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Quadratic Poisson Algebras
and Their Infinite
Dimensional Extensions
by

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Quadratic Poisson Algebras and Their Infinite Dimensional Extensions

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1 Introduction

Quadratic Poisson algebras arose in the study of classical integrable equations. These equations can be integrated via a complicated change of variables which maps the dynamical variables to certain new variables called scattering data. It turns out that the Poisson brackets of the scattering data satisfy quadratic relations uniquely specified in terms of some structure constants [1]. These Poisson brackets satisfy the Jacobi identity provided that the structure constants satisfy the celebrated classical Yang-Baxter (YB) equation. The scattering data form a Lie group, therefore in this case the Poisson manifold has the additional structure of a Lie group. Drinfel'd [2] established a connection between solutions of the classical YB equation and bi-algebras. The enveloping algebra of bi-algebras gives rise to a Poisson-Hopf algebra.

Sklyanin [3] in the same pioneering paper where he introduced the classical Sklyanin algebra also introduced general quadratic Poisson algebras (without assuming a Lie group structure for the underline Poisson manifold). However, the equation he derived for the relevant structure constants, is a sufficient but not a necessary condition for the fulfillment of the Jacobi identity.

Faddeev and his school studied the quantization of classical integrable equations [4]. It became apparent that upon quantization, some of the basic structures undergo "quantum deformations". In particular the commutative Poisson-Hopf algebras become noncommutative Hopf algebras [5]. In this way a new mathematical entity, the quantum group was born [6], and beautiful connections with braid groups were established [7].

The study of integrable systems also revived interest in infinite dimensional Hamiltonian systems. In particular it became apparent that the Hamiltonian operators associated with integrable equations and the formalism of the calculus of variations can be used to generate infinite dimensional algebras [8]. These algebras include the Heisenberg-Virasoro algebra as well as the classical limit of the Fateev-Zamolodchikov algebra [9].

In §2 we define Poisson algebras and present several examples of interesting Poisson algebras. In particular we present a certain generalization of the classical Sklyanin algebra (see equation (2.19)). In §3 we study quadratic Poisson algebras. In particular we show that a necessary and sufficient condition for the quadratic relations $\{u_l, u_m\} = c_{lm}^{\alpha\beta} u_\alpha u_\beta$ (summation over repeated indices is assumed throughout this paper) to form a Poisson algebra is that the structure constants $c_{lm}^{\alpha\beta} \in \mathbb{C}$ satisfy $c_{lm}^{\alpha\beta} = -c_{ml}^{\beta\alpha}$ as well as a quadratic equation which is a symmetrization of the classical YB equation (see Theorem 3.1). This new equation can be thought of as a generalization of the classical modified YB equation (see equation (3.18)). In §4 we investigate the Poisson algebras arising in the calculus of variations. In particular we study infinite dimensional algebras constructed from linear and quadratic Hamiltonian operators, with the aid of linear functionals of the form $\int u(x) e^{i\lambda x} dx$. The algebras constructed from linear Hamiltonian operators are well known. The algebras constructed from quadratic Hamiltonian operators are the infinite dimensional analogues of the quadratic Poisson algebras studied in §3. We also show that it is possible to obtain such algebras starting from constant Hamiltonian operators (such as $\frac{d}{dx}$) provided one uses nonlinear functionals, instead of linear functionals. For example, using this approach it is possible to obtain the Heisenberg-Virasoro and the classical Fateev-Zamolodchikov algebras from the constant Hamiltonian operators $\frac{d}{dx}$ and $\text{diag}(1, 1) \frac{d}{dx}$ respectively. In the general case, for scalar equations, one has to study nonlinear functionals of the form $\int u^{n_0} (u^{(1)})^{n_1} \dots (u^{(j)})^{n_j} e^{i\lambda x} dx$, where $u^{(j)} = \frac{\partial^j u}{\partial x^j}$ and $n_0, n_1, \dots, n_j \in \mathbb{Z}$. We point out that the Gel'fand-Dikii [10] formalism reduces the problem to studying the algebras associated

with such functionals to the problem of studying symmetric polynomials.

2 Poisson Algebras

Definition 2.1 A **Poisson algebra** is a commutative, associative algebra over a real or a complex field with an additional operation called Poisson bracket. Let P and $\{, \}$ denote a Poisson algebra, and a Poisson bracket respectively. A Poisson bracket

$$\{, \} : P \times P \rightarrow P, \quad (2.1)$$

satisfies the following axioms:

(i) bilinearity

$$\{c_1 f_1 + c_2 f_2, f_3\} = c_1 \{f_1, f_3\} + c_2 \{f_2, f_3\}, \quad (2.2)$$

where $f_i \in P$, $i = 1, 2, 3$ and $c_i \in \mathbb{C}$, $i = 1, 2$;

(ii) skew-symmetry

$$\{f_1, f_2\} = -\{f_2, f_1\}; \quad (2.3)$$

(iii) Leibnitz identity

$$\{f_1 f_2, f_3\} = f_1 \{f_2, f_3\} + \{f_1, f_3\} f_2; \quad (2.4)$$

(iv) Jacobi identity

$$\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} = 0. \quad (2.5)$$

In this paper we shall use the following convention: If an expression involves cyclic permutation of indices we shall often write only one term. For example equation (2.5) will also be denoted as

$$\{\{f_1, f_2\}, f_3\} + cp(1, 2, 3) = 0. \quad (2.6)$$

Definition 2.2 A **free Poisson algebra** associated with $u_1, \dots, u_r$ is the Poisson algebra generated by $u_1, \dots, u_r$ without any additional relations among its elements.

For example associated with u_1, u_2 there exists the free Poisson algebra consisting of polynomials of the infinitely many generators, u_1, u_2 and their repeated Poisson brackets, i.e.

$$u_1, u_2, u_{12} \doteq \{u_1, u_2\}, \{u_1, u_{12}\}, \text{ etc.}$$

Factor Poisson algebras are obtained by imposing relations among the generators of a free Poisson algebra.

An example of a factor Poisson algebra is the Cartan Poisson algebra. This algebra is generated by two free Poisson algebras with elements $e_1, \dots, e_r$ and $f_1, \dots, f_r$, when the following relations are valid,

$$\{e_i, f_j\} = \delta_{ij} h_j, \quad \{h_i, h_j\} = 0, \quad \{h_i, e_j\} = a_{ij} e_j, \quad \{h_i, f_j\} = -a_{ij} f_j; \quad i, j = 1, \dots, r;$$

in these equations no summation is assumed over repeated indices, $a_{ij} \in \mathbb{C}$, and δ_{ij} denotes the Kronecker δ . Another example of a factor Poisson algebra is the Kac-Moody Poisson algebra. This is a factor algebra of the Cartan Poisson algebra when the following relations are valid,

$$a_{ij} \in \mathbb{N}; a_{ii} = 2; a_{ij} \leq 0, i \neq j; \text{ if } a_{ij} = 0 \text{ then } a_{ji} = 0; (adg)^{1-a_{ij}} = 0, i \neq j; i, j = 1, \dots, r;$$

g in these equations is either e or f and $(adg)b = \{g, b\}$.

An important factor Poisson algebra is obtained when the Poisson brackets of $u_1, \dots, u_r$ are functions of $u_1, \dots, u_r$ only.

Definition 2.3 $P_h(u_1, \dots, u_r)$ denotes the Poisson algebra obtained from the free Poisson algebra associated with $u_1, \dots, u_r$, when the following condition is imposed,

$$\{u_l, u_m\} - h_{lm}(u_1, \dots, u_r) = 0, \quad l, m = 1, \dots, r. \quad (2.7)$$

The function h_{lm} is a polynomial, a rational, an algebraic, or generally a smooth function of its arguments.

Proposition 2.1 A necessary and sufficient condition for $P_h(u_1, \dots, u_r)$ to be a Poisson algebra is that h_{ij} is skew symmetric and satisfies the Schouten condition, i.e.

$$h_{lm} = -h_{ml}, \quad \frac{\partial h_{lm}}{\partial u_s} h_{sn} + cp(l, m, n) = 0, \quad l, m, n = 1, \dots, r, \quad (2.8)$$

where in equation (2.8) as well as throughout this paper summation over repeated indices is assumed unless otherwise stated.

The proof of this proposition is a direct consequence of the properties of the Poisson bracket.

Remark 2.1 Poisson brackets arose in classical mechanics [11]: Let M be an r -dimensional manifold, called the phase space, with local coordinates $x_1, \dots, x_r$. Let P be the algebra of smooth functions $f(x_1, \dots, x_r)$. Let the Poisson bracket between two such functions be defined by

$$\{f_1, f_2\} = h_{lm}(x_1, \dots, x_r) \frac{\partial f_1}{\partial x_l} \frac{\partial f_2}{\partial x_m}, \quad l, m = 1, \dots, r,$$

where h_{lm} is a smooth function of its arguments. The above equation defines a Poisson bracket iff h_{lm} satisfies equations (2.8). If $\det(h_{lm}) \neq 0$ the inverse of the matrix (h_{lm}) , which is denoted by (h^{lm}) , exists. Then equations (2.8) are precisely the conditions that the manifold M is symplectic, i.e. the conditions that the 2-form $h^{lm} dx_l \wedge dx_m$ is non-degenerate and closed. In this case it is always possible at least locally, using a smooth change of variables, to find new coordinates $\tilde{x}_l$ such that the symplectic 2-form becomes $d\tilde{x}_l \wedge d\tilde{x}_m$ (Darboux's theorem).

An important role in the study of Poisson algebras is played by the so-called Casimir functions.

Definition 2.4 The smooth function $F(u_1, \dots, u_r)$ is a **Casimir function** of the Poisson algebra $P_h(u_1, \dots, u_r)$, iff

$$\{u_i, F\} = 0, \quad i = 1, \dots, r. \quad (2.9)$$

We now give several examples of Poisson algebras.

Example 2.1 (The linear homogeneous case.) The elements $u_1, \dots, u_r$, equipped with the bracket defined by

$$\{u_l, u_m\} = c_{lm}^s u_s, \quad l, m, s = 1, \dots, r, \quad c_{ij}^s \in \mathbb{C}, \quad (2.10)$$

generate a Poisson algebra iff

$$c_{lm}^\alpha = -c_{ml}^\alpha, \quad \text{and} \quad c_{lm}^s c_{sn}^\alpha + cp(l, m, n) = 0, \quad l, m, n, s, \alpha = 1, \dots, r. \quad (2.11)$$

The equations (2.11) are precisely the equations defining the structure constants of some Lie algebra. Thus, there exists an isomorphism between a Lie algebra and a linear homogeneous Poisson algebra.

Example 2.2 (The linear case.) The elements $u_1, \dots, u_r$ equipped with the bracket defined by

$$\{u_l, u_m\} = c_{lm}^s u_s + c_{lm}, \quad l, m, s = 1, \dots, r, \quad c_{lm}^s, \quad c_{lm} \in \mathbb{C}, \quad (2.12)$$

generate a Poisson algebra iff c_{lm}^s satisfy equations (2.11) and in addition c_{lm} satisfy

$$c_{lm} = -c_{ml}, \quad \text{and} \quad c_{lm}^s c_{sn}^i + cp(l, m, n) = 0, \quad i, j, k, s = 1, \dots, r. \quad (2.13)$$

If the constants c_{lm}^s are the structure constants of a Lie algebra L , then equations (2.13) imply that the constants c_{lm} form a two-dimensional cocycle on L (i.e. functions of the form $a^s u_s + b$ form a one-dimensional central extension of L by the cocycle c_{lm}).

Example 2.3 The elements u_1, u_2, u_3 equipped with the brackets

$$\{u_1, u_2\} = f \frac{\partial F}{\partial u_3}, \quad \{u_2, u_3\} = f \frac{\partial F}{\partial u_1}, \quad \{u_3, u_1\} = f \frac{\partial F}{\partial u_2}, \quad (2.14)$$

where f and F are arbitrary smooth functions of u_1, u_2, u_3 , form a Poisson algebra. Furthermore, F is a Casimir function for this algebra.

Indeed, let

$$\{u_1, u_2\} = P^{(3)}, \quad \{u_2, u_3\} = P^{(1)}, \quad \{u_3, u_1\} = P^{(2)}, \quad (2.15)$$

where $P^{(j)}$, $j = 1, 2, 3$ are smooth functionals of u_1, u_2, u_3 . Then the Schouten condition is satisfied iff

$$\frac{\partial P^{(1)}}{\partial u_2} P^{(2)} - \frac{\partial P^{(3)}}{\partial u_2} P^{(1)} + cp(1, 2, 3) = 0, \quad (2.16)$$

where no summation over repeated indices is assumed.

If $P^{(j)} = f \frac{\partial F}{\partial u^{(j)}}$, $j = 1, 2, 3$, then equations (2.16) and (2.9) are satisfied identically.

The Poisson algebra (2.14) contains several interesting particular cases. For example letting $F = u_1 u_3 - u_2^2$, it follows that the equations

$$\{u_1, u_2\} = f u_1, \quad \{u_2, u_3\} = f u_3, \quad \{u_3, u_1\} = -2f u_2, \quad (2.17)$$

define a Poisson algebra for any smooth function f . In the particular case of $f = u_2$, this algebra becomes the semiclassical limit of the $SL(2)$ quantum group [22].

The Poisson algebra (2.14) can also be used to obtain a Poisson algebra with four generators. Indeed, equations (2.14) imply that the equations

$$\{u_1, u_2\} = f \frac{\partial F}{\partial u_3}, \quad \{u_1, f\} = \frac{1}{2} \left[\frac{\partial(f)^2}{\partial u_2} \frac{\partial F}{\partial u_3} - \frac{\partial(f)^2}{\partial u_3} \frac{\partial F}{\partial u_2} \right], \quad (2.18)$$

supplemented by similar expressions obtained by cyclic permutation, define a Poisson algebra for the four generators u_1, u_2, u_3, f . This algebra also contains several interesting cases. For example, letting

$$F = -(u_1^n + u_2^n + u_3^n), \quad f^2 = \alpha_1 u_1^2 + \alpha_2 u_2^2 + \alpha_3 u_3^2, \quad n \in \mathbb{Z}, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C},$$

it follows that the equations

$$\{u_1, u_2\} = -n f u_3^{n-1}, \quad \{u_1, f\} = n u_2 u_3 (\alpha_3 u_2^{n-2} - \alpha_2 u_3^{n-2}), \quad n \in \mathbb{Z}, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}, \quad (2.19)$$

supplemented by similar expression obtained by cyclic permutation, define a Poisson algebra. In the particular case of $n = 2$, this algebra becomes the celebrated classical Sklyanin algebra,

$$\{u_1, u_2\} = -2f u_3, \quad \{u_1, f\} = 2(\alpha_3 - \alpha_2) u_2 u_3. \quad (2.20)$$

It is interesting to note that since f is a function of u_1, u_2, u_3 , and F is a Casimir, it is possible to generate the Poisson algebra (2.19) using only two independent generators u_1 and u_2 . In particular the Sklyanin algebra can be written as

$$\{u_1, u_2\} = -2u_3 f; \quad u_1^2 + u_2^2 + u_3^2 = 1, \quad f^2 = (\alpha_1 - \alpha_3) u_1^2 + (\alpha_2 - \alpha_3) u_2^2.$$

Remark 2.2 Equation (2.16) can be written as

$$w \wedge dw = 0, \quad w \doteq P^{(j)} du_j, \quad (2.21)$$

which is the integrability condition of the Pfaffian system $w = 0$. Hence there exists a 1-form η such that

$$dw = \eta \wedge w. \quad (2.22)$$

The pair (w, η) of these differential 1-forms defines a foliation of co-dimension 1. The Godbillon-Vey form [19] of this foliation is the 3-form $\Gamma^0(\eta) = \eta \wedge d\eta$, which has the properties that it is closed, and that its cohomology class (for a given foliation) does not depend on the associated pair of forms. Actually

$$\Gamma^0(\eta') - \Gamma^0(\eta) = d\Gamma^1(w, \eta; w', \eta'); \Gamma^1(w, \eta; w', \eta') = \eta' \wedge \eta - d \ln \left| \frac{w'}{w} \right| \wedge (\eta' + \eta).$$

The local solution of equation (2.21) is $P = f dF$, where f and F are 0-forms (F is a conserved quantity and $\frac{1}{f}$ is the integrating factor of the Pfaffian system). Therefore, any Poisson algebra with three generators is given locally by equations (2.14). If η is exact, then the solution (2.14) also holds globally. However, if $d\eta \neq 0$, then the non-triviality of the Godbillon-Vey form prevents the integration of these local results. In this case, applying the operator d to equation (2.22), it follows that $d\eta \wedge w = 0$, i.e.

$$d\eta = w \wedge \Omega, \quad (2.23)$$

where Ω is some 1-form, independent of w . Applying the operator d to the above equation, it follows that

$$d\Omega = -\eta \wedge \Omega, \quad (2.24)$$

which implies $\Omega \wedge d\Omega = 0$, i.e. Ω can also be used to define a Poisson algebra. Equations (2.22)-(2.24) are the Maurer-Cartan equations for $SL(2, R)$. Hence the matrix of 1-forms

$$\Gamma = \begin{pmatrix} \frac{\eta}{2} & \Omega \\ w & -\frac{\eta}{2} \end{pmatrix},$$

forms an $SL(2, R)$ -valued connection with vanishing curvature [20], $d\Gamma + \Gamma \wedge \Gamma = 0$. Examples corresponding to nontrivial Godbillon-Vey class are discussed in [20]. A classification of all quadratic Poisson algebras with three generators is given in [31].

In the case of $d\eta = 0$ it is possible by a change of variables to transform the Poisson algebra to one with $dw = 0$. However, this equivalence holds only locally since the Jacobian of transformation will, in general, contain singularities [21].

Remark 2.3 Casimirs can be used to obtain complicated Poisson algebras from simple ones. Consider for example the Poisson algebra (2.10) associated with $sl(3)$. This linear algebra has eight generators. However, using the two Casimirs of $sl(3)$ one can define a Poisson algebra with six independent generators. The Poisson brackets of this algebra are algebraic functions of $u_1, \dots, u_6$ [18].

3 Quadratic Poisson Algebras

Definition 3.1 The Poisson algebra $P_h(u_1, \dots, u_r)$ is called homogeneous quadratic iff h_{lm} is a homogeneous quadratic polynomial, i.e.

$$\{u_l, u_m\} = c_{lm}^{\alpha\beta} u_\alpha u_\beta, \quad l, m, \alpha, \beta = 1, \dots, r, \quad c_{lm}^{\alpha\beta} \in \mathbb{C}. \quad (3.1)$$

Theorem 3.1 (i) Equation (3.1) defines a homogeneous quadratic Poisson algebra iff the structure constants $c_{lm}^{\alpha\beta} \in \mathbb{C}$ are skew symmetric,

$$c_{lm}^{\alpha\beta} = -c_{ml}^{\beta\alpha}, \quad l, m, \alpha, \beta = 1, \dots, r, \quad (3.2)$$

and satisfy the symmetrized classical skew symmetric YB equation

$$S^{\alpha\beta\gamma} Y_{lmn}^{\alpha\beta\gamma}(c) \doteq Y_{lmn}^{\alpha\beta\gamma}(c) + Y_{lmn}^{\alpha\gamma\beta}(c) + cp(\alpha, \beta, \gamma) = 0, \quad (3.3)$$

where all the indices run from 1 to r and

$$Y_{lmn}^{\alpha\beta\gamma}(c) \doteq c_{sm}^{\alpha\beta} c_{ln}^{\gamma s} + c_{ls}^{\alpha\beta} c_{mn}^{\gamma s} + cp(\alpha, \beta, \gamma; l, m, n). \quad (3.4)$$

In equation (3.4) the cyclic permutation of the sets α, β, γ and l, m, n is performed simultaneously.

(ii) An invariant form of equations (3.1) and (3.3) is given by

$$\{u \otimes u\} = c(u \otimes u), \quad (3.5)$$

and

$$S([c_{12}, c_{13}] + [c_{12}, c_{23}] + [c_{13}, c_{23}]) = 0, \quad (3.6)$$

where

$$c_{12} = c \otimes I, c_{23} = I \otimes c, c_{13} = (I \otimes \pi)(c \otimes I)(I \otimes \pi), \quad (3.7)$$

I is the unit $r \times r$ matrix ($I_l^\alpha = \delta_l^\alpha$), π is the permutation matrix ($\pi_{lm}^{\alpha\beta} = \delta_m^\alpha \delta_l^\beta$), $\otimes$ denotes the usual tensor product, $(c)_{lm}^{\alpha\beta} = c_{lm}^{\alpha\beta}$, and S denotes symmetrization with respect to upper indices.

Proof

(i) The Schouten condition (2.8) implies $F_{lmn}^{\alpha\beta\gamma}(c)u_\alpha u_\beta u_\gamma = 0$, where $F_{lmn}^{\alpha\beta\gamma}$ is defined by

$$F_{lmn}^{\alpha\beta\gamma}(c) = (c_{lm}^{s\alpha} + c_{lm}^{\alpha s}) c_{sn}^{\beta\gamma} + (c_{mn}^{s\alpha} + c_{mn}^{\alpha s}) c_{sl}^{\beta\gamma} + (c_{nl}^{s\alpha} + c_{nl}^{\alpha s}) c_{sm}^{\beta\gamma}, \quad (3.8)$$

and all indices run from 1 to r . Hence,

$$F_{lmn}^{\alpha\beta\gamma}(c) + F_{lmn}^{\alpha\gamma\beta}(c) + cp(\alpha, \beta, \gamma) = 0. \quad (3.9)$$

This equation contains 36 terms. These terms can be combined in such a way that they equal to the 36 terms obtained by the symmetrization of $Y_{lmn}^{\alpha\beta\gamma}(c)$.

Let $c_{lm}^{\alpha\beta}$ be a solution of equation (3.3) which also satisfies the symmetry $c_{lm}^{\alpha\beta} = -c_{ml}^{\beta\alpha}$. Use this $c_{lm}^{\alpha\beta}$ to define the bracket 3.1. Then, this bracket is well defined since it is skew symmetric and satisfies the Jacobi identity.

Conversely, assume that the bracket defined in equation (3.1) is a Poisson bracket. Then the symmetry of the Poisson bracket implies

$$(c_{lm}^{\alpha\beta} + c_{ml}^{\beta\alpha}) u_\alpha u_\beta = 0, \quad \text{or} \quad (c_{lm}^{\alpha\beta} + c_{ml}^{\beta\alpha}) + (c_{ml}^{\alpha\beta} + c_{lm}^{\beta\alpha}) = 0.$$

Thus

$$c_{lm}^{\alpha\beta} = -c_{ml}^{\beta\alpha} + f_{lm}^{\alpha\beta}, \quad (3.10)$$

where f satisfies $f_{lm}^{\alpha\beta} + f_{ml}^{\beta\alpha} = 0$. Without loss of generality we can consider only those c 's which satisfy $c_{lm}^{\alpha\beta} = -c_{ml}^{\beta\alpha}$, since these c 's and those satisfying (3.10) generate the same Poisson algebra. Indeed,

$$f_{lm}^{\alpha\beta} u_\alpha u_\beta = \frac{1}{2} (f_{lm}^{\alpha\beta} + f_{ml}^{\beta\alpha}) u_\alpha u_\beta = 0.$$

(ii) Let

$$Y(c) = [c_{12}, c_{13}] + [c_{12}, c_{23}] + [c_{13}, c_{23}]. \quad (3.11)$$

Then

$$(c_{12})_{lmn}^{\alpha\beta\gamma} = c_{lm}^{\alpha\beta} \delta_n^\gamma, (c_{13})_{lmn}^{\alpha\beta\gamma} = c_{ln}^{\alpha\gamma} \delta_m^\beta, (c_{23})_{lmn}^{\alpha\beta\gamma} = c_{mn}^{\beta\gamma} \delta_l^\alpha.$$

Substituting these expressions into equation (3.11) and assuming skew symmetry, it follows that the $_{lmn}^{\alpha\beta\gamma}$ component of $Y(c)$ is given in equation (3.4). QED.

Remark 3.1 Let $Y_{lmn}^{\alpha\beta\gamma}(c)$ be defined by equation (3.4). Then equations (3.2) and (3.3) imply that $Y_{lmn}^{\alpha\beta\gamma}(c)$ satisfies

$$(i) \quad Y_{lmn}^{\alpha\beta\gamma}(c) = Y_{mnl}^{\beta\gamma\alpha}(c) = Y_{nlm}^{\gamma\alpha\beta}(c), \quad (3.12a)$$

$$(ii) \quad Y_{lmn}^{\alpha\beta\gamma}(c) = -Y_{mln}^{\beta\alpha\gamma}(c), \quad (3.12b)$$

$$(iii) \quad Y_{lmn}^{\alpha\beta\gamma}(c) + Y_{lmn}^{\alpha\gamma\beta}(c) + cp(\alpha, \beta, \gamma) = 0. \quad (3.12c)$$

A solution of these equations is given by

$$Y_{lmn}^{\alpha\beta\gamma}(c) = \sum_j \left\{ (A_j)_m^\alpha (A_j)_n^\beta (A_j)_l^\gamma - (A_j)_n^\alpha (A_j)_m^\gamma (A_j)_l^\beta \right\}, \quad (3.13)$$

where the A_j 's are arbitrary $r \times r$ matrices. The classification of the solutions of the above equations will be discussed elsewhere.

Remark 3.2 The following invariant formulation of the quadratic Poisson algebra was suggested to the authors by M.A. Semenov-Tian-Shanskii. Equation (3.1) can be written as

$$\{\varphi, \psi\}(u) = \text{tr}(D\varphi c(D\psi)), \quad u \in V, \quad (3.14)$$

where φ and ψ are smooth functions of $u_1, \dots, u_r$, and $D\varphi$ is the matrix differential, i.e.

$$D\varphi(u) = u \otimes d\varphi, \quad u \in V \otimes V^*. \quad (3.15)$$

Equation (3.14) follows from

$$\{\varphi, \psi\} = \frac{\partial \varphi}{\partial u_k} \frac{\partial \psi}{\partial u_l} \{u_k, u_l\} = c_{kl}^{\alpha\beta} u_\alpha u_\beta \frac{\partial \varphi}{\partial u_k} \frac{\partial \psi}{\partial u_l},$$

using $(D\varphi)_\beta^\alpha = u_\beta \frac{\partial \varphi}{\partial u_\alpha}$ and $(c(u))_\beta^\alpha = c_{\beta\beta'}^{\alpha\alpha'} u_{\alpha'}^{\beta'}$. The Jacobi identity can be written in the form

$$\text{tr}(B(D\varphi_1(u), D\varphi_2(u))D\varphi_3(u)) = 0, \quad (3.16)$$

where $B(x, y)$ is defined by

$$B(x, y) = [c(x), c(y)] - c[c(x), y] + c[c(y), x]. \quad (3.17)$$

This is a consequence of the fact that $(B(x, y))_l^\alpha = Y_{lmn}^{\alpha\beta\gamma} x_\beta^m y_\gamma^n$.

In the above notation the classical YB equation is given by $B(x, y) = 0$, while the solution (3.13) can be written as

$$[c(x), c(y)] - c[c(x), y] + c[c(y), x] = \sum_j (A_j x A_j y A_j - A_j y A_j x A_j). \quad (3.18)$$

Equation (3.18) is a generalization of the modified YB equation, which corresponds to the case that A_j is a scalar.

Definition 3.2 The Poisson algebra $P_h(u_1, \dots, u_r)$ is called quadratic iff h_{lm} is a quadratic polynomial, i.e.

$$\{u_l, u_m\} = c_{lm}^{\alpha\beta} u_\alpha u_\beta + \tau_{lm}^\gamma u_\gamma, \quad c_{lm}^{\alpha\beta}, \tau_{lm}^\gamma \in \mathbb{C}, \quad (3.19)$$

where $l, m, \alpha, \beta, \gamma = 1, \dots, r$.

Theorem 3.2 Equation (3.19) defines a quadratic Poisson algebra iff τ_{lm}^γ are the structure constants of a Lie algebra, $c_{lm}^{\alpha\beta}$ are the structure constants of a homogeneous quadratic Poisson algebra, and

$$(c_{sn}^{\alpha\beta} + c_{sn}^{\beta\alpha}) \tau_{lm}^s + (c_{lm}^{s\beta} + c_{lm}^{\beta s}) \tau_{sn}^\alpha + (c_{lm}^{s\alpha} + c_{lm}^{\alpha s}) \tau_{sn}^\beta = 0. \quad (3.20)$$

Remark 3.3 Given τ_{lm}^γ , equation (3.20) is a linear equation for $c_{lm}^{\alpha\beta}$. The classification of those solutions $c_{lm}^{\alpha\beta}$ which also satisfy (3.3) will be discussed elsewhere.

Remark 3.4 Homogeneous quadratic Poisson algebras were first introduced in the classical paper of Sklyanin [3]. However, he considered equations

$$c_{lm}^{\alpha\beta} = c_{lm}^{\beta\alpha}, \quad c_{lm}^{\alpha\beta} = -c_{ml}^{\alpha\beta}, \quad c_{lm}^{s\alpha} c_{sn}^{\beta\gamma} + c_{mn}^{s\alpha} c_{sl}^{\beta\gamma} + c_{nl}^{s\alpha} c_{sm}^{\beta\gamma} = 0, \quad (3.21)$$

instead of equations (3.2) and (3.3). Equation (3.21c) is a sufficient but not necessary conditions for the existence of a quadratic algebra. Actually it can be checked explicitly that the constants $c_{lm}^{\alpha\beta}$ corresponding to the classical Sklyanin algebra, satisfy equation (3.3) but do not satisfy equation (3.21c). The Sklyanin algebra corresponds to $c_{14}^{23} = \alpha_3 - \alpha_2$, $c_{24}^{31} = \alpha_1 - \alpha_3$, $c_{34}^{12} = \alpha_2 - \alpha_1$, $c_{12}^{43} = -1$, $c_{23}^{41} = -1$, $c_{31}^{42} = -1$ (see equation (2.20)).

We recall that Gel'fand and Dorfman [16] have pointed out the Hamiltonian nature of the classical YB equation.

Remark 3.5 There exist particular cases for which equation (3.3) reduces to the classical YB equation. We now discuss some of these cases. Let $r = N^2$. Then the u 's can be written as elements of an $N \times N$ matrix so they can be denoted by u_i^α ; equation (3.1) now becomes

$$\{u_i^\alpha, u_m^\beta\} = c_{lm\alpha_1\beta_1}^{\alpha\beta l_1 m_1} u_{l_1}^{\alpha_1} u_{m_1}^{\beta_1}, \quad (3.22)$$

where all the indices run from 1 to r , and the constants $c_{lm\alpha_1\beta_1}^{\alpha\beta l_1 m_1}$ satisfy an equation obtained from equation (3.3) by doubling the indices. It turns out that the particular choices of $c_{lm\alpha_1\beta_1}^{\alpha\beta l_1 m_1}$, given by $r_{\alpha_1\beta_1}^{\alpha\beta} \delta_l^{l_1} \delta_m^{m_1}$, and by $r_{lm}^{l_1 m_1} \delta_{\alpha_1}^\alpha \delta_{\beta_1}^\beta$, reduce equation (3.3) to the YB equation. Indeed, the brackets

$$\{u_l^\alpha, u_m^\beta\} = r_{\alpha_1\beta_1}^{\alpha\beta} u_l^{\alpha_1} u_m^{\beta_1} \quad \text{and} \quad \{u_l^\alpha, u_m^\beta\} = r_{lm}^{l_1 m_1} u_{l_1}^\alpha u_{m_1}^\beta, \quad (3.23)$$

define a Poisson algebra if $r_{lm}^{\alpha\beta}$ satisfies the skew symmetric classical YB equation. The r.h.s. of equations (3.23) are the component forms of $r(u \otimes u)$ and $(u \otimes u)r$ respectively. Hence these equations are usually written in the symbolic form

$$\{u \otimes u\} = r(u \otimes u) \quad \text{and} \quad \{u \otimes u\} = (u \otimes u)r.$$

The group theoretic nature of these brackets was elucidated by Drinfel'd [2] who showed that they are right- and left-invariant Poisson brackets of the linear group (see also [17]). Similarly, he showed that the bracket $\{u \otimes u\} = [r, u \otimes u]$ (which is also a particular case of equation (3.22)) defines on the linear group a Lie-Poisson group structure.

4 Poisson Algebras Arising from Variational Calculus

Let $\mathcal{R}$ denote the ring of smooth functions of x, u , and the x -derivatives of u . We construct the space of functionals, which we denote by Ω^0 , by introducing in $\mathcal{R}$ a relation of equivalence: $f_1 \sim f_2$ iff $f_1 - f_2$ is an "exact derivative", i.e. $f_1 - f_2 = \frac{dg}{dx}$ where $\frac{dg}{dx}$ is the total derivation of g w.r.t. x i.e. $\frac{dg}{dx} = g_x + g_u u^{(1)} + g_{u_x} u^{(2)} + \dots$, where $u^{(j)} = \frac{\partial^j u}{\partial x^j}$. An equivalence class is called a functional. The set of all functionals belongs to the coset $\mathcal{R}/\partial\mathcal{R}$, where $\partial\mathcal{R}$ consists of all elements of $\mathcal{R}$ which are exact derivatives. The mapping assigning to each $f \in \mathcal{R}$, its equivalence class $\tilde{f} \in \mathcal{R}/\partial\mathcal{R}$, is called a formal integral and is denoted by $\tilde{f} = \int f(x, u, u^{(1)}, u^{(2)}, \dots) dx$. It is easy to prove that $\int f_x g dx = - \int f g_x dx$, which we call integration by parts.

The Lagrangian in classical mechanics is a particular functional of such form which depends only on the first derivative. The idea of allowing f to depend on an arbitrary number of derivatives was introduced by Gel'fand and Dikii [24]. Functionals of the above form are called functionals of degree zero. A functional is of degree one if its variational derivative is of degree zero. The eigenvalues and the traces of differential operators are examples of functionals of degree one. Functionals of degree higher or equal than one are called nonlocal functionals. Although nonlocal functionals are important, for example, in localizing topological invariants for manifolds or for polygons (see for example [25]-[28]) here we shall only study the algebra of local functionals.

Let f be polynomial in $u, u^{(1)}, u^{(2)}, \dots$, i.e. $f = \sum a_{n_0, n_1, \dots, n_j}(x) u^{n_0} (u^{(1)})^{n_1} \dots (u^{(j)})^{n_j}$, where $n_0, n_1, \dots, n_j \in \mathbb{N}$. Then f can be decomposed into the basic expressions

$$E_{n_0, n_1, \dots, n_j}(\lambda) = u^{n_0} (u^{(1)})^{n_1} \dots (u^{(j)})^{n_j} e^{i\lambda x}, \quad \lambda \in \mathbb{R} \text{ or } \lambda \in \mathbb{N}, n_0, n_1, n_1, \dots, n_j \in \mathbb{N}. \quad (4.1)$$

For f we have the operations of addition, of differentiation

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \sum_{j=0}^N u^{(j+1)} \frac{\partial f}{\partial u^{(j)}}, \quad u^{(0)} = u, \quad (4.2)$$

and of the variational derivative,

$$\frac{\delta f}{\delta u} = \sum_{j=0}^N (-1)^j \left(\frac{d}{dx} \right)^j \frac{\partial f}{\partial u^{(j)}}. \quad (4.3)$$

A theory of infinite dimensional Hamiltonian systems is reviewed in [15]: Let Ω^1 and V denote the space of 1-forms and of vector fields respectively. A Hamiltonian operator, denoted by h is a map from Ω^1 to V such that the Poisson bracket

$$\{\tilde{f}_1, \tilde{f}_2\} = \langle h d\tilde{f}_1, d\tilde{f}_2 \rangle, \quad \tilde{f}_i \in \Omega^0, \quad i = 1, 2, \quad (4.4)$$

is well defined. This is the case iff h is skew symmetric and satisfies the Schouten condition. Using

$$\langle \xi, v \rangle = \int \xi v dx, \quad \xi \in \Omega^1, \quad v \in V, \quad \text{and} \quad d\tilde{f} = \frac{\delta f}{\delta u}, \quad (4.5)$$

equation (4.4) becomes

$$\{\tilde{f}_1, \tilde{f}_2\} = \int \frac{\delta f_1}{\delta u} h \frac{\delta f_2}{\delta u} dx. \quad (4.6)$$

In the case that u is an r -dimensional vector with components u_j , $1 \leq j \leq r$, h is an $r \times r$ matrix, and equation (4.6) becomes

$$\{\tilde{f}_1, \tilde{f}_2\} = \int \frac{\delta f_1}{\delta u_i} h_{ij} \frac{\delta f_2}{\delta u_j} dx. \quad (4.7)$$

4.1 Linear Poisson Algebras

Certain Hamiltonian operators can be used to generate infinite dimensional Lie algebras. This is always the case for Hamiltonians linear in the u 's.

Proposition 4.1.1 Let h_{ij} be a linear function of $u_1, \dots, u_r$. Define $L_l(\lambda)$ by

$$L_l(\lambda) = \int u_l e^{i\lambda x} dx, \quad l = 1, \dots, r, \quad \lambda \in \mathbb{R}. \quad (4.1.1a)$$

h_{ij} is a Hamiltonian operator, i.e. the bracket (4.7) is well defined, iff the bracket

$$\{L_l(\lambda), L_m(\mu)\} \doteq \int e^{i\lambda x} h_{lm} e^{i\mu x} dx, \quad l, m = 1, \dots, r, \quad \lambda, \mu \in \mathbb{R}, \quad (4.1.1b)$$

defines the structure of a Lie algebra.

A similar result is valid if $\lambda, \mu \in \mathbb{N}$ and the integral is replaced by a sum.

Proof Let $f_1 = L_l(\lambda)$, $f_2 = L_m(\mu)$. Thus

$$\frac{\delta f_1}{\delta u_j} = \delta_l^j e^{i\lambda x}, \quad \frac{\delta f_2}{\delta u_j} = \delta_m^j e^{i\mu x},$$

and equation (4.7) reduces to equation (4.1.1b). Since h_{lm} is linear in the u 's, using integration by parts it is always possible to express the r.h.s. of equation (4.1.1b) as a linear function of $L_\alpha(\lambda + \mu)$.

Example 4.1.1 The operator

$$h_{lm} = c_{lmjk}^\alpha (-\partial_x)^j u_\alpha \partial_x^k, \quad c_{lmjk}^\alpha \in \mathbb{C}, \quad (4.1.2)$$

where $l, m, \alpha = 1, \dots, r$; $j = 1, \dots, N_1$; $k = 1, \dots, N_2$; is a Hamiltonian operator, iff the bracket

$$\{L_l(\lambda), L_m(\mu)\} = (i)^{j+k} \lambda^j \mu^k c_{lmjk}^\alpha L_\alpha(\lambda + \mu), \quad \lambda, \mu \in \mathbb{R} \text{ or } \mathbb{N}, \quad (4.1.3)$$

where $l, m, \alpha = 1, \dots, r$; $j = 1, \dots, N_1$; $k = 1, \dots, N_2$; defines a Lie algebra. This is the case iff $c_{lmjk}^\alpha = -c_{mlkj}^\alpha$, and $\varphi_{lm}^\alpha(\lambda, \mu) \doteq (i)^{j+k} c_{lmjk}^\alpha \lambda^j \mu^k$ satisfies the equation

$$\varphi_{lm}^\alpha(\lambda, \mu) \varphi_{\alpha n}^\beta(\lambda + \mu, \nu) + c p(l, m, n; \lambda, \mu, \nu) = 0; \quad (4.1.4)$$

where $l, m, n, \alpha = 1, \dots, r$; $j = 1, \dots, N_1$; $k = 1, \dots, N_2$.

Example 4.1.2 The operator

$$h_{lm} = c_{lm}^\alpha u_\alpha + c_{lm} + g_{lm} \partial_x, \quad c_{lm}^\alpha, c_{lm}, g_{lm} \in \mathbb{C}; \quad l, m, \alpha = 1, \dots, r, \quad (4.1.5)$$

is a Hamiltonian operator, iff the bracket

$$\{L_l(\lambda), L_m(\mu)\} = c_{lm}^\alpha L_\alpha(\lambda + \mu) + c_{lm} \delta(\lambda + \mu) + i \mu g_{lm} \delta(\lambda + \mu); \quad \lambda, \mu \in \mathbb{R} \text{ or } \mathbb{N}, \quad (4.1.6)$$

where $l, m, \alpha = 1, \dots, r$; defines a Lie algebra. This is the case iff the constants c_{lm}^α and c_{lm} satisfy equations (2.11) and (2.13), and if $c_{lm}^\alpha g_{\alpha n} = -c_{ln}^\alpha g_{\alpha m}$. That is, c_{lm}^α are the structure constants of some finite dimensional Lie algebra with an invariant scalar product g_{lm} , and c_{lm} is a cocycle on this algebra.

Remark 4.1.1 The bracket (4.1.3) was first introduced in [16]. The bracket (4.1.6) is called a bracket of hydrodynamic type and was introduced in [30]. Equation (4.1.1b) with $h = \partial_x^3 + u \partial_x + \partial_x u + c$, $c \in \mathbb{C}$ gives rise to the Heisenberg-Virasoro algebra.

Remark 4.1.2 Particular examples of the above algebras include current algebras. For example, equation (4.1.6) with $c_{lm} = g_{lm} = 0$, and with the c_{lm}^α corresponding to $sl(2)$ becomes

$$[e_\lambda, f_\mu] = h_{\lambda+\mu}, \quad [h_\lambda, e_\mu] = 2e_{\lambda+\mu}, \quad [h_\lambda, f_\mu] = -2f_{\lambda+\mu},$$

$$[e_\lambda, e_\mu] = 0, \quad [f_\lambda, f_\mu] = 0, \quad [h_\lambda, h_\mu] = 0, \quad \lambda, \mu \in \mathbb{N}.$$

4.2 Quadratic Poisson Algebras

We first note that every finite quadratic Poisson algebra can be used to generate an infinite dimensional quadratic Poisson algebra:

Proposition 4.2.1

(i) The algebra

$$\{u_l, u_m\} = c_{lm}^{\alpha\beta} u_\alpha u_\beta; \quad l, m, \alpha, \beta = 1, \dots, r \quad (4.2.1)$$

is a quadratic Poisson algebra (i.e. the constants $c_{lm}^{\alpha\beta}$ satisfy equations (3.2) and (3.3)), iff the Poisson bracket

$$\{L_l(\lambda), L_m(\mu)\} = c_{lm}^{\alpha\beta} \int L_\alpha(\tau) L_\beta(\lambda + \mu - \tau) d\tau, \quad \lambda, \mu \in \mathbb{R}, \quad (4.2.2)$$

is well defined, i.e. it is skew symmetric and satisfies the Jacobi identity.

(ii) A similar correspondence exists between

$$\{u_l^\alpha, u_m^\beta\} = c_{l_1 m_1}^{\alpha\beta} u_l^{l_1} u_m^{m_1}, \quad l, m, \alpha, \beta, l_1, m_1 = 1, \dots, r, \quad (4.2.3)$$

and

$$\{L_l^\alpha(\lambda), L_m^\beta(\mu)\} = c_{l_1 m_1}^{\alpha\beta} \int L_{l_1}^{l_1}(\tau) L_{m_1}^{m_1}(\lambda + \mu - \tau) d\tau. \quad (4.2.4)$$

(iii) Similar results are valid if $\lambda, \mu \in \mathbb{N}$ and the integrals are replaced by sums.

Proof

(i) Let $L_l(\lambda) = \int u_l(x) e^{i\lambda x} dx$. Then equation (4.1.1b) yields

$$\{L_l(\lambda), L_m(\mu)\} = c_{lm}^{\alpha\beta} \int u_\alpha(x) u_\beta(x) e^{i(\lambda+\mu)x} dx = c_{lm}^{\alpha\beta} \int u_\alpha(x) e^{i\tau x} u_\beta(y) e^{i(\lambda+\mu-\tau)y} dx dy d\tau,$$

which is equation (4.2.2). The Jacobi identity for the bracket (4.2.2) implies

$$(c_{lm}^{\alpha\beta} c_{sn}^{\beta\gamma} + cp(l, m, n)) \int d\tau d\tau_1 L_\alpha(\tau) L_\beta(\lambda + \mu + \nu - \tau - \tau_1) L_\gamma(\tau_1) = 0,$$

which yields the equations $S^{\alpha\beta\gamma} Y_{lmn}^{\alpha\beta\gamma}(c) = 0$ (see equation (3.3)) for the constants $c_{l_1 m_1}^{\alpha\beta}$ (note that the terms under the integral are commutative and symmetric within a change of variables).

Similar considerations apply to the bracket (4.2.4).

Proposition 4.2.2 The operator

$$h_{lm}^{\alpha\beta} = c_{l_1 m_1 j k}^{\alpha\beta} (-\partial_x)^j u_l^{l_1} u_m^{m_1} \partial_x^k, \quad (4.2.5)$$

where $\alpha, \beta, l, m, l_1, m_1 = 1, \dots, r$; $j = 1, \dots, N_1$; $k = 1, \dots, N_2$, is a Hamiltonian operator, iff the Poisson bracket

$$\{L_l^\alpha(\lambda), L_m^\beta(\mu)\} = (i)^{j+k} c_{l_1 m_1 j k}^{\alpha\beta} \lambda^j \mu^k \int L_l^{l_1}(\tau) L_m^{m_1}(\lambda + \mu - \tau) d\tau,$$

is well defined, i.e. it is skew symmetric and satisfies the Jacobi identity. This is the case iff $c_{lmjk}^{\alpha\beta} = -c_{mlkj}^{\beta\alpha}$, and $\varphi_{lm}^{\alpha\beta}(\lambda, \mu) \doteq (i)^{j+k} c_{lmjk}^{\alpha\beta} \lambda^j \mu^k$ satisfies

$$\varphi_{ls}^{\alpha\beta}(\lambda, \mu) \varphi_{mn}^{\gamma\delta}(\lambda + \mu - \tau, \nu) + \varphi_{sm}^{\alpha\beta}(\lambda, \mu) \varphi_{ln}^{\gamma\delta}(\lambda + \mu - \tau_1, \nu) + cp(\alpha, \beta, \gamma; l, m, n; \lambda, \mu, \nu) = 0, \quad (4.2.6)$$

where $\alpha, \beta, l, m, n = 1, \dots, r; j = 1, \dots, N_1; k = 1, \dots, N_2$.

Proof Let $L_l^\alpha(\lambda) = \int u_l^\alpha(x) e^{i\lambda x}$. Then equation (4.1.1b) and integration by parts yield

$$\{L_l^\alpha(\lambda), L_m^\beta(\mu)\} = \varphi_{l_1 m_1}^{\alpha\beta}(\lambda, \mu) \int L_l^{l_1}(\tau) L_m^{m_1}(\lambda + \mu - \tau) d\tau.$$

Thus

$$\begin{aligned} \{\{L_l^\alpha(\lambda), L_m^\beta(\mu)\}, L_n^\gamma(\nu)\} &= \varphi_{l_1 m_1}^{\alpha\beta}(\lambda, \mu) \int \int L_l^{l_1}(\tau) \varphi_{m_2 n_2}^{m_1 \gamma}(\lambda + \mu - \tau, \nu) L_m^{m_2}(\tau_1) L_n^{n_2}(\lambda + \mu + \nu - \tau - \tau_1) d\tau d\tau_1 \\ &\quad + \varphi_{l_1 m_1}^{\alpha\beta}(\lambda, \mu) \int \int L_m^{m_1}(\lambda + \mu - \tau') \varphi_{l_2 n_2}^{l_1 \gamma}(\tau', \nu) L_l^{l_2}(\tau'_1) L_n^{n_2}(\tau' + \nu - \tau'_1) d\tau' d\tau'_1. \end{aligned}$$

Letting $\tau'_1 = \tau$, $\lambda + \mu - \tau' = \tau_1$, the r.h.s. of the above equation becomes

$$\begin{aligned} \int \int & \left[\varphi_{l_1 s}^{\alpha\beta}(\lambda, \mu) \varphi_{m_1 n_1}^{\gamma\delta}(\lambda + \mu - \tau, \nu) + \varphi_{sm_1}^{\alpha\beta}(\lambda, \mu) \varphi_{l_1 n_1}^{\gamma\delta}(\lambda + \mu - \tau_1, \nu) \right] \\ & \cdot L_l^{l_1}(\tau) L_m^{m_1}(\tau_1) L_n^{n_1}(\lambda + \mu + \nu - \tau - \tau_1) d\tau d\tau_1. \end{aligned}$$

Thus the Jacobi identity implies equation (4.2.6).

An example of a quadratic algebra arising from the calculus of variations is the semiclassical limit of the Fateev-Zamolodchikov algebra [9]. This algebra can be obtained from a certain quadratic Hamiltonian operator which is the second Hamiltonian operator associated with the Boussinesq equation (this operator is a particular case of a more general operator of the Gel'fand-Dikii type).

4.3 Nonlinear Functionals

It turns out that it is possible to generate the same algebra by using two different Hamiltonian operators. For example, it was mentioned earlier that the Heisenberg-Virasoro algebra can be generated by the Hamiltonian operator $\partial_x^3 + u\partial_x + \partial_x u + c\partial_x$, with the aid of a linear functional. On the other hand, the same algebra can be generated by the Hamiltonian operator ∂_x provided one uses a quadratic functional:

Example 4.3.1 Let $L(\lambda) = i \int [u^2 + (c_1 \lambda + c_2)u] e^{i\lambda x} dx$, $c_1, c_2 \in \mathbb{C}$. Then equation (4.6) with $h = \partial_x$ implies that the bracket

$$\{L(\lambda), L(\mu)\} = 2(\lambda - \mu)L(\lambda + \mu) - ic_1^2 \lambda^3 \delta(\lambda + \mu) + ic_2^2 \lambda \delta(\lambda + \mu), \quad \lambda, \mu \in \mathbb{R} \text{ or } \mathbb{N}, \quad (4.3.1)$$

defines a Lie algebra. This is actually a linear combination of the Heisenberg and Virasoro algebras. We note that the nonlinear functional used here can be obtained from a linear one via a Bäcklund transformation.

From the above example it follows that even elementary Hamiltonian operators like ∂_x , can give rise to interesting Lie algebras. Another example of such a construction is given below.

Example 4.3.2 Let $L_n(\lambda) = i \int u^n e^{i\lambda x} dx$, $n \neq 1$. Then equation (4.6) with $h = \partial_x$ implies that the bracket

$$\{L_n(\lambda), L_{\hat{n}}(\hat{\lambda})\} = \frac{n\hat{n}}{n + \hat{n} - 2} [\hat{\lambda}(n - 1) - \lambda(\hat{n} - 1)] L_{n+\hat{n}-2}(\lambda + \hat{\lambda}), \quad n, \hat{n} \neq 1, \quad \lambda, \hat{\lambda} \in \mathbb{R} \text{ or } \mathbb{N} \quad (4.3.2)$$

defines a Lie algebra.

To derive equation (4.3.2) we note that $\delta L_n(\lambda)/\delta u = i n u^{n-1} e^{i\lambda x}$, and the integrand of equation (4.6) becomes

$$-i n \hat{n} \hat{\lambda} u^{n+\hat{n}-2} e^{i(\lambda+\hat{\lambda})x} - \frac{n\hat{n}(\hat{n}-1)}{n + \hat{n} - 2} (u^{n+\hat{n}-2})_x e^{i(\lambda+\hat{\lambda})x}.$$

Using integration by parts this equation implies equation (4.3.2).

It is also possible, using nonlinear functionals, to generate quadratic algebras starting from a constant Hamiltonian operator. As an example we shall derive the classical limit of the Fateev-Zamolodchikov algebra using $h = i \text{diag}(1, 1) \partial_x$.

Example 4.3.3 (The classical Fateev-Zamolodchikov algebra). Let

$$L(\lambda) = \int [-2i\lambda u_1 + (u_1)^2 + (u_2)^2] e^{i\lambda x} dx, \quad (4.3.3)$$

and

$$W(\lambda) = \int [-\lambda^2 u_2 - i\lambda u_1 u_2 + 2u_1(u_2)_x + 2(u_1)^2 u_2 - 2(u_2)^3] e^{i\lambda x} dx. \quad (4.3.4)$$

Then equation (4.7) with $h_{11} = h_{22} = i\partial_x$, $h_{12} = h_{21} = 0$, implies that the brackets

$$\{L(\lambda), L(\mu)\} = 4\lambda^3 \delta(\lambda + \mu) + 2(\lambda - \mu)L(\lambda + \mu), \quad \lambda, \mu \in \mathbb{R}, \quad (4.3.5a)$$

$$\{L(\lambda), W(\mu)\} = 2(2\lambda - \mu)W(\lambda + \mu), \quad (4.3.5b)$$

$$\{W(\lambda), W(\mu)\} = \lambda^5 \delta(\lambda + \mu) + (\lambda - \mu)(\lambda^2 + \mu^2 - \frac{1}{2}\lambda\mu)L(\lambda + \mu) + 2(\lambda - \mu) \int L(\lambda - \tau)L(\mu + \tau) d\tau, \quad (4.3.5c)$$

define a Poisson algebra. A similar result is valid if $\lambda, \mu \in \mathbb{Z}$ and the integral is replaced by a sum.

The derivation of equations (4.3.5) is tedious but straightforward. The origin of the nonlocal term in (4.3.5c) is the term

$$2(\lambda - \hat{\lambda}) \int \left[(2u_1)_x + (u_1)^2 + (u_2)^2 \right]^2 e^{i(\lambda + \hat{\lambda})x} dx.$$

Remark 4.3.1 In the general case one must consider functionals of the form $E_{n_0, n_1, \dots, n_j}(\lambda)$ defined by equation (4.1). The computation of the Poisson bracket associated with two such arbitrary expressions is in general complicated. This computation can be carried out effectively using the general formalism introduced by Gel'fand-Dikii [10]. This formalism associates with $E_{n_0, n_1, \dots, n_j}(\lambda)$ the symmetric polynomial $\tilde{E}_{n_0, n_1, \dots, n_j}(\xi_1, \dots, \xi_N, \zeta^\lambda)$, where $N = n_0 + n_1 + \dots + n_j$. For example,

$$u^2 u^{(1)} e^{2ix} \leftrightarrow \frac{1}{3!} S_3(\xi_1^0 \xi_2^0 \xi_3^0) \zeta^2, u^3 (u^{(1)})^2 u^{(2)} e^{i\lambda x} \leftrightarrow \frac{1}{6!} S_6(\xi_1^0 \xi_2^0 \xi_3^0 \xi_4^1 \xi_5^1 \xi_6^2) \zeta^\lambda,$$

where S denotes symmetrization. The advantage of this map is that the action of $\frac{d}{dx}$ and of $\frac{\delta}{\delta u}$ on $\tilde{E}(\xi_1, \dots, \xi_N, \zeta^\lambda)$ is now simplified and is given by the following equations:

$$\left(\frac{d}{dx} \tilde{E} \right) (\xi_1, \xi_2, \dots, \xi_N, \zeta^\lambda) = (\xi_1 + \xi_2 + \dots + \xi_N + i\lambda) \tilde{E},$$

$$\left(\frac{\delta}{\delta u} \tilde{E} \right) (\xi_1, \dots, \xi_{N-1}, \zeta^\lambda) = N \tilde{E}(\xi_1, \dots, \xi_{N-1}, -(\xi_1 + \dots + \xi_{N-1} + i\lambda)).$$

The proof of these formulae for $\lambda = 0$ is given in [10]. The generalization of the proof for $\lambda \neq 0$ is straightforward. This association implies that the problem of classification of the infinite dimensional algebras generated by Poisson brackets is intimately related to the study of symmetric polynomials.

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