \( b^{\alpha \beta} \) plays the role of the antighost.

There are other types of canonical transformations that are often important. The general type of a canonical transformation \( \{ \Phi, \Phi^* \} \) to \( \{ \hat{\Phi}, \hat{\Phi}^* \} \) where the field-field part is invertible (this does not include the above transformation (16)), can be generated from a (fermionic) function \( F(\Phi, \Phi^*) \):

\[
\hat{\Phi}^A = \frac{\partial F(\Phi, \Phi^*)}{\partial \Phi_A} \quad \Phi_A^* = \frac{\partial F(\Phi, \Phi^*)}{\partial \Phi_A^*} .
\]  

(18)

We will here illustrate the use of such a transformation, not for gauge fixing (that will be done below), but to achieve a simplification in the extended action of \( d = 2 \) gravity.
Consider the transformation from \( \{ \Phi \} = \{ X^\mu, g_{\alpha \beta}, \xi^\alpha, c \} \) and their antifields to \( \{ \Phi \} = \{ \bar{X}^\mu, \bar{h}_{++}, \bar{h}_{--}, \bar{\epsilon}, \bar{c}^\alpha, \bar{\bar{c}} \} \) and the corresponding antifields generated by (an integral over \( d^2 x \) is implied)

\[
F = \bar{X}^\mu X^\mu + \bar{\epsilon}^* \sqrt{\bar{g}} + \bar{\bar{c}}^* (c^* \sqrt{\bar{g}} + \partial_\alpha \xi^\alpha \sqrt{\bar{g}}) \\
+ h^{++*} \frac{g_{++}}{g_{++} + \sqrt{\bar{g}}} + h^{--*} \frac{g_{--}}{g_{--} + \sqrt{\bar{g}}} \\
+ c_+^* \left( \xi^+ + \frac{g_{--}}{g_{++} + \sqrt{\bar{g}}} \xi^- \right) + c_-^* \left( \xi^- + \frac{g_{++}}{g_{--} + \sqrt{\bar{g}}} \xi^+ \right).
\]

This leads to

\[
S = \frac{1}{1 - h_{++} h_{--}} \left( -\nabla_+ X^\mu \cdot \nabla_- X^\nu \eta_{\mu \nu} + X^\mu c^\alpha \nabla_\alpha X^\mu \right) \\
+ h^{++*} \nabla_+ c^- + h^{--*} \nabla_- c^+ + \bar{\epsilon}^* \bar{\bar{c}} \\
- c_+^* \partial_+ c^+ - c_-^* \partial_- c^-,
\]

where \( \nabla_+ = \partial_+ - h_{++} \partial_+ + \lambda (\partial_+ h_{++}) \) (\( \lambda \) is the spin: lower + indices, or upper – indices ...). Here the term \( \bar{\epsilon}^* \bar{\bar{c}} \) exhibits a trivial system: \( \bar{\epsilon} \bar{\bar{c}} = \bar{c} \) and \( \bar{\epsilon}^* c^* = c^* \). The cohomology in the quartet \( \{ \epsilon, \bar{\epsilon}, \epsilon^*, \bar{\epsilon}^* \} \) is trivial, it can therefore be omitted. The reader may also consult section I.3.2 for a review on the applications of canonical transformations, and section I.3.4 on the trivial systems. In the new basis gauge fixing is again performed just by the transformation \( h^{++*} = b^{++} \) and \( h^{--} = -b_{++}^{--} \).

The canonical transformation of the form (18) is often used for gauge fixing. Consider \( F \) of the particular form

\[
F = \Phi A \Phi^*_A + \Psi(\Phi).
\]

The first term by itself produces just the identical transformation. The second is the so-called gauge fermion \( \Psi \), which was needed in [1], although it should be clear now that this method is a particular case. For Maxwell theory for example, the minimal extended action would be just \( S = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + A^\mu \partial^\mu c \). One can first add an auxiliary field term \(-\frac{1}{2} b^* b^* \).

In the cohomology language this is just again a trivial system \( S b = b^* \) and \( S b^* = 0 \). Then the canonical transformation we need for gauge fixing is of the type (21) with \( \Psi = b \partial_\mu A^\mu \).

The reader can check that this gives the usual gauge-fixed action. Note that, although we had to introduce a trivial system before the canonical transformation, even in this simple example there is still a simplification w.r.t. the usual BRST procedure. In BRST one first introduces also an auxiliary field \( \lambda \). This is necessary to have a nilpotent BRST–operator off shell. However the algebra of \( S \) is always closed.

We have related the physical observables to the cohomology of the operator \( S \). The essential principle is that we always use canonical transformations, and add or delete trivial systems. A canonical transformation does not change the cohomology of this operator, and neither do the trivial systems. In a gauge–fixed basis we may again define a BRST operator by (9), where the fields and antifields are now those of the gauge–fixed basis. Then one can prove that the cohomology of \( S \) is equivalent to the ‘weak’ cohomology of \( \Omega \) (using the new field equations of all the fields). This proves that the BRST cohomology gives the
physical observables again, but also shows the advantage of the $S$-cohomology: it includes the information in the field equations, as $S\Phi^* = \frac{\delta S}{\delta \Phi}$.

5 Quantum theory and Pauli–Villars regularisation

The general principles of the quantum treatment in the BV framework have been described in section II.4, to which we refer for an exposition of our method. The advantage of the Pauli–Villars (PV) regularisation is that it has a Lagrangian formulation that interfaces nicely with the BV framework. We will now describe some additional aspects of the PV procedure, using $d = 2$ gravity as an illustration.\(^5\) The quantum theory, in one loop approximation, is based on the action

$$S_T = S(z) + S_{PV}(z, \dot{z}) + S_M(z, \Phi).$$

In this formula, the massless PV extended action and the mass terms for the PV fields are given by

$$S_{PV} = \frac{1}{2} \dot{z}^\alpha \left( \frac{\partial}{\partial z^\alpha} S \frac{\partial}{\partial \dot{z}^\beta} \right) \dot{z}^\beta$$

and

$$S_M = -\frac{1}{2} M^2 \Phi^A T_{AB}(z) \Phi^B,$$

where $T_{AB}(z)$ is largely arbitrary but must be invertible. Note that this last condition implies that, formally, in the limit $M \to \infty$, the PV Fields are cohomologically trivial since

$$(S, \Phi^*_A) = M^2 T_{AB} \Phi^B.$$  

In the example of $d = 2$ gravity, we get from (1)

$$S_{PV} = -\frac{1}{2} \sqrt{g} g^{\alpha \beta} \partial_\alpha X^\mu \cdot \partial_\beta X_\mu$$

$$+ g_{\alpha \beta} \sqrt{g} \left( g^{\alpha \gamma} g^{\beta \epsilon} - \frac{1}{2} g^{\alpha \beta} g^{\gamma \epsilon} \right) \partial_\alpha X^\mu \cdot \partial_\beta X_\mu$$

$$- \frac{1}{2} g_{\gamma \epsilon} g_{\alpha \beta} \sqrt{g} \left( g^{\alpha \gamma} g^{\beta \epsilon} g^{\delta \beta} - \text{traces} \right) \partial_\alpha X^\mu \cdot \partial_\beta X_\mu$$

$$+ X^\mu_\alpha \xi^\alpha \partial_\alpha X^\mu + X^\mu_\beta \xi^\beta \partial_\beta X^\mu + X^\mu_\gamma \partial_\gamma X^\mu$$

$$+ g^{\alpha \beta} \left( \xi^\gamma \partial_\gamma g_{\alpha \beta} + 2 g_{\gamma (a} \partial_{\beta)\xi^\gamma} + c g_{\alpha \beta} \right)$$

$$+ g^{* \alpha \beta} \left( \xi^\gamma \partial_\gamma g_{\alpha \beta} + 2 g_{\gamma (a} \partial_{\beta)\xi^\gamma} + c g_{\alpha \beta} \right)$$

$$- \xi^\alpha \partial_\alpha \xi^\beta - \xi^\beta \partial_\beta \xi^\alpha - \xi^\gamma \partial_\gamma \xi^\alpha - \xi^\alpha \partial_\alpha \xi^\gamma$$

$$- \xi^\beta \partial_\beta \xi^\gamma - \xi^\gamma \partial_\gamma \xi^\beta + c \xi^\alpha \partial_\alpha \xi^\gamma - c \xi^\beta \partial_\beta \xi^\gamma.$$  

Gauge fixing is now performed by the canonical transformation (16) together with

$$g^{* \alpha \beta} = -\bar{b}^{\alpha \beta}, \quad g_{\alpha \beta} = \bar{b}_{\alpha \beta}^*.$$  

Note that here we introduced the PV system already before gauge fixing. To keep the correspondence between the PV action and the action of the ordinary fields as in (23), their canonical transformations must be related. This question is treated in more detail in (I.42).

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\(^5\)The notation used here and in the sequel differs slightly from the one used in (I) and (II), viz. we denote the PV–partner of a Field by underscoring its symbol.
The PV mass term should be quadratic in the PV partners of the fields in the gauge-fixed basis. As an example we take

$$S_M = -\frac{1}{2} M^2 g^{\mu \nu} \nabla^2 \Phi + \frac{1}{2} M g \epsilon_{\alpha \beta} \Phi_{\gamma} \epsilon^{\gamma} + \frac{1}{2} M \Phi^a \partial^b g_{\alpha \beta} \epsilon^{\gamma} \epsilon_{\gamma \delta} \epsilon^{\delta \epsilon} .$$

To illustrate the arbitrariness in the choice of mass term, which will also play a role in section 6, we introduced an arbitrary real parameter \(t\) in the mass term for the scalars. If necessary one introduces several copies of PV fields with multiplicities \(C_i\), and imposes the conditions

$$\sum_i C_i = 1 \quad \text{and} \quad \sum_i C_i (M_i)^{2n} = m^{2n} \quad \text{for} \quad n = 1, 2, \ldots, n_{\text{max}}.$$  

The divergences of the theory manifest themselves as terms of the form \(\sum_i C_i (M_i)^{2n} \log M_i^2\), and have to be absorbed by renormalisation before the limit \(M_i \rightarrow \infty\) is taken. An example will be seen in the next section.

6 Anomalies

How anomalies occur in the formal path integral has been explained in section (I.4) and the first part of section II.5.1. Concerning cohomology let us add that even in the presence of anomalies there is a nilpotent operator

$$S_\gamma F = e^{-\frac{i}{\hbar} W} \frac{\hbar}{i} \Delta e^{\frac{i}{\hbar} W} F = (W, F) + \frac{\hbar}{i} \Delta F + \mathcal{A} F ,$$

where \(W\) is the quantum action \(S + \hbar M + \ldots\), and \(\Delta\) is acting from the left:  

$$\Delta = (-)^A \frac{\partial}{\partial \Phi^A} \frac{\partial}{\partial \Phi^*} .$$

The nilpotency of \(\Delta\) immediately implies the nilpotency of \(S_\gamma\). The anomaly itself is \(\mathcal{A} = S_\gamma 1\), and is trivially invariant under \(S_\gamma\). If another quantity \(M\) can be found such that \(\mathcal{A} = S_\gamma M\), then the anomaly can formally be removed by a redefinition \(W' = W + \frac{\hbar}{i} \log(1 - M)\). Note however that in general this additional contribution to the action can not be written as a local term. Note also that in \(\hbar\) expansions of expressions like (29) it has often been assumed that \(\Delta S\) gives only terms of order \(\hbar^0\) [1, 5]. This may not always be true in a regulated theory.

In sections I.5.1, I.5.2 and II.5.1 it has been shown why these formal expressions need regularisation, and how this can be performed by using PV regulators [4, 5]. One ends up with a regulated determination of \(\Delta S\) which depends on the choice of \(T\) in the PV mass term of (23). (i.e. on the regularisation), on the choice of basis (gauge fixing), ... but it always takes a value in the same cohomology class: for two such choices \((a)\) and \((b)\) we have

$$\Delta^{(b)} S = \Delta^{(a)} S + (F, S) \quad \text{with} \quad F \text{ local} .$$

Rather than repeating the general exposition, we will sketch here the application to the scalar loops in \(d = 2\) gravity.

The anomaly at one loop arises from

$$\mathcal{A} = \frac{1}{2\pi}(S_T, S_T) = \frac{1}{\pi}(S + S_{PV}, S_M) .$$

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\(^6\)Important properties are \(\Delta F G = \Delta F \cdot G + (-)^F F \Delta G + (-)^F (F, G)\) and \(\Delta^2 = 0\).
We only treat here the integration over the "matter" PV fields $X$, for the purpose of illustration. From (28) we then only take into account the first term, and will in particular consider two values for $t$:

$$\text{at } t = 0 : \quad A_0 = \frac{M^2}{\hbar} \left[ X^\mu X_\mu \partial_\alpha \xi^\alpha - 2X^\mu \xi^\alpha \partial_\alpha X_\mu \right]$$

$$\text{at } t = 1 : \quad A_1 = \frac{M^2}{\hbar} \sqrt{g} \left[ -c X^\mu X_\mu - 2X^\mu \xi^\alpha \partial_\alpha X_\mu \right].$$

(33)

After integration over the PV fields the off–diagonal terms will vanish and ($\partial_\alpha \sqrt{g} g^{\alpha \beta} \partial_\beta$)

$$X^\mu X_\mu \to \frac{\hbar}{M^2 g^{1/2}}.$$ 

(34)

We then use heat kernel techniques. The method was reviewed in section 5 of (1); one applies (1.47) and (1.50) with

$$t = 0 : \quad J = \partial_\alpha \xi^\alpha \quad \mathcal{R} =$$

$$t = 1 : \quad J = -c \quad \mathcal{R} = \frac{1}{\sqrt{g}},$$

(35)

leading to

$$A_0 = \frac{1}{8\pi} \left( M^2 \log M^2 - \frac{1}{5} \sqrt{g} R + \frac{1}{12} \log g \right) \left( -\partial_\alpha \xi^\alpha \right)$$

$$A_1 = \frac{1}{8\pi} \left( M^2 \log M^2 \sqrt{g} - \frac{1}{5} \sqrt{g} R \right) c.$$

(36)

At $t = 0$ our mass term is indeed invariant under Weyl transformations, and not under general $d = 2$ coordinate transformations. At $t = 1$ we have the reverse situation. This explains the occurrence of the Weyl ghosts for $t = 1$ and the diffeomorphism ghosts for $t = 0$. Note that for $M \to \infty$ there is a diverging part, which is however a total derivative for $t = 0$, and, as will be clear below, is even locally $\mathcal{S}$–exact for $t = 1$. The two expressions for the anomaly are related by $A_1 - A_0 = SM_1$ with

$$M_1 = \frac{1}{8\pi} \left( M^2 \log M^2 \sqrt{g} - \frac{1}{12} \log g \sqrt{g} R + \frac{1}{48} \log g \log g \right),$$

(37)

showing explicitly that the change in regularisation preserves the cohomological class of the anomaly.

As already stressed in the previous lectures, the anomaly depends on $g_{\alpha \beta} = \eta_{\alpha \beta} + b^*_\alpha \beta$, i.e. on the antifields of the gauge–fixed basis.

Knowing that in general the anomaly is an element of the cohomology of $\mathcal{S}$, it is interesting to investigate what the possible classes are, a priori, without doing the actual anomaly calculation. To this we now turn.

7 Cohomological analysis of $d = 2$ gravity and background charges

In collaboration with F. Brandt [6], we have performed an analysis of the cohomology of $\mathcal{S}$ for local 2–forms modulo $d$, or $H^2(\mathcal{S} \mid d)$. The starting point of this analysis is the symmetry algebra, without assumptions about the classical action. Technically, we assumed knowledge of $S^1$ and $S^2$ only, i.e. the second and third line of (1).
There is a relation with the local cohomology of 0-forms

\[ H^2(S \mid d) \sim H^{g+2}(S), \]

using descent equations \((\omega^g_2\text{ is an }f\text{-form of ghost number } g)\)

\[ S\omega^g_2 + d\omega^{g+1}_1 = 0; \quad S\omega^{g+1}_1 + d\omega^{g+2}_0 = 0; \quad S\omega^{g+2}_0 = 0. \]

For this equivalence it is necessary that the reparametrisation ghosts are present. The rather general results of [13], implying e.g. that derivatives of ghosts are not present in the cohomology, cannot be applied directly to the case of \(d=2\) gravity, because they rest on the assumption that all local symmetries have independent gauge fields, which is not the case for the Weyl symmetry in our setup. We will limit ourselves to a description of the results.

We find that a non-trivial cohomology exists for ghost numbers \(g = -1, 0, \ldots, 4\). Let us first mention the most general functional with ghost number 0 which does not depend on antifields. This functional can be used as a classical action, replacing the first line of (1):

\[ S_{cl} = \frac{1}{2} \sqrt{g} \frac{\partial}{\partial y} G_{\mu\nu}(X) \partial_{x} X^{\mu} \cdot \partial_{y} X^{\nu} + B_{\mu
u}(X) \partial_{+} X^{\mu} \cdot \partial_{-} X^{\nu}. \]

Here \(G_{\mu\nu}(X)\) and \(B_{\mu\nu}(X)\) are arbitrary, but not all physically different: the following changes relate solutions that are cohomologically equivalent:

\[ G \sim G'_{\mu\nu} = G_{\mu\nu} + 2\partial_{x} f_{\mu} \Gamma_{\mu\nu\rho} f_{\rho}, \]
\[ B \sim B'_{\mu\nu} = B_{\mu\nu} + 2\partial_{x} h_{\mu\nu} + H_{\mu\nu\rho} f_{\rho}, \]

where

\[ f_{\mu} = G_{\mu\nu} f^{\nu}; \quad \Gamma_{\mu\nu,\rho} = \partial_{x} G_{\nu,\rho} - \frac{1}{2} \partial_{x} G_{\mu\nu}; \quad H_{\mu\nu\rho} = 3\partial_{x} h_{\mu\rho}. \]

The functions \(B_{\mu}\) and \(f^{\mu}\) are arbitrary, and the latter can be interpreted as a target space reparametrisation \(X^{\mu} \rightarrow X^{\mu} + f^{\mu}\).

In addition there are antifield-dependent solutions depending on arbitrary covariantly constant functions \(f^{(\pm)p}(X)\),

\[ D^{\pm}f^{(\pm)p}_{\nu} \equiv \partial_{x} f^{(\pm)p}_{\nu} - \Gamma^{\pm}_{\mu\nu,\rho} f^{(\pm)p}_{\rho} = 0 \quad \text{with} \quad \Gamma^{\pm}_{\mu\nu,\rho} = \Gamma_{\mu\nu,\rho} \pm \frac{1}{2} H_{\mu\nu\rho}. \]

These solutions are, using the notations of (19) and \(y = h_{++} h_{--}\),

\[ M(f^{(+)p}, f^{(-)p}) = X^{\mu} \left( \partial_{+} \xi^{+} + h_{-+} \partial_{-} \xi^{+} \right) \cdot f^{(+)p}_{\mu} - 2 \frac{1}{1-y} \partial_{-} h_{-+} \cdot \nabla_{+} X^{\mu} \cdot f^{(+)p}_{\mu} + X^{\mu} \left( \partial_{-} \xi^{-} + h_{++} \partial_{+} \xi^{-} \right) \cdot f^{(-)p}_{\mu} - 2 \frac{1}{1-y} \partial_{+} h_{++} \cdot \nabla_{-} X^{\mu} \cdot f^{(-)p}_{\mu}. \]

Before discussing the meaning of these terms, we give also the solutions for ghost number 1. There are 2 special solutions that contain no arbitrary parameters apart from an overall factor:

\[ A_{\pm} = c_{\pm} \partial_{x}^{3} h_{\pm\pm}. \]

Also, there are solutions depending on arbitrary functions \(f^{(\pm)p}_{\mu\nu}(X)\):

\[ \omega(f^{(+)p}_{\mu\nu}, f^{(-)p}_{\mu\nu}) = \frac{1}{1-y} \nabla_{+} X^{\mu} \cdot \nabla_{-} X^{\nu} \cdot \left[ f^{(+)p}_{\mu\nu} \left( \partial_{+} \xi^{+} + h_{-+} \partial_{-} \xi^{+} \right) + f^{(-)p}_{\mu\nu} \left( \partial_{-} \xi^{-} + h_{++} \partial_{+} \xi^{-} \right) \right]. \]
Again, some of these solutions are cohomologically equivalent: for arbitrary $H^+_\mu(X)$,
\begin{equation}
\begin{split}
    f_{\mu}^+ & \sim f_{\mu}^+ + D^+_{\mu} H^+_{\mu}; \\
    f_{\mu}^- & \sim f_{\mu}^- + D^-_{\mu} H^-_{\mu}.
\end{split}
\end{equation}

All these solutions of ghost number 1 appear when discussing the anomalies of general $\sigma$-models with classical action (40). In our analysis for the action (1) we met the particular combination
\begin{equation}
    A_+ + A_- \cong \frac{1}{2} c \sqrt{g} R
\end{equation}
where $\cong$ means an equality up to $S$-exact terms. In chiral gravity one obtains separately $A_+$ and/or $A_-$ as anomaly.

Now we come to the meaning of the other solutions of ghost number 0, eq.(44). First we give the antibracket\footnote{Note that $f^{(+)}_{\mu} g^{(+)}_{\mu}$ is a constant due to (43), which is valid also for $g$.}
\begin{equation}
    \left( M(f^{(+)}), M(g^{(+)}), M(g^{(+)}) \right) \cong -4 f^{(+)}_{\mu} g^{(+)}_{\mu} A_+ - 4 f^{(-)}_{\mu} g^{(-)}_{\mu} A_-. \end{equation}

This implies that, although $M$ is a local integral with ghost number zero, it cannot be used as an extra part of the extended action, as this breaks the master equation. However, the right hand side of (49) allows another interpretation. If we modify the action by adding an $M(f^{(+)}_+, f^{(-)}_-)$-term that is formally of order $\hbar^{1/2}$, then $(M, M)$ will contribute to the anomaly on the same level as the one-loop diagrams, modifying the coefficient of the anomaly, and possibly canceling it. This term therefore becomes an important addition for the quantum theory. It is the proper generalisation of what is usually called a "background charge" term. This is apparent when, for the chiral case, one puts $h^{++} = 0$, drops the corresponding $\xi$-ghost, and specializes to $G_{\mu\nu} = -\frac{1}{2} \delta_{\mu\nu}$. It is then another example of a term $M_{1/2}$, as treated in section II.5.3, presented there in the case of chiral $W_3$ to cancel the anomalies [14] (see [15, 9] for their inclusion in the BV formalism). For our non–chiral case, again one has to include both chiralities, and add an appropriate $S$–trivial term. This leads to the dilaton term in the $\sigma$-model.

8 Conclusions

In this review we described some of the ongoing work in the BV quantisation program, with emphasis on the cohomology of the operator $S \equiv (S, \cdot)$. It appears in various places:

- the local cohomology at ghost number 0 gives the classical physical observables.
- the cohomology of integrals at ghost number 0 gives the possible actions, and is also important for counterterms in the renormalisation procedure [16].
- the cohomology of integrals at ghost number 1 gives the possible anomalies.
- Antifield dependent anomalies at ghost number 0 correspond to background charges, possibly allowing a cancellation of anomalies.

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References