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Building Blocks for Topology Change

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Abstract

Topology changing spacetimes are investigated by breaking them into their constituent parts. It is shown that, for a large class of spacetimes, topology change between kink zero (eg. spacelike) surfaces must produce singularities regardless of the existence of closed timelike curves. A relationship between the kink number and causality is also described.

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1 Introduction

The idea that the topology of spacetime may change in quantum gravity can be traced back to Wheeler [1] who speculated that quantum fluctuations in the spacetime metric could produce wormholes. One difficulty in examining this proposition is that topology change is heavily restricted in the classical theory. Indeed, the well known theorem of Geroch [2] states

Theorem 1. (Geroch) *Let M be a compact Lorentzian manifold with everywhere spacelike boundary. If M admits a non-vanishing timelike vector field and there are no closed timelike curves in M , then M is topologically $\sigma \times I$ for some closed three-manifold σ .*

Thus topology change between spacelike surfaces implies that there are singularities in the Lorentz metric and/or closed timelike curves. One of the aims of this work is to determine the type of topology change that remains free of singularities.

The recent interest in kinking vector fields has produced some new results which further restrict classical topology change. Gibbons and Hawking [3, 4] derived the obstruction to spin-Lorentz cobordism, and hence showed how the existence of a spin structure could prevent certain types of topology change. It was also conjectured [4] that there might be a relationship between kink number and causality, however, Chamblin and Penrose [5] showed that these concepts were independent when they proved

Theorem 2. (Chamblin and Penrose) *Let M be a compact manifold with $\partial M \neq \emptyset$, and let g_L be a Lorentz metric on M . Suppose that M contains closed timelike curves, then the metric g_L may be continuously deformed into a new metric g'_L so that M has no closed timelike curves. Since the deformation of the metric is continuous it follows that $\text{kink}(\partial M, g_L) = \text{kink}(\partial M, g'_L)$.*

In this paper we adopt the new approach of analysing a topology change situation by breaking it into its constituent parts. We will call these parts *building blocks*. The strength of this approach is that it makes no assumptions except global Lorentz structure. We show that, for a large class of spacetimes, topology change between kink zero (eg. spacelike) surfaces must

produce singularities regardless of the existence of closed timelike curves. Furthermore, we show that in certain circumstances there is a relationship between kinking and causality.

2 Building Blocks

2.1 Morse theory

Morse Theory [6] describes the relationship between the topological properties of a manifold M and the extremal properties of a function $f: M \rightarrow \mathbb{R}$. A useful account of this theory is given in [7].

Definition. Let $f: M \rightarrow \mathbb{R}$ be a smooth function on a manifold M . A point $p \in M$ is called a *critical point* of f if the induced map $f_*: T_p(M) \rightarrow T_{f(p)}(\mathbb{R})$ is zero.

Definition. Let p be a critical point of a smooth function $f: M \rightarrow \mathbb{R}$, and let $\{x^i\}$ be a coordinate patch about p . Then p is called *non-degenerate* if and only if the matrix $\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)(p)$ is non-singular. Non-degeneracy is independent of the coordinate system $\{x^i\}$.

Definition Let $f: M \rightarrow \mathbb{R}$ be a smooth (non-constant) function on a manifold M , then f is called a *Morse function* if all the critical points of f are non-degenerate.

If M is a manifold with boundary $\partial M \cong \sigma_1 \cup \dots \cup \sigma_n$, then we also require that the critical points of a Morse function f do not lie in ∂M , and further, that f is single valued on each connected component of the boundary (ie. $f(\sigma_i) = a_i$ for some $a_i \in \mathbb{R}$). For compact smooth manifolds (with or without boundary) the existence of Morse functions is guaranteed.

As an example, consider the situation pictured in figure 1. The manifold M is topologically a torus with two disks removed. In this case, the Morse function $f: M \rightarrow \mathbb{R}$, gives (with respect to some arbitrary Riemannian metric)

the minimum distance $f(p)$ from the plane P to a point $p \in M$. We note that there is a critical point at each occurrence of topology change; this observation will be made precise later.

The proof of the following theorem of Morse (which is the basis for Geroch's work [2]) is given in [6, 7].

Theorem 3. (Morse) *Let $f: M \rightarrow \mathbb{R}$ be a smooth function, and define $M^a = \{p \in M | f(p) \leq a\}$. Suppose that $f^{-1}[a, b]$ is compact and contains no critical points of f , then $M^a \cong M^b$.*

2.2 Definition of a building block

Definition. Let Σ be a compact manifold with boundary $\partial\Sigma = \sigma_1 \cup \dots \cup \sigma_n$, where the σ_i denote the connected components of the boundary. Then we will call Σ a *building block* if there exists a Morse function $f: \Sigma \rightarrow \mathbb{R}$ with at most one critical point.

Definition. Let M be a compact manifold, then a *set of building blocks* for M is a collection of building blocks $\{\Sigma_i\}$ which can be joined across their boundaries to form M .

Lemma. *Let M be a compact manifold, then there exists a set of building blocks for M .*

Proof. Given any compact manifold M then a Morse function $f: M \rightarrow \mathbb{R}$ has a finite number of critical points. The manifold may be arbitrarily sliced up into sections which contain at most one critical point. Each of these sections is by definition a building block. ■

3 Building Blocks and Topology Change

Throughout this paper, we define $\text{kink}(\partial M, \mathbf{v})$ as in [4]. If $\mathbf{v}$ is a non-singular timelike vector field with respect to some Lorentz metric g_L (or if some such

v is implied) then we will often use the notation $\text{kink}(\partial M, g_L)$ instead of $\text{kink}(\partial M, v)$. The proofs of the theorems in this section use the relationship between the Euler characteristic and the kink number, which with our conventions is given by

$$\chi(M) = \text{kink}(\partial M, v) + \sum \text{ind}(v) . \quad (3.1)$$

3.1 Trivial and non-trivial building blocks

A counter-example was provided in [4] which showed that Geroch's theorem (theorem 1) could not be extended to the case when the boundary three-manifolds had zero kink (rather than just being spacelike). However, if we restrict our attention to spacetimes which are building blocks then this type of extension is possible.

Theorem 4. *Let Σ be a building block endowed with a non-singular Lorentz metric g_L such that $\text{kink}(\partial \Sigma, g_L) = 0$, then Σ is topologically $\sigma \times I$ for some closed three-manifold σ .*

Proof. Let $f: \Sigma \rightarrow \mathbb{R}$ be a smooth function on Σ , chosen so that if $\partial \Sigma = \sigma_1 \cup \dots \cup \sigma_n$ then $f(\sigma_1) = a$ and for $i = 2, \dots, n$ $f(\sigma_i) = b$ where $(a \neq b)$. Define a vector field $\mathbf{n}$ on M by $\mathbf{n} = \nabla f$. Clearly, $\text{kink}(\partial M, \mathbf{n}) = 0$ since $\mathbf{n}$ is normal to ∂M . However, since $\text{kink}(\partial \Sigma, g_L) = 0$ we know that $\chi(M) = 0$ and so $\sum \text{ind}(\mathbf{n}) = 0$. It follows that f has no critical points, and then the result follows by theorem 3. ■

It is simple to construct counter-examples to show that the reverse statement of theorem 1 does not hold (ie. $\exists$ spacetimes with closed timelike curves which do not change topology between spacelike surfaces). However, it is possible to prove the reverse statement of theorem 4.

Theorem 5. *Let Σ be a manifold which is topologically $\sigma \times I$ for some closed three-manifold σ and let g_L be a Lorentz metric on Σ . Then Σ is a building block and $\text{kink}(\partial \Sigma, g_L) = 0$.*

Proof. Clearly Σ is a building block. Furthermore, since $\chi(\sigma \times I) = \chi(\sigma)\chi(I)$ and σ is closed, we have that $\chi(\Sigma) = 0$, and hence $\text{kink}(\partial \Sigma, g_L) = 0$. ■

Thus we see that building blocks may be split into two classes.

Definition. Let Σ be a building block, then we will call Σ *trivial* if $\chi(\Sigma) = 0$, otherwise we will call Σ *non-trivial*.

Clearly, if M is a manifold that does not exhibit topology change then all sets of building blocks for M will consist solely of trivial building blocks. However, if M is a topology changing manifold, then there exists at least one set of building blocks for M consisting only of non-trivial building blocks. The latter follows since joining a trivial building block to a non-trivial building block gives a non-trivial building block.

Summarising the results so far, we have seen that in order to produce non-singular topology change between kink zero surfaces, we must use non-trivial building blocks. However, it is not possible to achieve such topology change using a single non-trivial building block as, by definition, a non-trivial building block can only have kink zero boundary if it has an interior singularity.

Thus, unless singularities are permitted, many simple topology change scenarios are ruled out. For example, a kink zero S^3 evolving directly into kink zero $S^3 \cup S^3$, or a kink zero S^3 evolving directly into kink zero $S^1 \times S^2$ must both be singular spacetimes. In fact, any isolated occurrence of topology change between kink zero surfaces will be singular.

3.2 Building topology change

Although kink zero topology change cannot be achieved with a single building block, we may produce kink zero topology change situations by stacking building blocks together. However, there is a further result which limits how such stacking takes place.

Theorem 6. *Let M be a compact manifold with boundary endowed with a non-singular Lorentz metric g_L . Suppose that $\text{kink}(\partial M, g_L) = k$, then any set $\{\Sigma_i\}$ of building blocks for M must satisfy*

$$\sum_i \chi(\Sigma_i) = k . \quad (3.2)$$

Proof. This is really just a standard property of the Euler characteristic, however for completeness a proof in terms of kinks is provided. Let $\mathbf{t}$ be a non-singular timelike vector field on M given by g_L . Break the spacetime M up into the individual building blocks then for each Σ_i we can write

$$\chi(\Sigma_i) = \text{kink}(\partial\Sigma_i, \mathbf{t}) . \quad (3.3)$$

Let σ be any connected component of $\partial\Sigma_i$ which is not a connected component of ∂M , and let $\mathbf{n}$ denote the outward pointing unit normal to σ , then σ is also a connected component of $\partial\Sigma_j$ (for some $j \neq i$) with outward pointing unit normal $-\mathbf{n}$. Hence any kinks will appear in pairs with opposite signs except for those on ∂M where we are given that they sum to k . Thus summing (3.3) over i gives the desired result. ■

Corollary. *Given a compact manifold M endowed with a non-singular Lorentz metric and which ‘changes topology’ between kink zero surfaces. Let $\{\Sigma_i\}$ be a set of building blocks for M , then at least one Σ_i must satisfy $\chi(\Sigma_i) > 0$ and at least one Σ_i must satisfy $\chi(\Sigma_i) < 0$.*

Proof. We have seen that the set $\{\Sigma_i\}$ consists of at least two elements. Furthermore not all the elements of $\{\Sigma_i\}$ can satisfy $\chi(\Sigma_i) = 0$ otherwise by theorem 4 there would be no topology change on M . Thus at least one $\Sigma_j \in \{\Sigma_i\}$ satisfies $\chi(\Sigma_j) \neq 0$ and hence by the theorem 6 there exists at least one $\Sigma_k \in \{\Sigma_i\}$ such that $\chi(\Sigma_k)$ has opposite sign to $\chi(\Sigma_j)$. ■

Consider, for example the formation of a wormhole on a universe, as shown in figure 2. If the initial and final surfaces are kink zero then by this corollary there must be singularities as this scenario may be broken into two identical building blocks. More generally, any topology change that just consists of ‘branching’ suffers from the same problem. Similar arguments suffice to show that a spacetime with identical initial and final surfaces but with internal topology change, must be made of at least three building blocks.

4 Discussion and Conclusion

We have seen that non-singular topology change between kink zero surfaces can be achieved with a minimum of two different non-trivial building blocks. Let us examine this result in more detail. Assume that we are able to build a topology changing spacetime with ‘initial’ surface σ_1 and ‘final’ surface σ_2 , ($\sigma_1 \not\cong \sigma_2$), from two different non-trivial building blocks, as shown in figure 3. The building blocks are joined across a third common surface σ_3 (σ_3 need not be connected). Since the building blocks are non-trivial, $\sigma_1 \not\cong \sigma_3 \not\cong \sigma_2$, and furthermore, $\text{kink}(\sigma_3, g_L) \neq 0$. Thus we see that kink zero topology change is possible only via extra interpolating topologies, and at least one of these interpolating topologies must have non-zero kink.

Now let us consider non-singular topology change between spacelike surfaces. By Geroch’s theorem (theorem 1) there must exist closed timelike curves, and we have also seen that there must exist an interpolating topology between the two spacelike surfaces with non-zero kink. Hence there is a relationship between kinking and causality, but it is not as strong as the original conjecture given in [4]. Let M be a non-singular spacetime with everywhere spacelike boundary then

Unfortunately, it is simple to construct counter-examples which show that the reverse statement is not true.

Although it is possible to produce non-singular topology change between kink zero surfaces we have seen that the resulting spacetimes must be quite complex, consisting of at least two different types of building block. These are not the situations which are usually considered when topology changing processes are discussed. Thus, if simple kink zero topology change is required, the correct path is to study singular spacetimes (see eg. [8]). Although a singular timelike vector field does not necessarily constitute a curvature singularity, it does mean that the deterministic nature of ordinary classical physics breaks

down.

In retrospect, it seems unnatural (given that non-singular topology change between kink zero surfaces must have internal kinking) to restrict to kink zero boundary when discussing topology change. In fact, with the appropriate choice of kink on the boundary, topology change is always non-singular, and furthermore, there are not necessarily any closed timelike curves.

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Figure 1. A manifold M (a torus with two disks removed) and a Morse function $f: M \rightarrow \mathbb{R}$. In this example, $f(p)$ is defined to be the minimum distance from a point $p \in M$ to the plane P .

Figure 2. A wormhole forming on the main body of the universe. If the initial and final surfaces are kink zero then this process is impossible without the introduction of singularities (since this process is the result of joining two identical building blocks).

Figure 3. The minimum requirement for kink zero topology change consists of two non-trivial building blocks ($\sigma_1 \not\cong \sigma_2$). There must be an interpolating topology σ_3 such that $\sigma_1 \not\cong \sigma_3 \not\cong \sigma_2$ and $\text{kink}(\sigma_3, g_L) \neq 0$.