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We study the distribution of shadows cast in a self-affine landscape. For incidental parallel light forming an angle θ with the basal plane of the landscape, we find that the percentage of the landscape which is illuminated scales as $\delta_M = (\tan \theta)^{-1/(1-H)}$, where H is the Hurst exponent. The distribution of shadow lengths measured along a cut in the landscape parallel to the incidental light has the form $N(\delta, \theta) = (1/\delta\delta_M)\phi(\delta/\delta_M)$, where δ is the length of a shadow, and ϕ is a scaling function whose behavior for small arguments is $\phi(x) \propto x^{-H}$, so that $N(\delta, \theta) \propto \delta^{-(1+H)}$ for small δ . We support and illustrate our results with numerical simulations. They form the basis for an efficient method to study self-affine surfaces, and should be of particular interest for example in the planetary sciences and fractography.

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Self-affine geometry has been shown to be ubiquitous in nature^[1–3]. Examples range from fracture surfaces^[4–6], surfaces created in various growth models^[7–9] proposed for describing deposition, erosion, corrosion, etching, sputtering, *etc.* Apart from specific physical modelling, the geometry of self-affine surfaces has also been used in order to generate realistic artificial landscapes.^[10]

The subject of this letter is the statistical distribution of shadows in such self-affine landscapes. If light comes in at an angle, mountains in a real landscape — or asperities in a fracture surface — will cast shadows with a large distribution of sizes when the incident angle is small. We address here the question for the geometry illustrated in Figures 1: what is the functional form of the histogram of shadow lengths δ measured along a straight line in the landscape, both with respect to δ and to the angle θ which the light forms with the landscape's basal plane? From the shadow patterns such as those shown in Figure 2, is it possible to extract some geometrical information about the statistics of the roughness?

As developed below, we find a scaling form

$$N(\delta, \theta) = \left(\frac{1}{\delta \delta_M} \right) \phi \left(\frac{\delta}{\delta_M} \right), \quad (1)$$

where N is the number of shadows of length δ *per unit length*, H is the Hurst (or roughness) exponent of the landscape, ϕ is a scaling function and δ_M is a cutoff (*i.e.* largest typical shadow length), behaving as

$$\delta_M \propto (\tan \theta)^{-1/(1-H)}. \quad (2)$$

The scaling function ϕ has the following behavior

$$\begin{cases} \phi(x) \propto x^{-H} & \text{for } x \ll 1, \\ \phi(x) \rightarrow 0 & \text{for } x \gg 1, \end{cases} \quad (3)$$

while $\phi(x)$ vanishes faster than any power-law for large arguments $x \gg 1$.

The surfaces we study are characterized by a single valued height $z = h(x, y)$. We focus on the particular case of self-affine surfaces which are isotropic along their mean plane (x, y) , *i.e.* they are statistically invariant under the scale transformation

$$\begin{cases} x \rightarrow \lambda x, \\ y \rightarrow \lambda y, \\ z \rightarrow \lambda^H z, \end{cases} \quad (4)$$

where λ is any scaling factor. The Hurst exponent H appearing in the latter transformation is generally in the range $0 < H < 1$. A mere consequence of the latter invariance is that the rms height difference between two points distant by d along the (x, y) plane satisfies

$$\langle (h(x, y) - h(x + d, y))^2 \rangle^{1/2} = Ad^H \quad (5)$$

where the constant A sets the roughness scale. In the following, we will set the coordinate axis such that the light direction is in the (x, z) plane. The shadow patterns in a plane $y = y_0$ are only determined by the profile $h(x, y_0)$, we will resort in the following to two-dimensional self-affine profiles and drop the y dependence of the physical quantities appearing in the equations.

Let us first analyse the scaling of the upper cut-off δ_M . Let us consider a point $x = x_0$ on the surface which is at the rightmost end of a shadow zone, with the convention that the light goes from left to right. The opposite end of the shadow is at $x = x_1$ and the extent of the shadow is $\delta = x_0 - x_1$. For all intermediate x , $x_1 \leq x \leq x_0$, we have the following property

$$h(x) \leq h(x_0) + (x_0 - x) \tan(\theta) \quad (6)$$

where the equality is only fulfilled for $x = x_0$ or $x = x_1$. From Eq.(5), it is clear that δ cannot be arbitrarily large, since the height difference between the two ends of the shadow has to be of order $\delta \tan(\theta)$ and this scales as δ^H . This provides an estimate of the upper cut-off δ_M of δ above which a shadow zone has a vanishing probability to occur. Using Eqs. (5) and (6), we can write the following scaling

$$A\delta_M^H = \delta_M \tan(\theta) , \quad (7)$$

or extracting δ_M , we obtain Eq.(2).

Since δ_M sets a natural scale for the shadow zones, we propose to write the distribution of shadow length δ as given by Eq.(1). It simply means that the cumulative distribution of shadows per unit profile length is of the form $(1/\delta_M)\Psi(\delta/\delta_M)$, *i.e.* it depends only on

the reduced variable δ/δ_M , up to a factor $(1/\delta_M)$ chosen so that it has the dimension of an inverse length. Therefore the density of probability $N(\delta, \theta) = (1/\delta_M^2)\Psi'(\delta/\delta_M)$ can be cast into the form (1), with $\phi(x) = x\Psi'(x)$.

The probability that the height difference Δh between two points exceeds the rms value given in Eq.(5) drops faster than any power-law for large Δh . More precisely, in some cases, such as for the analysis of fracture surfaces^[10,11] it has been proposed that the relative height difference — normalised to its rms value — is distributed as a Gaussian. The same property holds for artificially generated self-affine surfaces^[12] where a Gaussian distribution is not imposed *a priori* but appears at large distances. In these cases in particular, but also more generally for any self-affine surface, the scaling function ϕ is expected to decay exponentially fast for $\delta/\delta_M \gg 1$.

For the other limit of a small shadow zone, no intermediate length scale appears between the upper cut-off δ_M and the lower cut-off ϵ either coming from the microstructure of the medium or from the numerical discretization. Therefore, we assume that $\phi(x)$ can be described by a power-law for small x . We introduce an exponent α such that

$$\phi(x) \propto x^{-\alpha} \quad \text{for} \quad x \ll 1. \quad (8)$$

With these two limiting behaviors, we can compute the scaling of any moment of the distribution. Let β be the moment order we wish to compute.

$$\begin{aligned} \mathcal{M}_\beta = \langle \delta^\beta \rangle &= \left(\frac{1}{\delta_M} \right) \int_\epsilon^\infty \delta^{\beta-1} \phi\left(\frac{\delta}{\delta_M}\right) d\delta \\ &= \delta_M^{\beta-1} \int_{\epsilon/\delta_M}^\infty u^{\beta-1} \phi(u) du. \end{aligned} \quad (9)$$

For $\beta > \alpha$, the above integral is well behaved and thus it tends to a constant when ϵ/δ_M approaches 0. On the other hand, for $\beta < \alpha$, the integral Eq.(9) is governed by its lower limit, and thus $\mathcal{M}_\beta \propto \delta_M^{\alpha-1} \epsilon^{\beta-\alpha}$.

$$\begin{cases} \mathcal{M}_\beta \propto \delta_M^{\beta-1} & \text{for } \beta > \alpha, \\ \mathcal{M}_\beta \propto \delta_M^{\alpha-1} \epsilon^{\beta-\alpha} & \text{for } \beta < \alpha. \end{cases} \quad (10)$$

In order to determine the scaling behavior of ϕ , we only need to compare the scaling of a low order moment with an independent estimate. Such a condition is naturally provided by the scaling of the zeroth order moment. Indeed, $\mathcal{M}_0$ simply counts the number of disconnected shadows per unit length. Let us now consider the illuminated regions between shadows. Their size distribution is expected to decay exponentially fast — much more quickly than the shadow length — since they are formed by facets of the profile which should all have a slope larger than $-\theta$, which is a local property. Let ℓ be their mean length. When θ approaches 0, ℓ should have a very weak variation with θ (no singular behavior is to be expected). Therefore the mean exposed length L^* for a profile of length L should be simply the number of illuminated regions (equal to the number of shadow zones), $L\mathcal{M}_0$ times the mean length ℓ , $L^* = L\ell\mathcal{M}_0$. It is also possible to compute L^* differently. Since all light rays will hit the profile at some point, the exposed length will be $L^* = L \tan(\theta)$. Equating these two estimates of L^* gives $\mathcal{M}_0 \propto \tan(\theta)$. Using Eq.(2) we obtain the scaling of the zeroth order moment as

$$\mathcal{M}_0 \propto \delta_M^{H-1} \quad (11)$$

Identifying Eq.(10) — for $\beta = 0 < \alpha$ — and Eq.(11) gives the value of α ,

$$\alpha = H, \quad (12)$$

consistent with the previous hypothesis $\beta = 0 < \alpha = H$.

Let us also note that the result (12) is somewhat surprising. Indeed, after reinserting the low δ behavior of ϕ in Eq.(1), we obtain the limit power-law behavior of N as $N(\delta, \theta) \propto \delta^{-(1+H)}\delta_M^H$. It has been shown rigorously in a recent work^[13] that starting from any point on a self-affine profile, the distribution of distances d along x of the current point to the nearest point having the same height is distributed as a power-law $d^{-(2-H)}$. Although it might have been guessed that the two problems were similar to each other, the above obtained result proves that this is not the case — unless one considers a random walk

where $H = 1/2$ and thus $1 + H = 2 - H$. The difference between these two problems stems from the extra condition which is imposed on a shadow: Its two end points have to be illuminated.

We have tested the above predicted scaling by computing the statistical distribution of “shadows” over artificial self-affine profiles generated using an algorithm due to Voss^[12]. The length of the profiles was chosen to be $L = 2^{14} = 16384$. Three typical values of the roughness exponent were used: $H = 0.3, 0.5$ and 0.7 . The height of the profiles was normalized so that the average height difference $\langle |h(x) - h(x+1)| \rangle$ between consecutive points was set to one. The tangent of the angle of incidence of the light was chosen small compared to one. It varied in the range $10^{-2} < \tan(\theta) < 4 \times 10^{-1}$ depending on the value of the Hurst exponent. For each studied case (fixed H and θ) a total of 500 independent profiles were generated to extract the distributions.

Figures 3, 4, and 5 show three distributions of shadow length δ for the three chosen values of Hurst exponent. The distributions have been recorded using a logarithmic binning, so that $\delta N(\delta) = \phi$ is directly measured as a function of $\log(\delta)$. For each data set, we have divided the shadow length by the upper cut-off δ_M as given through Eq.(2). For each figure, four different incident angles have been used so as to check the scaling of the upper cut-off together with the scaling form Eq.(1). We do observe a good data collapse for $H = 0.3$ and $H = 0.5$. The largest $H = 0.7$ shown on Fig.5 gives a poorer fit. Different reasons can be invoked. First, a distinct bump can be seen for the smallest shadow lengths. The very same shape can be seen for all angles, appearing over less than a decade in δ . Such an effect can be obtained by adding to a power-law distributed variable an independent variable whose distribution is exponential with a mean value of a few units compared to the discretization size. Thus this observation suggests that small-scale details of the self-affine profile may affect the quality of the scaling without further consequences. Past this bump, a reasonable collapse can be seen, up to the largest shadow lengths. There again, the collapse appears to be poor. The deviation from the expected behavior is partly

due to the fact that for large H , the upper cut-off scales very quickly with θ . Thus large H profiles are very sensitive to finite size effects, and even profiles of length 16384 are small. The characteristic drop of the data points close to their maximum value does suggest that this is solely due to a finite size effect. Moreover as θ approaches its maximum considered value the upper cut-off is essentially due to the saturation due to δ_M more than by a finite size effect, because of the fact that the value of δ at which a decay of the distribution appears is clearly different from the system size. The predicted power-law behavior of ϕ for small argument is shown on the three figures 3-5 as a dotted line. We do observe a nice agreement between the fit and the numerical data for the three H values.

The analysis we have proposed in this article provides a new tool to analyse the self-affine nature of a profile. It is worth emphasizing here that the numerical analysis of the scaling invariance of profiles and surfaces is not a simple technical issue. A number of artefacts can appear which depend crucially both on the profile itself (length, height scale, small scale round-off, ...) and on the technique used. A systematic study of those effects has been reported in Ref.[14].

Let us note that if we wished to reconstruct the complete height surface $h(x, y)$, from shadow patterns, we would require a continuous series of incident angles. However, in order to analyse the Hurst exponent of the profile (and to check the validity of a self-affine character) a single snapshot is sufficient, with a scale limitation δ_M which only depends on the incident angle. That such a partial information is enough is not a surprise. A similar observation can be made from the analysis of cuts through a profile or surface. However, a clear advantage of the shadow analysis is that it can be performed experimentally easily, and that it is a non-destructive test, in contrast to cut analysis which require experimentally to cover the surface with another material to strengthen it and to polish the solid until a sufficient number of asperities appears (see *e.g.* Ref.[15] where this procedure is discussed in detail).

For larger scale analysis, such as landscape topography, the same shadow technique can easily be applied. Moreover, instead of considering the complete statistical distribution

of shadow length, one can also resort to an analysis of the total reflected light intensity over a length L (assuming an isotropic diffusion of light on the surface). One might measure the fraction of the line of length L which is illuminated. This fraction f is related to the first moment $\mathcal{M}_1$ by the relation

$$f = \frac{\delta_M}{L} \mathcal{M}_1 \propto \delta_M \propto (\tan \theta)^{-1/(1-H)} . \quad (13)$$

Again, this provides a simple way of measuring the Hurst exponent directly. Moreover, if a long wavelength bias is superimposed to the rough profile (*e.g.* a sphere with a rough self-affine surface), this bias may be analysed from the average reflected intensity in different regions. Let us also note that as can be seen from Fig. 2, the shadow patterns of a two dimensional surface reveal a pronounced anisotropy, which is not that of the surface itself (which is isotropic in the case of Fig. 2), but rather arises from the directionality of the light source. This point may benefit from a systematic analysis, but we have not addressed this question here.

Finally, apart from the geometrical analysis of surfaces, we can also mention other potential applications of the analysed properties in contact and friction problems where similar problems of shadowing appear. The present analysis provides the key to address these problems statistically.

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Figure Captions

Figure 1: Schematical description of the shadowing problem. The surface is supposed to be self-affine characterized by a roughness exponent H . An incident light forming an angle θ with the basal plane produces a series of shadows zones (shown in grey below the surface) alternating with exposed regions. The purpose of the present study is to analyse the statistics of these “shadows”.

Figure 2: An example of the spatial distribution of the shadow zones over a self-affine surface of size 512×512 , and roughness exponent $H = 0.8$.

Figure 3: Statistical distribution of the scaling function ϕ as a function of the normalized shadow length δ/δ_M in log-log scale. The value of the Hurst exponent is $H = 0.3$. Three values of θ are plotted together with the predicted power-law as a dotted line (Symbols are $\bullet$: $\theta = 0.0125$, $\triangle$: $\theta = 0.025$, $+$: $\theta = 0.05$ and $\diamond$: $\theta = 0.1$).

Figure 4: Same as Figure 3 with $H = 0.5$ (Symbols are $\bullet$: $\theta = 0.025$, $\triangle$: $\theta = 0.05$, $+$: $\theta = 0.1$ and $\diamond$: $\theta = 0.2$).

Figure 5: Same as Figure 3 with $H = 0.7$ (Symbols are $\bullet$: $\theta = 0.05$, $\triangle$: $\theta = 0.1$, $+$: $\theta = 0.2$ and $\diamond$: $\theta = 0.4$).

Figure 1

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“Shadows in self-affine landscapes”

Figure 2

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Figure 3

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Figure 4

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Figure 5

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