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for the Korteweg-deVries Equation

by

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June 1993

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An Initial-Boundary Value Problem for the Korteweg-deVries Equation

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1 Introduction

For integrable equations, a method exists for solving the initial-value problem on the infinite line for decaying initial data. For evolution equations in one spatial variable, this method reduces the solution of the Cauchy problem to the formulation of a certain classical mathematical problem called the Riemann-Hilbert (RH) problem. A RH problem can be solved via a linear integral equation. The asymptotic analysis of this integral equation shows that for large time and for $\frac{x}{t} = O(1)$, any initial data will decompose into a number of solitons. This type of result, which is mathematically and physically very interesting, cannot be obtained by the usual PDE techniques. A distinguished property of integrable equations is that they can be written as the compatibility condition of a pair of linear eigenvalue equations, called the Lax pair [1]. The associated RH problem is essentially determined by the x -part of the Lax pair; the t -part of the Lax pair plays only an auxiliary role. In the case of the Korteweg-deVries (KdV) equation, the relevant RH problem is formulated in the complex k -plane with a jump on $Im(k) = 0$.

Many physical problems are naturally formulated as boundary-value problems. Such a physical problem arises, for example, in a certain laboratory study of water waves [2]; this situation can be modeled by the Korteweg-deVries (KdV) equation on a semi-infinite line, with $u(x, 0) = 0$ and $u(0, t)$ given. This problem is well posed [3]; furthermore, numerical studies [4,5] indicate soliton generation, which is in agreement with the experimental observations. But an analytical confirmation of these numerical results remains open.

Recently a method has been developed [6], for solving initial-boundary value problems on the half-infinite line for decaying initial and boundary data. This method also reduces the solution of the initial-boundary value problem to the solution of a single RH problem. However, for the formulation of this RH problem, both the x - and the t -parts of the Lax pair play an important role. Actually, it is the t -part which determines where, in the complex k -plane, the jumps occur. We will show that for the KdV equation the jumps of the RH problem occur on $Im(k^3) = 0$, which is a reflection of the fact that the t -part of the Lax pair contains k^3 , which in turn is a consequence of the fact that the KdV involves a third derivative in x . As it was pointed out in [7], the analysis of initial-boundary value problems for integrable equations such as the KdV equation, in addition to techniques from exact integrability it also requires the essential use of general PDE techniques. In the case of initial value problems, the exact methods could be used to establish existence of global solutions as well as to study the properties of the solutions. In contrast, for initial-boundary value problems exact methods are used only to study the properties of solutions.

We consider the KdV equation in the form

$$u_t + u_{xxx} + 6uu_x = 0, \quad x, t \in [0, \infty), \quad (1.1)$$

where $u(x, 0) = q(x)$ and $u(0, t) = v(t)$ are given. We assume that

$$q(x), \quad v(t), \quad (1.2)$$

and a sufficient number of their derivatives decays for large x and large t ; the precise functional class will be investigated elsewhere. We also assume that

$$q(0) = v(0). \quad (1.3)$$

The KdV can be written as the compatibility condition of the following Lax pair for the scalar function $\mu(x, t, k)$,

$$\mu_{xx} + (u + k^2)\mu = 0, \quad (1.4)$$

$$\mu_t = u_x\mu + (4k^2 - 2u)\mu_x. \quad (1.5)$$

We have developed the following linearization scheme for the solution of the initial-boundary value problem of the KdV. Given $u(x, 0)$ construct $s_1^+(k) = \frac{1}{2}w(0, k) - \frac{1}{2ik}w_x(0, k)$ and $s_2^+(k) = \frac{1}{2}w(0, k) + \frac{1}{2ik}w_x(0, k)$ where $w(x, k)$ is the solution of equation (1.4) with $u(x, t)$ replaced by $u(x, 0)$ satisfying the boundary condition $\lim_{x \rightarrow \infty} w(x, k) \exp(-ikx) = 1$. Define $b(k)$ by $b = s_1^+/\bar{s}_2^+$. Let II denote the domain of the complex k -plane bounded by the rays $\arg k = \pi/3$ and $\arg k = 2\pi/3$. Let $c(k)$, $\arg k = \frac{\pi}{3}$ and $\arg k = \frac{2\pi}{3}$ be the boundary value of a function meromorphic for $k \in II$, with poles at the zeros of $s_2^+(k)$ and at the points $\{k_j\}_1^N$, $k_j \in i\mathbb{R}^+$, and such that $c(k) \rightarrow 0$ as $k \rightarrow \infty$. Having $s_1^+(k)$, $s_2^+(k)$ and $c(k)$ solve a RH problem for the sectionally meromorphic function $\tilde{Z}_p(x, t, k)$ with the jumps shown in Figure 1.1. This function can have poles only at $\{k_j\}_1^N$. Define $\hat{Z}_p(x, t, k)$ by

$$\hat{Z}_p(x, t, k) = I + \frac{iy(x, t)}{2k} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \tilde{Z}_p(x, t, k); \quad y(x, t) \doteq \lim_{k \rightarrow \infty} (k\tilde{Z}_p)_{12}. \quad (1.6)$$

Then determine $u(x, t)$ by $u(x, t) = 2i\partial_x \left[\lim_{k \rightarrow \infty} (k\hat{Z}_p(x, t, k))_{11} \right]$. The points $\{k_j\}_1^N$ have the meaning of the discrete spectrum of the matrix version (2.1.16) of equation (1.5) evaluated at $x = 0$ and

supplemented by the boundary condition $\nu_1 s_2^+(k) - \nu_2 s_1^+(k) = 0$ at $t = 0$. These zeros give rise to solitons which always move away from the boundary.

Figure 1.1

The RH problem associated with the initial-boundary value problem of the KdV. The x, t dependence enters only through $\theta(x, t) = 2i(kx + 4k^3t)$.

The domains formed by the rays imaginary $k^3 = 0$ will be denoted by $I, \dots, VI$.

In the particular case of $u(x, 0) = 0$, the RH problem reduces to the one depicted in Figure 1.2; it is specified by the boundary values of $c(k)$ for $\arg k = \frac{\pi}{3}$ and $\arg k = \frac{2\pi}{3}$, by $\{k_j\}_1^N$ and by $\{c_j\}_1^N$, where c_j denotes the residue of $c(k)$ at k_j .

Figure 1.2

The RH problem associated with the initial-boundary value problem of the KdV with $u(x, 0) = 0$. The x and t dependents enter only through $\theta(x, t) = 2i(kx + 4k^3t)$.

Unfortunately, although we have a complete characterization of the analytic properties of $c(k)$, we have not found an effective way of computing $c(k)$ in terms of $u(0, t)$. In spite of this we can give an effective description of the long time asymptotics. The study of the large t behavior of $u(x, t)$ reduces to the study of the large t behavior of the RH problem of Figure 1.1. Because the x, t dependence of this RH problem is rather simple, it is possible to give an effective asymptotic description of $u(x, t)$ as $t \rightarrow \infty$. It turns out that the asymptotics of $u(x, t)$ is given by the usual N -soliton solution of the KdV equation.

Discussion The linearization scheme developed in this paper can be summarized as follows: Given $u(x, 0)$, construct $b(k)$. Then, if $c(k)$ is **any** suitably decaying function meromorphic for $k \in \mathbb{C}$, the solution of the RH problem of Figure 1.1 generates the solution $u(x, t)$ corresponding to initial data $u(x, 0)$ and **some** boundary data $u(0, t)$. The main limitation of our result is that for given $u(x, 0)$ and $u(0, t)$ we cannot construct $c(k)$ by solving a linear problem. Nevertheless, we claim that for **any** given $u(x, 0)$ and $u(0, t)$ satisfying (1.2), the corresponding function $c(k)$ exists; in other words, the RH problem of Figure 1.1 solves the initial-boundary value problem (1.1) for **general** initial-boundary data.

The above RH problem is quite natural. To appreciate this we first recall the RH problem that corresponds to the KdV with $x \in (-\infty, \infty)$. This RH problem is depicted in Figure 1.3.

$$\begin{array}{c} + \\ \hline - \end{array} \left(\begin{array}{cc} 1 & -b(k)e^{-\theta} \\ \overline{b(k)}e^{\theta} & 1 - |b(k)|^2 \end{array} \right).$$

Figure 1.3

Comparing the RH problems of Figure 1.1 and Figure 1.3 we see that the jumps for $k \in \mathbb{R}$ are identical. The jumps for $\arg k = \pi/3$ and $\arg k = 2\pi/3$ can not have a nonzero entry in the 12 position since $e^{-\theta}$ is unbounded for $k \in \mathbb{C}^+$. The jumps for $k \in \mathbb{C}^-$ follow by symmetry considerations. The fact that $c(k)$ has analytic continuation for $k \in II$ can also be easily understood. At $t = 0$, the RH problem of Figure 1.1 must be reduced to the one that defines $u(x, 0)$. At $t = 0$, the term e^{θ} has analytic continuation in II . In this way, at $t = 0$ one finds the RH problem of Figure 1.3 with θ replaced by $2ikx$. This RH problem corresponds precisely to $u(x, 0)$.

The fact that $u(x, 0)$, $u(0, t)$ and $c(k)$ are related in a nonlinear way is a reflection of the fact that $u_x(0, t)$ and $u_{xx}(0, t)$ depend nonlinearly on $u(x, 0)$ and $u(0, t)$. To appreciate this we first recall the solution of the linearized problem

$$u_t + u_{xxx} = 0, \quad x, t \in [0, \infty), \quad (1.7)$$

where $u(0, t)$ and $u(x, 0)$ are given and decaying for large t and large x . In order to draw comparisons with the nonlinear problem we use a Fourier transform to solve equation (1.7),

$$\hat{u}(k, t) = \int_0^\infty dx e^{ikx} u(x, t). \quad (1.8)$$

The evolution of the Fourier data $\hat{u}(k, t)$ is given by

$$\hat{u}_t + ik^3 \hat{u} = u_{xx}(0, t) - ik u_x(0, t) - k^2 u(0, t). \quad (1.9)$$

In this equation $u(0, t)$ is known but $u_x(0, t)$ and $u_{xx}(0, t)$ are unknown. This apparently ominous situation can be bypassed by using the fact that the solution $\hat{u}(k, t)$ of (1.9) is analytic in the upper half of the k complex plane. It turns out that this requirement implies

$$\hat{u}(k, 0) = \int_0^\infty dt e^{ik^3 t} [-u_{xx}(0, t) + ik u_x(0, t) + k^2 u(0, t)]. \quad (1.10)$$

Letting $k = e^{i\pi/6} \sqrt{\rho}$, $\rho > 0$, and $k = e^{5i\pi/6} \sqrt{\rho}$, $\rho > 0$, equation (1.10) yields the equations

$$\hat{u}(e^{i\pi/6} \sqrt{\rho}, 0) = \int_0^\infty dt e^{-\rho t} [-u_{xx}(0, t) + i e^{i\pi/6} \sqrt{\rho} u_x(0, t) + e^{i\pi/3} \rho u(0, t)], \quad (1.11)$$

and

$$\hat{u}(-e^{-i\pi/6} \sqrt{\rho}, 0) = \int_0^\infty dt e^{-\rho t} [-u_{xx}(0, t) - i e^{-i\pi/6} \sqrt{\rho} u_x(0, t) + e^{-i\pi/3} \rho u(0, t)]. \quad (1.12)$$

If $u \in \mathbb{R}$, equation (1.12) is the complex conjugate of equation (1.11). Given $u(x, 0)$ and $u(0, t)$, equations (1.11) and (1.12) can be solved for $u_x(0, t)$ and $u_{xx}(0, t)$.

It is important to notice that if $u(0, t)$, $u_x(0, t)$, and $u_{xx}(0, t)$ are arbitrary decaying functions, then the rhs of equation (1.10) will be analytic for $k \in I \cup III \cup V$. However, in order for these functions to be the boundary values of the solution of equation (1.7), it is necessary and sufficient that the rhs of equation (1.10) has analytic continuation across the rays $\arg k = \pi/3$ and $\arg k = 2\pi/3$ ($\hat{u}(k, 0)$ is analytic for k in the upper half of the complex plane).

We now discuss the nonlinear problem. Let $(\hat{\psi}_1(t, k), \hat{\psi}_2(t, k))^T$ be the solution of the vector equation

$$\hat{\psi}_t(t, k) + (4ik^3 \sigma_3 - 2ku(0, t) \sigma_2 - u_x(0, t) \sigma_1 - \frac{q(0, t)}{k} (i\sigma_3 - \sigma_2)) \hat{\psi}(t, k) = 0, \quad (1.13)$$

where

$$q(0, t) \doteq \frac{1}{2} u_{xx}(0, t) + u^2(0, t),$$

satisfying the boundary condition $\lim_{t \rightarrow \infty} [(\hat{\psi}_1, \hat{\psi}_2) \exp(-4ik^3 t)] = (0, 1)^T$. Let $r(k) \doteq \hat{\psi}_1(0, k)/\hat{\psi}_2(0, k)$. It turns out that for arbitrary decaying functions $u(0, t)$, $u_x(0, t)$, and $u_{xx}(0, t)$, the function $r(k)$ is a meromorphic function for $k \in I \cup III \cup V$ and $r(k) \rightarrow 0$ as $k \rightarrow \infty$. However, if $u(0, t)$, $u_x(0, t)$ and $u_{xx}(0, t)$ are the boundary values of the KdV then in addition $r(k)$ satisfies

$$r(k) = \frac{s_1^+(k)}{s_2^+(k)}, \quad k \in I \cup III, \quad (1.14)$$

where s_1^+ and s_2^+ are determined from $u(x, 0)$ and are analytic for k in the upper half complex plane. Equation (1.14) is the analogue of equation (1.10). It shows that although the relationship between $u(x, 0)$, $u(0, t)$, $u_x(0, t)$, and $u_{xx}(0, t)$ is highly nonlinear, its reflection in the k -plane (scattering space) is rather simple: $r(k)$ has analytic continuation across the rays $\arg k = \pi/3$ and $\arg k = 2\pi/3$.

There exists an invertible correspondence between the “potential” $Q(t) \doteq \{u(0, t), u_x(0, t), u_{xx}(0, t)\}$ and the scattering data $r(k)$: Given $Q(t)$, equation (1.13) implies $r(k)$. Conversely given a meromorphic function $r(k)$, one can find $Q(t)$ by solving a RH problem with the jump $r(k)$ for $k^3 \in \mathbb{R}$. This provides an effective way for deriving triplets of functions $u(0, t)$, $u_x(0, t)$, and $u_{xx}(0, t)$ **compatible** with a given $u(x, 0)$. Indeed, given $u(x, 0)$ one first computes $r(k)$ for $k \in I \cup III$ from equation (1.14). Let $r(k)$, $k \in V$ be any suitably decaying meromorphic function. Then the solution of the above RH problem yields $u(0, t)$, $u_x(0, t)$, and $u_{xx}(0, t)$.

Unfortunately, given $u(0, t)$ and $u(x, 0)$ we cannot compute $r(k)$ for $k \in V$ by solving a linear problem. This is a consequence of the fact that now we have a “mixed” problem where one gives part of the potential, i.e. $u(0, t)$, and part of the scattering data, i.e. $r(k)$, for $k \in I \cup III$. It turns out that this problem can be formulated as a **nonlinear** RH problem and will be discussed elsewhere.

It was mentioned earlier that the analysis of equation (1.1) requires an essential use of general PDE techniques. This follows from the fact that in order to study the map between $Q(t)$ and $r(k)$ one needs apriori estimates for $u_x(0, t)$ and $u_{xx}(0, t)$. The uniqueness and existence of a global solution for the KdV on the quarter plane is established in [3]. This result is based on L_2 estimates. However, the methods of exact integrability are based on L_1 estimates. It is therefore crucial for us to extend the results of [3] from L_2 to L_1 . This poses significant technical difficulties which for the case of the nonlinear Schrödinger equation were discussed in [7]-[8]. The situation of the KdV is more complicated. This and other rigorous aspects of the present formalism will be discussed elsewhere.

Notation We shall introduce functions which are sectionally meromorphic in the complex k -plane cut along either $Im(k) = 0$ or $Im(k^3) = 0$, see Figure 1.4. Most of these functions have a pole at $k = 0$; the terminology punctuated complex k -plane will refer to the exclusion of $k = 0$. $a^+(k)$ denotes a function analytic for $k \in \mathbb{C}^+$ (upper half of the complex k -plane) with the possible exclusion of $k = 0$; similarly $a^-(k)$ denotes a function analytic for $k \in \mathbb{C}^-$ with the possible exclusion of $k = 0$. $\hat{a}^+(k)$ denotes a function analytic for $k \in I \cup III \cup V$ with the possible exclusion of $k = 0$; $\hat{a}^-(k)$ denotes a function analytic for $k \in II \cup IV \cup V$ with the possible exclusion of $k = 0$. The subscript p denotes meromorphicity instead of analyticity. If the matrix A is denoted by $A = (A^+, A^-)$, then $A^+ = (A_1^+, A_2^+)^T$ and $A^- = (A_1^-, A_2^-)^T$ denote the first and second columns of the matrix A . $I = diag(1, 1)$, $\sigma_3 = diag(1, -1)$. Overbar denotes complex conjugation.

Figure 1.4
Regions of analyticity of the complex k -plane

Summary of Results In §2 we present the relevant formalism. (a) In §2.1 we analyze the x -part of the Lax pair, i.e. equation (1.4). We define the scalar eigenfunctions $s(x, t, k)$, $v(x, t, k)$, and $w(x, t, k)$ as the solutions of equation (1.4) specified by the boundary conditions $s(0, t, k) = 0$ and $s_x(0, t, k) = -2ik$, $v(0, t, k) = 2$ and $v_x(0, t, k) = 0$, $\lim_{x \rightarrow \infty} e^{-ikx} w(x, t, k) = 1$, respectively. We define the 2×2 matrices $\phi(x, t, k)$ and $\psi(x, t, k)$ as the matrix eigenfunctions corresponding to the scalar eigenfunctions $v-s$ and w respectively (see equation (2.1.17)). We define the matrices $\Phi(x, t, k)$ and $\Psi(x, t, k)$ by the equations $\Phi(x, t, k) = \phi(x, t, k) \exp(ikx\sigma_3)$ and $\Psi(x, t, k) = \psi(x, t, k) \exp(ikx\sigma_3)$. These functions are analytic in the punctured complex k -plane cut along $\text{Im}(k) = 0$, they tend to the unit matrix as $k \rightarrow \infty$, and they have unit determinant. Actually $\Psi = (\Psi^+, \Psi^-)$, and the matrix $\Phi = (\Phi^+, \Phi^-)$ is an entire function with respect to k . The eigenfunctions Φ and Ψ satisfy certain symmetry conditions involving $k \rightarrow -k$, $k \rightarrow \bar{k}$, and complex conjugation (see equation (2.1.23)). The behavior of $\psi(x, t, k)$ and $\phi(x, t, k)$ at $k = 0$ is given by

$$\frac{ia(x, t)}{k} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \text{as } k \rightarrow 0, \quad (1.15)$$

where $a = \alpha(x, t)$, $\beta(x, t)$ for ψ , ϕ respectively. We define the 2×2 matrix χ_p by

$$\chi_p^-(x, t, k) = (\Psi^-(x, t, k), \Phi^-(x, t, k)/\overline{\psi_2^+(0, t, \bar{k})}), \quad \chi_p^+(x, t, k) = (\Phi^+(x, t, k)/\psi_2^+(0, t, k), \Psi^+(x, t, k)), \quad (1.16)$$

for $k \in \mathbb{C}^-$, and $k \in \mathbb{C}^+$ respectively. The matrix χ_p has unit determinant and satisfies a certain jump condition,

$$\chi_p^-(x, t, k) = \chi_p^+(x, t, k)F(x, t, k), \quad k \in \mathbb{R}, \quad (1.17)$$

where $F(x, t, k)$ is a 2×2 matrix involving $\psi_1^+(0, t, k)$ and $\psi_2^+(0, t, k)$.

(b) In §2.2 we use the t part of the Lax pair to analyze $\psi(0, t, k)$. This function satisfies a matrix version of equation (1.5) evaluated at $x = 0$ (see equation (2.2.1)). We define the 2×2 matrices $\hat{\Phi}(t, k)$ and $\hat{\Psi}(t, k)$ by the requirement that they satisfy the same equation as $\psi(0, t, k)$, and that $\hat{\Phi}(0, k) = I$, $\lim_{t \rightarrow \infty} \hat{\Psi}(t, k) = I$. These functions are analytic in the complex k -plane cut along $\text{Im}(k^3) = 0$, they tend to the unit matrix as $k \rightarrow \infty$, and they have unit determinant. Actually $\hat{\Psi} = (\hat{\Psi}^-, \hat{\Psi}^+)$, and the matrix $\hat{\Phi} = (\hat{\Phi}^+, \hat{\Phi}^-)$ is an entire function with respect to k . The eigenfunctions $\hat{\Phi}$ and $\hat{\Psi}$ satisfy the same symmetry condition as those satisfied by Φ and Ψ . We define the matrix $\hat{Y}_p(t, k)$ by

$$\left(\frac{\hat{\Phi}^+(t, k)}{s_2^+(k)}, \psi^+(0, t, k) \right), \left(\frac{\hat{\Psi}^-(t, k)}{\rho(k)}, \psi^+(0, t, k) \right), \left(\psi^-(0, t, k), \frac{\hat{\Psi}^+(t, k)}{\rho(k)} \right), \left(\psi^-(0, t, k), \frac{\hat{\Phi}^-(t, k)}{s_2^+(k)} \right), \quad (1.18)$$

for $k \in I \cup III$, $k \in II$, $k \in V$, $k \in IV \cup VI$ respectively, where the scalars $s_2^+(k)$ and $\rho(k)$ are defined by

$$s_2^+(k) \doteq \psi_2^+(0, 0, k), k \in \mathbb{C}^+; \rho(k) \doteq \hat{\Psi}_1^-(0, k)s_2^+(k) - \hat{\Psi}_2^-(0, k)s_1^+(k), k \in II; s_1^+(k) \doteq \psi_1^+(0, 0, k), k \in \mathbb{C}^+. \quad (1.19)$$

The matrix $\hat{Y}_p$ has unit determinant and satisfies the jump condition

$$\hat{Y}_p^-(t, k) = \hat{Y}_p^+(t, k)e^{-4ik^3t\sigma_3}G(k)e^{4ik^3t\sigma_3}, \quad k^3 \in \mathbb{R}, \quad (1.20)$$

where the 2×2 matrix $G(k)$ involves the scalars $b(k)$ and $c(k)$ which are defined by

$$b(k) \doteq \frac{s_1^+(k)}{s_2^+(k)}; \quad c(k) \doteq \frac{\hat{\Psi}_2^-(0, k)}{s_2^+(k)\rho(k)}, \quad \arg k = \frac{\pi}{3}, \quad \frac{2\pi}{3}. \quad (1.21)$$

(c) In §2.3 we formulate a RH problem whose solution gives the solution of the initial-boundary value problem of the KdV equation in the generic situation (see Remark 2.1 at the end of §2.2).

Let the 2×2 matrix $\hat{Z}_p(x, t, k)$ be defined by

$$\chi_p(x, t, k) \begin{pmatrix} 1 & 0 \\ \frac{(\hat{Y}_p)_{21}(t, k)}{(\hat{Y}_p)_{22}(t, k)} e^{2ikx} & 1 \end{pmatrix}, \quad k \in \mathbb{C}^+; \quad \chi_p(x, t, k) \begin{pmatrix} 1 & \frac{(\hat{Y}_p)_{12}(t, k)}{(\hat{Y}_p)_{11}(t, k)} e^{-2ikx} \\ 0 & 1 \end{pmatrix}, \quad k \in \mathbb{C}^-. \quad (1.22)$$

Then the meromorphic function $\hat{Z}_p(x, t, k)$, which has unit determinant, satisfies the RH problem

$$\hat{Z}_p^-(x, t, k) = \hat{Z}_p^+(x, t, k) e^{-i(kx+4k^3t)\sigma_3} G(k) e^{i(kx+4k^3t)\sigma_3}, \quad k^3 \in \mathbb{R}; \quad \hat{Z}_p \rightarrow I \text{ as } k \rightarrow \infty, \quad (1.23)$$

$$\lim_{k \rightarrow \infty} [k \hat{Z}_p]_{12} = 0, \quad k \rightarrow \infty; \quad \hat{Z}_p(x, t, k) \sim \frac{i\alpha(x, t)}{k} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad k \rightarrow 0, \quad k \in I \cup III. \quad (1.24)$$

The jumps of this RH problem are given in Figure 1.1. The possible poles of $\hat{Z}_p(x, t, k)$ can occur only at the zeros of $\rho(k)$ for $k \in II$, and at the complex conjugates of these zeros; the additional constraints on the residues of $\hat{Z}_p$ are presented in §2.4 (see equation (2.4.1)). It should be noted that although the function $\alpha(x, t)$ is unknown, the RH for $\hat{Z}_p$ has a unique solution, because of equation (1.24a). This solution can be found via equation (1.6).

In §3 we use the elegant approach of [14] to study the asymptotic behavior of $u(x, t)$ as $t \rightarrow \infty$.

2 The Formal Analysis

2.1 The x Problem

Let $s(x, t, k)$, $v(x, t, k)$, and $w(x, t, k)$ be the solutions of equation (1.4) satisfying the boundary conditions,

$$s(0, t, k) = 0; \quad v(0, t, k) = 2; \quad w(x, t, k) \sim e^{ikx}, \quad x \rightarrow \infty. \quad (2.1.1)$$

$$s_x(0, t, k) = -2ik; \quad v_x(0, t, k) = 0;$$

These eigenfunctions satisfy the following Volterra integral equations

$$s(x, t, k) = -2i \sin kx - \frac{1}{k} \int_0^x d\xi u(\xi, t) s(\xi, t, k) \sin k(x - \xi), \quad (2.1.2)$$

$$v(x, t, k) = 2 \cos kx - \frac{1}{k} \int_0^x d\xi u(\xi, t) v(\xi, t, k) \sin k(x - \xi), \quad (2.1.3)$$

$$w(x, t, k) = \exp(ikx) + \frac{1}{k} \int_x^\infty d\xi u(\xi, t) w(\xi, t, k) \sin k(x - \xi). \quad (2.1.4)$$

The integral equations for s and v imply the symmetry conditions

$$s(x, t, -k) = -s(x, t, k), \quad v(x, t, -k) = v(x, t, k). \quad (2.1.5)$$

The scattering data are defined by

$$F(t, k) \doteq 1 + \frac{1}{2ik} \int_0^\infty dx e^{ikx} u(x, t) s(x, t, k), \quad G(t, k) \doteq 1 + \frac{1}{2ik} \int_0^\infty dx e^{ikx} u(x, t) v(x, t, k). \quad (2.1.6)$$

The integral equations for s and v , and the symmetry conditions (2.1.5) yield

$$s \sim e^{-ikx} F(t, k) - e^{ikx} F(t, -k), \quad v \sim e^{-ikx} G(t, k) + e^{ikx} G(t, -k), \quad x \rightarrow \infty. \quad (2.1.7)$$

The scattering data F and G , in addition to characterizing the large x behavior of the eigenfunctions s and v , also determine the behavior of w at $x = 0$:

$$F(t, k) = w(0, t, k), \quad G(t, k) = \frac{w_x(0, t, k)}{ik}. \quad (2.1.8)$$

The derivation of equations (2.1.8) follows from the evaluation of the Wronskian of pairs of eigenfunctions at the origin and at infinity:

$$(sw_x - s_x w)(0, t, k) = 2ikw(0, t) = (sw_x - s_x w)(\infty, t, k) = 2ikF(t, k),$$

$$(vw_x - v_x w)(0, t, k) = 2w_x(0, t) = (vw_x - v_x w)(\infty, t, k) = 2ikG(t, k).$$

Similarly, evaluating $sv_x - s_x v$ at the origin and at infinity, it follows that

$$F(t, k)G(t, -k) + F(t, -k)G(t, k) = 2. \quad (2.1.9)$$

The eigenfunctions s, ψ , and $\psi(x, t, -k)$, are related by an equation involving x -independent coefficients. Evaluating this equation as $x \rightarrow \infty$, it follows that

$$s(x, t, k) = F(t, k)w(x, t, -k) - F(t, -k)w(x, t, k).$$

Defining S and W by

$$S = se^{ikx}, \quad W = we^{-ikx}, \quad (2.1.10)$$

the above equations yield

$$\frac{S(x, t, k)}{F(t, k)} - W(x, t, -k) = -\frac{F(t, -k)}{F(t, k)} W(x, t, k) e^{2ikx}, \quad k \in \mathbb{R}. \quad (2.1.11)$$

Equations (2.1.2), (2.1.4), (2.1.6), and (2.1.10) imply that the functions S, W , and F are $+$ functions, i.e. they are analytic in the upper half of the k -complex plane and they tend to 1 as $k \rightarrow \infty$ (these functions have a removable singularity at $k = 0$).

If $u \in \mathbb{R}$, there exist the additional symmetry conditions,

$$s(x, t, k) = \overline{s(x, t, -\bar{k})}, \quad v(x, t, k) = \overline{v(x, t, -\bar{k})}, \quad F(t, k) = \overline{F(t, -\bar{k})}. \quad (2.1.12)$$

For the analysis of the t - problem it will be convenient to work with matrix eigenfunctions. For this reason we rewrite the above results using matrix notations. Let σ_j , $j = 1, 2, 3$, denote the Pauli matrices, i.e.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.1.13)$$

It is straightforward to verify that if the scalar eigenfunction $\mu(x, t, k)$ satisfies the scalar Lax pair (1.4) and (1.5), then the matrix eigenfunction $\nu(x, t, k)$

$$\nu(x, t, k) = \begin{pmatrix} \mu(-k) - \frac{1}{ik} \mu_x(-k) & \mu(k) - \frac{1}{ik} \mu_x(k) \\ \mu(-k) + \frac{1}{ik} \mu_x(-k) & \mu(k) + \frac{1}{ik} \mu_x(k) \end{pmatrix}, \quad \mu(k) \doteq \mu(x, t, k), \quad (2.1.14)$$

satisfies the matrix Lax pair

$$\nu_x + ik\sigma_3\nu = Q\nu, \quad Q(x, t, k) \doteq \frac{u}{2k}(\sigma_2 - i\sigma_3); \quad (2.1.15)$$

$$\nu_t + 4ik^3\sigma_3\nu = \hat{Q}\nu, \quad \hat{Q}(x, t, k) \doteq 2ku\sigma_2 + u_x\sigma_1 + \frac{q}{k}(i\sigma_3 - \sigma_2), \quad q(x, t) \doteq \frac{1}{2}u_{xx} + u^2. \quad (2.1.16)$$

Let ψ and φ be the matrix eigenfunctions corresponding to the scalar eigenfunctions w and $v - s$, respectively, i.e.

$$\psi(x, t, k) = \frac{1}{2} \begin{pmatrix} w(-k) - \frac{1}{ik} w_x(-k) & w(k) - \frac{1}{ik} w_x(k) \\ w(-k) + \frac{1}{ik} w_x(-k) & w(k) + \frac{1}{ik} w_x(k) \end{pmatrix}, \quad w(k) = w(x, t, k), \quad (2.1.17)$$

and $\varphi(x, t, k)$ is defined by the r.h.s. of equation (2.1.17) with w replaced by $v - s$ and $\frac{1}{2}$ by $\frac{1}{4}$. These eigenfunctions, in addition to satisfying equation (2.1.15), also have the following properties:

(i)

$$\psi e^{ikx\sigma_3} \sim I, \quad x \rightarrow \infty; \quad \varphi(0, t, k) = I. \quad (2.1.18)$$

(ii)

$$\psi(x, t, k) = \frac{i}{2k} w_x(x, t, 0) \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + O(1), \quad k \rightarrow 0; \quad (2.1.19)$$

$$\varphi(x, t, k) = \frac{i}{4k} [v_x(x, t, 0) - s_x(x, t, 0)] \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + O(1), \quad k \rightarrow 0. \quad (2.1.20)$$

(iii) Let

$$\Psi \doteq \psi \exp(ikx\sigma_3), \quad \Phi \doteq \varphi \exp(ikx\sigma_3). \quad (2.1.21)$$

Then

$$\Phi = (\Phi^+, \Phi^-) = \begin{pmatrix} \Phi_1^+ & \Phi_1^- \\ \Phi_2^+ & \Phi_2^- \end{pmatrix}, \quad \Psi = (\Psi^-, \Psi^+) = \begin{pmatrix} \Psi_1^- & \Psi_1^+ \\ \Psi_2^- & \Psi_2^+ \end{pmatrix}, \quad \Phi, \Psi \rightarrow I \text{ as } k \rightarrow \infty. \quad (2.1.22)$$

This notation was explained in the introduction. These properties are a direct consequence of the properties of the scalar eigenfunction s, v , and ψ , which in turn follow from the analysis of the Volterra integral equations (2.1.2)-(2.1.4). It is important to notice that since s and v satisfy Volterra integral equations with compact support, Φ is an entire function in the punctured complex k -plane.

The eigenfunctions Φ and Ψ have unit determinant. Furthermore they satisfy the symmetry conditions

$$\Psi_1^+(x, t, k) = \Psi_2^-(x, t, -k) = \overline{\Psi_2^-(x, t, \bar{k})}, \quad \Psi_2^+(x, t, k) = \Psi_1^-(x, t, -k) = \overline{\Psi_1^-(x, t, \bar{k})}. \quad (2.1.23)$$

The eigenfunction ψ and φ are related through an x -independent constant matrix. This implies the relationship

$$\Phi(x, t, k) = \Psi(x, t, k) e^{-ikx\sigma_3} (\psi(0, t, k))^{-1} e^{ikx\sigma_3}, \quad k \in \mathbb{R},$$

between the eigenfunctions Φ and Ψ . Using this equation and defining χ_p by equation (1.16), one finds the jump condition (1.17), where the 2×2 matrix $F(x, t, k)$ is given by

$$F_{11} = 1, F_{22} = \frac{1}{|\psi_2^+(0, t, k)|}, F_{12} = \frac{-\psi_1^+(0, t, k)}{\psi_2^+(0, t, k)} \exp(-2ikx), F_{21} = \frac{\overline{\psi_1^+(0, t, k)}}{\psi_2^+(0, t, k)} \exp(2ikx). \quad (2.1.24)$$

2.2 The t Problem

The t -part of the matrix Lax pair (equation (2.1.16)) implies that

$$\psi_t(0, t, k) + 4ik^3[\sigma_3, \psi(0, t, k)] = \hat{Q}(t, k)\psi(0, t, k), \quad (2.2.1a)$$

where

$$\hat{Q}(t, k) \doteq 2ku(0, t)\sigma_2 + u_x(0, t)\sigma_1 + \frac{q(0, t)}{k}(i\sigma_3 - \sigma_2), \quad q(0, t) \doteq \frac{1}{2}u_{xx}(0, t) + u^2(0, t), \quad (2.2.1b)$$

The matrix $\hat{Q}(t, k)$, involves $u(0, t)$ which is known, but it also involves $u_x(0, t)$ and $u_{xx}(0, t)$ which are unknown. The main idea of [6] is to analyze equation (2.2.1) by formulating an inverse scattering problem.

Let $\hat{\Psi}(t, k)$ and $\hat{\Phi}(t, k)$ be the matrix solutions of equation (2.2.1) specified by the boundary conditions $\lim_{t \rightarrow \infty} \hat{\Psi} = I$, $\hat{\Phi}(0, k) = I$. The matrix valued function $\hat{\Phi}$ satisfies the Volterra integral equation

$$\hat{\Phi}(t, k) = I + \int_0^t d\tau e^{-4ik^3(t-\tau)\sigma_3} \hat{Q}(\tau, k) \hat{\Phi}(\tau, k) e^{4ik^3(t-\tau)\sigma_3}. \quad (2.2.2)$$

This equation and the analogous equation satisfied by $\hat{\Psi}(x, t, k)$ (where $\int_0^t$ is replaced by $-\int_t^\infty$) imply that

$$\hat{\Phi} = (\hat{\Phi}^+, \hat{\Phi}^-) = \begin{pmatrix} \hat{\Phi}_1^+ & \hat{\Phi}_1^- \\ \hat{\Phi}_2^+ & \hat{\Phi}_2^- \end{pmatrix}, \quad \hat{\Psi} = (\hat{\Psi}^-, \hat{\Psi}^+) = \begin{pmatrix} \hat{\Psi}_1^- & \hat{\Psi}_1^+ \\ \hat{\Psi}_2^- & \hat{\Psi}_2^+ \end{pmatrix}, \quad \hat{\Phi}, \hat{\Psi} \rightarrow I \text{ as } k \rightarrow \infty. \quad (2.2.3)$$

This notation is explained in the introduction. The matrices $\hat{\Phi}$ and $\hat{\Psi}$ have unit determinant, and they satisfy the same symmetry relations as those satisfied by Φ and Ψ , ie.

$$\hat{\Psi}_1^+(t, k) = \hat{\Psi}_2^-(t, -k) = \overline{\hat{\Psi}_2^-(t, \bar{k})}, \quad \hat{\Psi}_2^+(t, k) = \hat{\Psi}_1^-(t, -k) = \overline{\hat{\Psi}_1^-(t, \bar{k})}. \quad (2.2.4)$$

Also, for the same reason as for Φ , the matrix $\hat{\Phi}$ is an entire function in the punctured complex k -plane. It turns out that the following important relationships are valid between $\psi(0, t, k)$, $\hat{\Psi}(t, k)$ and $\hat{\Phi}(t, k)$

$$\psi^+(0, t, k) = \frac{\psi_2^+(0, 0, k)}{\hat{\Psi}_2^+(0, k)} \hat{\Psi}^+(t, k), \quad k \in I \cup III, \quad (2.2.5)$$

$$\psi^-(0, t, k) = \frac{\psi_1^-(0, 0, k)}{\hat{\Psi}_1^-(0, k)} \hat{\Psi}^-(t, k), \quad k \in IV \cup VI, \quad (2.2.6)$$

$$\psi^+(0, t, k) = \psi_2^+(0, 0, k) \hat{\Phi}^-(t, k) + \psi_1^+(0, 0, k) \hat{\Phi}^+(t, k) \exp(-8ik^3t), \quad k \in \mathbb{C}^+, \quad (2.2.7)$$

$$\psi^-(0, t, k) = \psi_1^-(0, 0, k) \hat{\Phi}^+(t, k) + \psi_2^-(0, 0, k) \hat{\Phi}^-(t, k) \exp(8ik^3t), \quad k \in \mathbb{C}^-, \quad (2.2.8)$$

To derive these equations we use the fact that ψ^-E^{-1} , ψ^+E , $\hat{\Psi}^-E^{-1}$, $\hat{\Phi}^-E$, $\hat{\Psi}^+E$, and $\hat{\Phi}^+E^{-1}$, where $E = \exp(4ik^3t)$, are related through equations involving constant coefficients. For example

$$\psi^+(0, t, k) = \alpha \hat{\Psi}^+(t, k) + \beta \hat{\Psi}^-(t, k) \exp(-8ik^3t), \quad k \in \mathbb{R},$$

where α and β are t -independent scalars. Using $\beta = \det(\psi^+(0, t, k), \hat{\Psi}^+(t, k)) \exp(8ik^3t)$, and the fact that the r.h.s. of this equation is a $+$ function going to zero as $t \rightarrow \infty$, it follows that $\beta = 0$. Then evaluating this equation at $t = 0$, equation (2.2.5) follows. Equations (2.2.6)-(2.2.8) are derived in a similar way. We also note that equations (2.2.6) and (2.2.8) follow from equations (2.2.5) and (2.2.7) using the underline symmetry conditions.

We now prove that the behavior of the functions $\hat{\Phi}$ and $\hat{\Psi}$ as $k \rightarrow 0$ is given by equation (1.15). The function $\hat{\Psi}(t, k)$ can be represented by

$$\hat{\Psi}(t, k) = \begin{pmatrix} 1 & \frac{i}{k} \\ 1 & -\frac{i}{k} \end{pmatrix} \tilde{\Psi}(t, k)$$

where $\tilde{\Psi}(t, k)$ satisfies the integral equation

$$\tilde{\Psi}(t, k) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -ik & ik \end{pmatrix} - \int_t^\infty d\tau \varepsilon(t, k) \varepsilon^{-1}(\tau, k) \tilde{Q}(\tau, k) \tilde{\Psi}(\tau, k) e^{4ik^3(t-\tau)\sigma_3},$$

$$\varepsilon(t, k) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -ik & ik \end{pmatrix} e^{-4ik^3 t \sigma_3}, \quad \tilde{Q}(t, k) = \begin{pmatrix} u_x(0, t) & -2u(0, t) \\ 2q(0, t) - 2k^2 u(0, t) & -u_x(0, t) \end{pmatrix}.$$

The kernel of this equation, and hence $\tilde{\Psi}(t, k)$, has no singularities at $k = 0$. Thus

$$\tilde{\Psi}(t, k) = \frac{i}{k} \begin{pmatrix} \hat{\alpha}(t) & \hat{d}(t) \\ -\hat{\alpha}(t) & -\hat{d}(t) \end{pmatrix} + O(1), \quad x \rightarrow 0,$$

$$\hat{\alpha}(t) = (\tilde{\Psi}(t, 0))_{21}, \quad \hat{d}(t) = (\tilde{\Psi}(t, 0))_{22}$$

. The symmetry condition $\hat{\Psi}(t, -k) = \sigma_1 \hat{\Psi}(t, k) \sigma_1$ on $Im \ k^3 = 0$, implies $\hat{\alpha} = \hat{d}$, which therefore shows that $\hat{\Psi}(t, k)$ satisfies equation (1.15). Similar considerations show that $\hat{\Phi}(t, k)$ also satisfies equation (1.15). We also note that the behavior (1.15) for the functions $\psi(x, t, k)$ and $\varphi(x, t, k)$ follows from equations (2.1.19) and (2.1.20). The symmetry conditions associated with $k \rightarrow \bar{k}$ imply that all functions $\alpha, \beta, \hat{\alpha}$, and $\hat{\beta}$ are real.

Let us introduce the notation

$$\psi(0) \doteq \psi(0, 0, k), \quad \hat{\Psi}(0) \doteq \hat{\Psi}(0, k), \quad e = \exp(8ik^3 t). \quad (2.2.9)$$

The t -RH problem will be formulated in terms of the matrix $\hat{Y}_p$ defined by

$$\left(\frac{\hat{\Phi}^+(t, k)}{\psi_2^+(0)}, \psi^+(0, t, k) \right), \quad \left(\frac{\hat{\Psi}^-(t, k)}{\rho(k)}, \psi^+(0, t, k) \right), \quad \left(\psi^-(0, t, k), \frac{\hat{\Psi}^+(t, k)}{\tilde{\rho}(k)} \right), \quad \left(\psi^-(0, t, k), \frac{\hat{\Phi}^-(t, k)}{\psi_1^-(0)} \right), \quad (2.2.10)$$

for $k \in I \cup III$, $k \in II$, $k \in V$, $k \in IV \cup VI$ respectively, where the scalar functions $\rho(k)$ and $\tilde{\rho}(k)$ are given by

$$\rho(k) \doteq \hat{\Psi}_1^-(0) \psi_2^+(0) - \hat{\Psi}_2^-(0) \psi_1^+(0), \quad \tilde{\rho}(k) \doteq \hat{\Psi}_2^+(0) \psi_1^-(0) - \hat{\Psi}_1^+(0) \psi_2^-(0) = \overline{\rho(\bar{k})}. \quad (2.2.11)$$

In defining $\hat{Y}_p$, we use when it is possible $\psi(0, t, k)$ instead of $\hat{\Phi}(t, k)$ and $\hat{\Psi}(t, k)$; the scalars appearing in equation (2.2.10), are chosen by the requirement that $\det \hat{Y}_p = 1$. Since the $\det \hat{Y}_p$ is t -independent, it follows that $\det \hat{Y}_p = 1$ for $k \in II \cup V$. For $k \in I \cup III$, $\det \hat{Y}_p$ is given by

$$\hat{\Phi}_1^+ \frac{\psi_2^+}{\psi_2^+(0)} - \hat{\Phi}_2^+ \frac{\psi_1^+}{\psi_2^+(0)} = \hat{\Phi}_1^+ \frac{\hat{\Psi}_2^+}{\hat{\Psi}_2^+(0)} - \hat{\Phi}_2^+ \frac{\hat{\Psi}_1^+}{\hat{\Psi}_2^+(0)},$$

where we have used equation (2.2.5). The rhs of this equation simplifies to 1 using the 22 component of the equation $(\hat{\Phi})^{-1}\hat{\Psi} = \exp(-4ik^3t\sigma_3)\hat{\Psi}(0)\exp(4ik^3t\sigma_3)$, i.e. equation (2.2.12b),

$$\hat{\Phi}_2^-\hat{\Psi}_1^--\hat{\Phi}_1^-\hat{\Psi}_2^- = \hat{\Psi}_1^-(0), \hat{\Phi}_1^+\hat{\Psi}_2^+-\hat{\Phi}_2^+\hat{\Psi}_1^+ = \hat{\Psi}_2^+(0), \hat{\Phi}_2^-\hat{\Psi}_1^+-\hat{\Phi}_1^-\hat{\Psi}_2^+ = \hat{\Psi}_1^+(0)e^{-1}, \hat{\Phi}_1^+\hat{\Psi}_2^--\hat{\Phi}_2^+\hat{\Psi}_1^- = \hat{\Psi}_2^-(0)e. \quad (2.2.12)$$

A similar analysis implies that $\det \hat{Y}_p = 1$ for $k \in IV \cup VI$.

The matrix $\hat{Y}_p$ satisfies the following jump conditions

$$\hat{Y}_p^-(t, k) = \hat{Y}_p^+(t, k)\hat{G}(t, k), \quad k^3 \in \mathbb{R}. \quad (2.2.13)$$

$$\arg k = 0, \pi : \quad \hat{G}_{11} = 1, \hat{G}_{22} = \frac{1}{\psi_2^+(0)\psi_1^-(0)}, \quad \hat{G}_{12} = -\frac{\psi_1^+(0)}{\psi_1^-(0)}e^{-1}, \hat{G}_{21} = \frac{\psi_2^-(0)}{\psi_2^+(0)}e. \quad (2.2.14)$$

$$\arg k = \frac{\pi}{3}, \frac{2\pi}{3} : \quad \hat{G}_{11} = \hat{G}_{22} = 1, \hat{G}_{12} = 0, \quad \hat{G}_{21} = \frac{\hat{\Psi}_2^-(0)}{(\psi_2^+(0))\rho(k)}e. \quad (2.2.15)$$

$$\arg k = \frac{4\pi}{3}, \frac{5\pi}{3} : \quad \hat{G}_{11} = \hat{G}_{12} = 1, \hat{G}_{21} = 0, \quad \hat{G}_{12} = -\frac{\hat{\Psi}_1^+(0)}{(\psi_1^-(0))\tilde{\rho}(k)}e^{-1}. \quad (2.2.16)$$

We now indicate how these jump conditions can be derived.

$\arg k = 0$: Substituting the expression for $\hat{Y}_p$ in equation (2.2.13), and using $\det \psi = \det \hat{\Phi} = 1$, one finds that $\hat{G}_{11}$ and $\hat{G}_{22}$ are given by equations (2.2.14a) and (2.2.14b), while

$$\hat{G}_{12} = \hat{\Phi}_1^- \frac{\psi_2^+}{\psi_1^-(0)} - \hat{\Phi}_2^- \frac{\psi_1^+}{\psi_1^-(0)}, \quad \hat{G}_{21} = \hat{\Phi}_1^+ \frac{\psi_2^-}{\psi_2^+(0)} - \hat{\Phi}_2^+ \frac{\psi_1^-}{\psi_2^+(0)}.$$

Using equations (2.2.5), (2.2.6) to eliminate ψ_2^+ , ψ_1^- , ψ_2^- , ψ_1^+ , and then using (2.2.12c) and (2.2.12d), these equations become

$$\hat{G}_{12} = -\frac{\psi_2^+(0)\hat{\Psi}_1^+(0)}{\psi_1^-(0)\hat{\Psi}_2^+(0)}e^{-1}, \quad \hat{G}_{21} = \frac{\psi_1^-(0)\hat{\Psi}_2^-(0)}{\psi_2^+(0)\hat{\Psi}_1^-(0)}e.$$

Using equations (2.2.5), (2.2.6) evaluated at $t = 0$, these equations simplify to those given in equation (2.2.14).

$\arg k = \frac{\pi}{3}$: Substituting the expressions for $\hat{Y}_p$ in equation (2.2.13), one finds $\hat{G}_{11} = 1$, $\hat{G}_{12} = 0$, and

$$\hat{G}_{22} = \hat{\Phi}_1^+ \frac{\psi_2^+}{\psi_2^+(0)} - \hat{\Phi}_2^+ \frac{\psi_1^+}{\psi_2^+(0)}, \quad \hat{G}_{21} = \frac{\hat{\Phi}_1^+ \hat{\Psi}_2^- - \hat{\Phi}_2^+ \hat{\Psi}_1^-}{\rho \psi_2^+(0)}.$$

Using equation (2.2.5) to eliminate ψ_2^+ and ψ_1^+ , and then using equation (2.2.12b) it follows that $\hat{G}_{22} = 1$. Using equation (2.2.12d), it follows that $\hat{G}_{21} = \hat{\Psi}_2^-(0)e/\rho\psi_2^+(0)$. The derivation of the jump matrices along the other contours is similar to the above and hence is omitted.

The number of independent functions appearing in the above jumps can be reduced by using the underline symmetry conditions. Some of these functions can be computed effectively in terms of initial data, while the rest, although depend in a nonlinear way on the boundary data, can be given an effective characterization in the k -complex plane.

Let us summarize the main formal results of this subsection.

Proposition 2.1 Let the matrix value function $\hat{Y}(t, k)$ be defined by equation (1.18). Then $\hat{Y}_p$ satisfies equation (1.20), where the jump matrices are depicted in Figure 1.1 with θ replaced by $8ik^3t$.

The matrix $\hat{Y}_p(t, k)$ has a pole at the origin. The behavior of $\hat{Y}_p$ as $k \rightarrow 0$ follows from the behavior of $\psi(x, t, k)$, $\phi(x, t, k)$, $\hat{\Psi}(t, k)$, $\hat{\Phi}(t, k)$ which is given by equations (1.15).

The functions $s_1^+(k)$ and $s_2^+(k)$ defined in (1.19) can be computed in terms of the initial data $u(x, 0)$. The function $c(k)$ defined in (1.21) depends in a nonlinear way on the boundary data $u(0, t)$. These functions, which are called the scattering data, have the following properties:

1. $s_1^+(k)$ and $s_2^+(k)$ are analytic for $k \in \mathbb{C}^+ \setminus \{0\}$; $s_1^+(k) \rightarrow 0$, and $s_2^+(k) \rightarrow 1$, as $k \rightarrow \infty$; $s_1^+(k) \sim \frac{i\alpha(0,0)}{k}$, $s_2^+(k) \sim \frac{-i\alpha(0,0)}{k}$, $k \rightarrow 0$.

2.

$$|s_2^+(k)|^2 - |s_1^+(k)|^2 = 1, \quad k \in \mathbb{R}. \quad (2.2.17)$$

3. $c(k)$ is meromorphic in II with possible poles at the zeros of $s_2^+(k)$ and of $\rho(k)$, $k \in II$; $c(k) \rightarrow 0$ as $k \rightarrow \infty$.

4.

$$b(-k) = \overline{b(k)}, \quad c(-k) = \overline{c(k)}, \quad b(0) = 1, \quad c(0) = c_0 \doteq \frac{\hat{\alpha}^2(0)}{\alpha^2(0,0)}, \quad (2.2.18)$$

where $\hat{\alpha}(t)$ and $\alpha(x, t)$ are related to the behavior of $\hat{\Psi}(t, k)$ and $\Psi(x, t, k)$ as $k \rightarrow 0$.

5. Zeros k_j of $\rho(k)$ are pure imaginary: $k_j \in i\mathbb{R}^+$.

Proof

It is straightforward to derive the jump matrices appearing in Figure 1.1 from equations (2.2.14)-(2.2.16), the notations introduced in (1.19), (1.21), the underline symmetry, and the relation $|s_2^+|^{-2} = 1 - |b|^2$, which in turn follows from $\det \psi(0) = 1$ and the underline symmetry.

The initial data $u(x, 0)$ can be used to compute $\psi(x, 0, k)$, which in turn implies $\psi(0, 0, k)$. Equation (2.2.17) follows from $\det \psi(0, t, k) = 1$. Hence all of the components of $\psi(0, 0, k)$ can be effectively calculated in terms of $u(x, 0)$. Although the scattering data $c(k)$ depends in a nonlinear way on the boundary data $u(0, t)$, its analytic structure can be characterized explicitly. The large k behavior of s_1^+ , s_2^+ , and $c(k)$ follows from the large k behavior of $\psi_1^+(0, 0, t)$, $\psi_2^+(0, 0, k)$, $\hat{\Psi}_2^-$, and $\hat{\Psi}_1^-$. The possible poles for $c(k)$ appear at the possible zeros of $s_2^+(k)$ and of $\rho(k)$. The symmetry equations (2.2.18a) and (2.2.18b) follow from (2.1.23) and (2.2.4); equations (2.2.18c) as well as the small k behavior of s_1^+ , s_2^+ are the direct consequence of equation (1.15). Regarding the last equation in (2.2.18) we first notice that the term of order k^{-2} in the asymptotics of $\rho(k)$ as $x \rightarrow 0$ disappears, so

$$\rho(k) = \frac{\rho_1}{k} + \rho_0 + o(1), \quad k \rightarrow 0. \quad (2.2.19)$$

This asymptotics representation is valid for $\pi/3 \leq \arg k \leq 2\pi/3$. On the other hand, if $\arg k = \pi/3, 2\pi/3$, we can use (2.2.5) and deduce that

$$\rho(k) = \hat{\Psi}_1^-(0)\psi_2^+(0) - \hat{\Psi}_2^-(0)\frac{\psi_2^+(0)}{\hat{\Psi}_2^+(0)}\hat{\Psi}_1^+(0) = \frac{s_2^+(k)}{\hat{\Psi}_2^+(0, k)} \det \hat{\Psi}(0, k) = \frac{s_2^+(k)}{\hat{\Psi}_2^+(0, k)}.$$

This implies $\rho_1 = 0$ and $\rho_0 = \alpha(0, 0)/\hat{\alpha}(0)$, which in turn led to (2.2.18d). This completes the proof of the Proposition 2.1 except the last point 5, which will be proven in the end of §3 (see corollary 3.2).

Remark 2.1 In this paper we consider the generic situation only, i.e. we assume that the functions $\alpha, \beta, \hat{\alpha}$, and $\hat{\beta}$ in equation (1.15) do not vanish. In other words, we assume that $k = 0$ is not a virtual level in the spectrum of either the x -operator (1.4) or the t -operator (1.5).

Remark 2.2 If one investigates the direct and inverse problem associated with equation (2.1.16) without assuming a relationship between $u(0, t)$, $u_x(0, t)$, and $u_{xx}(0, t)$, one finds that the basic scattering data is given by $r(k)$,

$$r(k) = \frac{\hat{\Psi}_1^+(0, k)}{\hat{\Psi}_2^+(0, k)}. \quad (2.2.20)$$

The only analytic restriction for $r(k)$ is that it is meromorphic in $I \cup III \cup V$. However, in our case, since $u(0, t)$, $u_x(0, t)$, and $u_{xx}(0, t)$ are the boundary values of the KdV equation, there exists a relationship between $u(0, t)$, $u_x(0, t)$, and $u_{xx}(0, t)$. It is quite interesting that although this relationship is highly nonlinear, its representation in the scattering space is very simple: $r(k)$ has analytic continuation for $k \in II$. Indeed, equation (2.2.5) evaluated at $t = 0$ becomes

$$\frac{\psi_1^+(0, 0, k)}{\psi_2^+(0, 0, k)} = \frac{\hat{\Psi}_1^+(0, k)}{\hat{\Psi}_2^+(0, k)}, \quad k \in I \cup III.$$

But $\psi^+(0, 0, k)$ is analytic in $I \cup II \cup III$, hence $r(k)$ has an analytic continuation across $\arg k = \pi/3$, $2\pi/3$. The function $c(k)$ is simply related to $r(k)$,

$$c(k) = \frac{\overline{r(\bar{k})}}{(s_2^+(k))^2 \left(1 - \frac{s_1^+(k)}{s_2^+(k)} \overline{r(\bar{k})}\right)}. \quad (2.2.21)$$

2.3 The RH Problem

In this section we formulate a RH problem, whose solution gives the solution of the initial-boundary value problem of the KdV equation.

Proposition 2.2 Let $\hat{Z}_p(x, t, k)$ be defined by equation (1.22). Then

- (i) $\hat{Z}_p(x, t, k)$ has unit determinant and it satisfies the RH problem (1.23), (1.24).
- (ii) $\hat{Z}_p(x, t, k)$ is a meromorphic function in the complex k -plane cut along $\text{Im}(k^3) = 0$. It has a pole at $k = 0$, where its behavior is given by equation (1.24b). It also may have poles only at the zeros of $\rho(k)$ for $k \in II$, and at the complex conjugate of these zeros.
- (iii) The eigenfunction $\Psi^+(x, t, k)$ and the potential $u(x, t)$ can be obtained from $\hat{Z}_p(x, t, k)$ by

$$\Psi^+(x, t, k) = \hat{Z}_p(x, t, k)(0, 1)^T, k \in \mathbb{C}^+; u(x, t) = 2i\partial_x \left[\lim_{k \rightarrow \infty} (k \hat{Z}_p(x, t, k))_{11} \right]. \quad (2.3.1)$$

Proof For convenience of writing we drop the subscript p in $\hat{Y}$.

(i) We first note that the jump matrix $F(x, t, k)$ defined in equation (2.1.24) can be written as

$$F(x, t, k) = \begin{pmatrix} 1 & 0 \\ \frac{\hat{Y}_{21}^-(t, k)}{\hat{Y}_{22}^+(t, k)} e^{2ikx} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\hat{Y}_{12}^+(t, k)}{\hat{Y}_{11}^-(t, k)} e^{-2ikx} \\ 0 & 1 \end{pmatrix}, \quad k \in \mathbb{R}. \quad (2.3.2)$$

Consider now the jump conditions for $\hat{Z}_p(x, t, k)$.

$\arg k = 0, \pi$: Equations (1.22), (1.17), and (2.3.2) yield

$$\begin{aligned} [\hat{Z}_p^+(x, t, k)]^{-1} \hat{Z}_p^-(x, t, k) &= \begin{pmatrix} 1 & 0 \\ -\frac{\hat{Y}_{21}^+}{\hat{Y}_{22}^+} e^{2ikx} & 1 \end{pmatrix} F(x, t, k) \begin{pmatrix} 1 & \frac{\hat{Y}_{12}^-}{\hat{Y}_{11}^-} e^{-2ikx} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{\hat{Y}_{21}^- - \hat{Y}_{21}^+}{\hat{Y}_{22}^+} e^{2ikx} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\hat{Y}_{12}^- - \hat{Y}_{12}^+}{\hat{Y}_{11}^-} e^{-2ikx} \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where $\hat{Y}_{jl}^\pm \equiv \hat{Y}_{jl}^\pm(t, k)$. Equation (1.20) implies

$$\hat{Y}_{21}^- - \hat{Y}_{21}^+ = \hat{Y}_{22}^+ \hat{G}_{21}(t, k), \quad \hat{Y}_{12}^- - \hat{Y}_{12}^+ = \hat{Y}_{11}^- \hat{G}_{12}(t, k).$$

Using these equations it follows that

$$[\hat{Z}_p^+(x, t, k)]^{-1} \hat{Z}_p^-(x, t, k) = \begin{pmatrix} 1 & 0 \\ \hat{G}_{21}(t, k) e^{2ikx} & 1 \end{pmatrix} \begin{pmatrix} 1 & \hat{G}_{12}(t, k) e^{-2ikx} \\ 0 & 1 \end{pmatrix} = e^{-i(kx+4k^3t)\sigma_3} G(k) e^{i(kx+4k^3t)\sigma_3}.$$

$\arg k = \pi/3, 2\pi/3$: Equation (1.22) yields

$$[\hat{Z}_p^+(x, t, k)]^{-1} \hat{Z}_p^-(x, t, k) = \begin{pmatrix} 1 & 0 \\ \left(\frac{\hat{Y}_{21}^-}{\hat{Y}_{22}^-} - \frac{\hat{Y}_{21}^+}{\hat{Y}_{22}^+} \right) e^{2ikx} & 1 \end{pmatrix}.$$

Equation (1.20) implies

$$\hat{Y}_{22}^- = \hat{Y}_{22}^+, \quad \hat{Y}_{21}^- - \hat{Y}_{21}^+ = \hat{Y}_{22}^+ \hat{G}_{21}(t, k).$$

Using these equations it follows that

$$[\hat{Z}_p^+(x, t, k)]^{-1} \hat{Z}_p^-(x, t, k) = \begin{pmatrix} 1 & 0 \\ \hat{G}_{21}(t, k) e^{2ikx} & 1 \end{pmatrix} = e^{-i(kx+4k^3t)\sigma_3} G(k) e^{i(kx+4k^3t)\sigma_3}.$$

The derivation of the jump conditions for $\arg k = 4\pi/3, 5\pi/3$ is similar to the above and hence is omitted.

The normalizing condition (1.24a) follows from the fact that $\hat{Z}_p(x, t, k) \exp(-ikx\sigma_3)$ satisfies equation (2.1.15), and the potential Q has no terms of zero order in k .

The behavior of $\hat{Z}_p$ at $k = 0$ follows from the behavior of χ_p and $\hat{Y}_p$ at $k = 0$: For $k \in I \cup III$ we have

$$\begin{aligned} \hat{Z}_p &= \left(\frac{1}{\psi_2^+(0, t, k)} \Phi^+, \Psi^+ \right) \begin{pmatrix} 1 & 0 \\ \frac{\hat{\Psi}_2^+ e^{2ikx}}{s_2^+ \psi_2^+(0, t, k)} & 1 \end{pmatrix} \sim \begin{pmatrix} -\frac{\beta}{\alpha(0, t)} & \frac{i\alpha}{k} \\ \frac{\beta}{\alpha(0, t)} & -\frac{i\alpha}{k} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\hat{\beta}k}{\alpha(0, 0)\alpha(0, t)} & 1 \end{pmatrix} \\ &= \left(O(1), \frac{i\alpha}{k} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \end{aligned}$$

which implies (1.24b).

(ii) Let $k \in I \cup III$. Using the definitions of $\chi_p(x, t, k)$ and $\hat{Y}_p(t, k)$ for $k \in I \cup III$, i.e.

$$\chi_p(x, t, k) = \left(\frac{\Phi^+(x, t, k)}{\psi_2^+(0, t, k)}, \Psi^+(x, t, k) \right), \quad \hat{Y}_{21}(t, k) = \frac{\hat{\Phi}_2^+(t, k)}{s_2^+(k)}, \quad \hat{Y}_{22}(t, k) = \psi_2^+(0, t, k),$$

equation (1.22) yields

$$\hat{Z}_p(x, t, k) = \left(\frac{1}{\psi_2^+(0, t, k)} \left(\Phi^+(x, t, k) + \frac{\hat{\Phi}_2^+(t, k)}{s_2^+(k)} \Psi^+(x, t, k) e^{2ikx} \right), \Psi^+(x, t, k) \right), \quad k \in I \cup III. \quad (2.3.3)$$

We shall show that if k_0 is a zero of $\psi_2^+(0, t, k)$ then

$$\Phi^+(x, t, k_0) + \frac{\hat{\Phi}_2^+(t, k_0)}{s_2^+(k_0)} \Psi^+(x, t, k_0) e^{2ik_0x} = 0, \quad k_0 \in I \cup III. \quad (2.3.4)$$

Hence the possible poles of $\hat{Z}_p(x, t, k)$ for $k \in I \cup III$, can occur only at the fixed possible zeros of $s_2^+(k)$. To derive equation (2.3.4) we first relate $\Phi^+(x, t, k)$ to $\Psi^+(x, t, k) e^{2ikx}$, and to $\Phi^-(x, t, k) e^{2ikx}$ through an equation with x -independent coefficients. Evaluating this equation at $x = 0$ it follows that

$$\Phi^+(x, t, k) = \frac{\Psi^+(x, t, k)}{\psi_1^+(0, t, k)} e^{2ikx} - \frac{\psi_2^+(0, t, k)}{\psi_1^+(0, t, k)} \Phi^-(x, t, k) e^{2ikx}, \quad k \in \mathbb{C}^+. \quad (2.3.5)$$

Equation (2.3.4) follows from equation (2.3.5) evaluated at $k = k_0$. Indeed, when $k = k_0$, $\psi_2^+(0, t, k_0) = 0$, hence equation (2.2.7) yields

$$\begin{aligned}\psi_1^+(0, t, k_0) &= s_2^+(k_0)\hat{\Phi}_1^-(t, k_0) + s_1^+(k_0)\hat{\Phi}_1^+(t, k_0)e^{-8ik_0^3t}, \\ 0 &= s_2^+(k_0)\hat{\Phi}_2^-(t, k_0) + s_1^+(k_0)\hat{\Phi}_2^+(t, k_0)e^{-8ik_0^3t}.\end{aligned}\tag{2.3.6}$$

These equations together with $\det \hat{\Phi} = 1$ imply $\psi_1^+(0, t, k_0)\hat{\Phi}_2^+(t, k_0) = -s_2^+(k_0)$.

Let $k \in II$. Instead of equation (2.3.3) we now find

$$\hat{Z}_p(x, t, k) = \left(\frac{1}{\psi_2^+(0, t, k)} \left(\Phi^+(x, t, k) + \frac{\hat{\Psi}_2^-(t, k)}{\rho(k)} \Psi^+(x, t, k) e^{2ikx} \right), \Psi^+(x, t, k) \right), \quad k \in II. \tag{2.3.7}$$

Again it can be shown that if $\psi_2^+(0, t, k_0) = 0$, $k_0 \in II$, then

$$\Phi^+(x, t, k_0) + \frac{\hat{\Psi}_2^-(t, k_0)}{\rho(k_0)} \Psi^+(x, t, k_0) e^{2ik_0x} = 0, \quad k_0 \in II. \tag{2.3.8}$$

Hence the possible poles of $\hat{Z}_p(x, t, k)$ for $k \in II$, can occur only at the fixed possible zeros of $\rho(k)$. Equation (2.3.8) follows from equation (2.3.5) using $\det \hat{Y}_p = 1$, $k \in II$, i.e.

$$\rho(k) = \hat{\Psi}_1^-(t, k)\psi_2^+(0, t, k) - \hat{\Psi}_2^-(t, k)\psi_1^+(0, t, k).$$

The absence of singularities of $\hat{Z}_p(x, t, k)$ at the possible zeros of $\psi_1^-(0, t, k)$ can be proven in a similar way; alternatively one can use the underline symmetry.

To finish with the proof of the statement (ii) it remains to notice that $s_2^+(k)$ cannot have zeros in $I \cup III$. Indeed, equation (2.2.7) shows that if $s_2^+(k_0) = 0$ for $k_0 \in I \cup III$, then

$$e^{8ik_0^3t}\psi^+(0, t, k_0) = s_1^+(k_0)\hat{\Phi}^+(t, k_0), \quad k_0 \in I \cup III.$$

This implies $\hat{\Phi}^+(t, k_0) \rightarrow 0$, $t \rightarrow \infty$ which contradict the fact that $\det \hat{\Phi} = 1$ for all $t > 0$. The function $s_2^+(k)$ participates in the definition of $\hat{Z}_p$ only in the regions I and III , hence, $\hat{Z}_p(x, t, k)$ can have poles only in the regions II and V of the complex k -plane. These poles occur at the zeros of $\rho(k)$ and $\overline{\rho(\bar{k})}$ respectively.

(iii) Equation (2.3.1a) follows from the definitions (1.16) and (1.22) of functions χ_p and $\hat{Z}_p$; equation (2.3.1b) follows from the large k asymptotics of equation (2.1.15). This completes the proof of the Proposition 2.2.

Remark 2.3 The above analysis is valid for the generic situation, i.e. we assume that all zeros of $\psi_2^+(0, t, k)$, $s_2^+(k)$, and $\rho(k)$ are simple, do not coincide with each other, do not lie on the rays $Imk^3 = 0$, and there is a finite number of them.

Remark 2.4 $s_2^+(k) \neq 0$ for $k \in \mathbb{R}$ and $\rho(k) \neq 0$ for $\arg k = \pi/3, 2\pi/3$. The first inequality follows from equation (2.2.17). The second inequality follows from the fact that s_1^+ and s_2^+ cannot vanish simultaneously and from equations $s_1^+(k) = \rho(k)\hat{\Psi}_1^+(0, k)$ and $s_2^+(k) = \rho(k)\hat{\Psi}_2^+(0, k)$ for $\arg k = \pi/3, 2\pi/3$ (see equation 2.2.5).

We note that the behavior of $\hat{Z}_p$ at $k = 0$ for $k \in I, III$, together with the jump conditions imply the behavior of $\hat{Z}_p$ at $k = 0$ in the entire complex k -plane. Using $\rho(0) = \alpha(0, 0)/\hat{\alpha}(0)$, it is straightforward to verify that $\hat{Z}_p$ approaches

$$\frac{i\alpha(x, t)}{k} \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad \frac{i\alpha(x, t)}{k} \begin{pmatrix} c_0 & 1 \\ -c_0 & -1 \end{pmatrix}, \quad \frac{i\alpha(x, t)}{k} \begin{pmatrix} 1 & c_0 \\ -1 & -c_0 \end{pmatrix}, \quad k \rightarrow 0 \quad (2.3.9)$$

for $k \in VI, IV, k \in II$, and $k \in V$ respectively.

The scalar $\alpha(x, t)$ appearing in the zero k behavior of the RH problem (1.23), (1.24) is unknown. In spite of this the RH problem has a unique solution. This is a consequence of the condition (1.24a). This will become apparent after this RH problem has been regularized, i.e. it is mapped to a new RH problem which is regular at $k = 0$.

Proposition 2.3 Let $\tilde{Z}_p(x, t, k)$ be the solution of the RH problem regularized at $k = 0$, i.e. $\tilde{Z}_p$ satisfies the same conditions as $\hat{Z}_p$, except equation (1.24) and it is assumed to be regular at $k = 0$. Then

$$\hat{Z}_p(x, t, k) = \left[I + \frac{iy(x, t)}{2k} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right] \tilde{Z}_p(x, t, k); \quad y(x, t) \doteq 2i \lim_{k \rightarrow \infty} (k\tilde{Z}_p)_{12}. \quad (2.3.10)$$

Proof The RH problem for the function $\tilde{Z}_p$ has the same jumps as the RH problem for $\hat{Z}_p$. It is obvious that condition (1.24a) and the additional conditions at the possible poles (see (2.4.1)) are satisfied by $\hat{Z}_p$ as defined by equation (2.3.10). Therefore, in order to prove that $\hat{Z}_p$ solves the RH problem one only needs to check that $\hat{Z}_p$ as defined by equation (2.3.10) has the correct behavior at $k = 0$. Let

$$\tilde{Z}_{pI}(x, t, 0) = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix}, \quad \tilde{\alpha}\tilde{\delta} - \tilde{\beta}\tilde{\gamma} = 1, \quad (2.3.11)$$

where the subscript I denotes that $k \in I$. Using the jump matrices to relate $\tilde{Z}_{pIV}$ to $\tilde{Z}_{pI}$ it follows that

$$\tilde{Z}_{pIV}(x, t, 0) = \tilde{Z}_{pI}(x, t, 0) \begin{pmatrix} 1 & 0 \\ c_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{\alpha} + \tilde{\beta} & -\tilde{\alpha} \\ \tilde{\gamma} + \tilde{\delta} & -\tilde{\gamma} \end{pmatrix}. \quad (2.3.12)$$

Equations (2.3.11) and (2.3.12), together with the symmetry condition $\tilde{Z}_{pIV}(x, t, 0) = \sigma_1 \tilde{Z}_{pI}(x, t, 0) \sigma_1$ (which is a reflection of $k \rightarrow -k$), imply

$$\begin{pmatrix} \tilde{\alpha} + \tilde{\beta} & -\tilde{\alpha} \\ \tilde{\gamma} + \tilde{\delta} & -\tilde{\gamma} \end{pmatrix} = \begin{pmatrix} \tilde{\delta} & \tilde{\gamma} \\ \tilde{\beta} & \tilde{\alpha} \end{pmatrix}.$$

This equation together with $\tilde{\alpha}\tilde{\delta} - \tilde{\beta}\tilde{\gamma} = 1$, yield $\tilde{\alpha} \neq 0$, and

$$\tilde{Z}_{pI}(x, t, 0) = \begin{pmatrix} \tilde{\alpha} & \frac{1-\tilde{\alpha}^2}{2\tilde{\alpha}} \\ -\tilde{\alpha} & \frac{1+\tilde{\alpha}^2}{2\tilde{\alpha}} \end{pmatrix}.$$

Using this expression in (2.3.10), one finds equation (1.24b) with $\alpha = y/2\tilde{\alpha}$.

It now remains to show that the RH problem (1.23), (1.24) has a unique solution. Let $\hat{Z}'_p$ be another solution of the RH problem whose behavior at $k = 0$ is associated with α' . Consider the product $\hat{Z}'_p(\hat{Z}_p)^{-1}$; since

$$(\hat{Z}_p)^{-1} \sim (\tilde{Z}_p)^{-1} \left(I - \frac{iy}{2k} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right), \quad k \rightarrow 0, \quad k \in I,$$

it follows that

$$(\hat{Z}_p)^{-1} \sim \frac{-i\alpha}{k} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad k \rightarrow 0, \quad k \in I.$$

Hence $(\hat{Z}'_p)(\hat{Z}_p)^{-1} = O(1/k)$, $k \rightarrow 0$. The singularities of $(\hat{Z}'_p)(\hat{Z}_p)^{-1}$ at the possible zeros of $\rho(k)$ cancel (see next section). This fact together with the normalizing condition as $k \rightarrow \infty$, and the symmetry $k \rightarrow -k$ implies

$$(\hat{Z}'_p)(\hat{Z}_p)^{-1} = I + \frac{1}{k} \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}.$$

The condition (1.24a) implies $B = 0$, and $\det \hat{Z}_p = 1$ implies $A = 0$. This completes the proof of the Proposition 2.3.

Remark 2.5 (The connection with the modified KdV). The transformation (2.3.10) is the Miura transformation between KdV and the modified KdV. Indeed, it is easy to verify that if ν satisfies equation (2.1.15), and ν is related to $\tilde{\nu}$ through equation (2.3.10), then $\tilde{\nu}$ satisfies the Dirac eigenvalue equation

$$\tilde{\nu}_x + ik\sigma_3\tilde{\nu} = y\sigma_1\tilde{\nu},$$

with $u = -y^2 - y_x$. Hence $y(x, t)$ satisfies the modified KdV equation

$$y_t + y_{xxx} + 6y^2y_x = 0.$$

This implies that the regularized RH for $\tilde{Z}_p(x, t, k)$ is essentially the RH problem for the initial-boundary value problem for the modified KdV equation. To be more precise, if one formulates an initial-boundary value problem for the MKdV one will find the RH for $\tilde{Z}_p(x, t, k)$ but without the restriction

$$b(0) = 1. \tag{2.3.13}$$

In other words KdV corresponds to a special reduction of the MKdV where the scattering data satisfy the constraint (2.3.13) (compare with the whole-line case [17]).

We conclude this subsection with a discussion of the appropriate data needed to solve the RH problem for $\hat{Z}_p$.

Proposition 2.4 The RH problem for $\hat{Z}_p$ is uniquely specified by the following data, called scattering data: (i) $c(k)$ for $\arg k = \pi/3, 2\pi/3$; (ii) $s_1^+(k)$ for $k \in \mathbb{R}$; (iii) the zeros of $s_2^+(k)$ for $k \in \mathbb{C}^+$; (iv) the zeros $\{k_j\}_1^N$ of $\rho(k)$ and the residues $\{c_j\}_1^N$ of $c(k)$ at these poles.

Proof Indeed, (ii) and (iii) together with the equation (2.2.17) and normalization $b(0) = 1$ yield s_2^+ through the solution of a singular scalar RH problem. Then, $c(k)$, s_1^+ , and s_2^+ specify all the

jump conditions. The possible poles of $\hat{Z}_p$ can occur at the possible zeros of $\rho(k)$. The additional conditions on the residues of $\hat{Z}_p$ which involve c_j are presented in the next section (equation (2.4.1)).

Remark 2.5 The function $c(k)$, $\arg k = \pi/3, 2\pi/3$ satisfies certain constraints. Since it is the boundary value of a function which is meromorphic and decreasing as $k \rightarrow \infty$ in the region II , $c(k)$ satisfies the following infinite set of constraints:

$$\frac{1}{2\pi i} \int_{\{\arg k = \frac{2\pi}{3}\} \cup \{\arg k = \frac{\pi}{3}\}} \frac{c(k')}{k' - k} dk' = \sum_{j=1}^N \frac{c_j}{k_j - k} + \sum_{j=1}^M \frac{c_j^+}{k_j^+ - k}, \quad \forall k \notin II,$$

or

$$\frac{1}{2\pi i} \int_{\{\arg k = \frac{2\pi}{3}\} \cup \{\arg k = \frac{\pi}{3}\}} c(k)(k-1)^{-n} dk = \sum_{j=1}^N c_j(k_j-1)^{-n} + \sum_{j=1}^M c_j^+(k_j^+-1)^{-n},$$

where k_j^+ , $j = 1, \dots, M$ are the zeros of $s_2^+(k)$ and c_j^+ are the corresponding residues of $c(k)$. As a set of independent parameters for $c(k)$ one can take the **whole** set of its poles and the corresponding residues $\{k_j, k_j^+, c_j, c_j^+\}$, supplemented by

$$\alpha(k) \doteq \operatorname{Re} \left\{ c(k) - \sum_{j=1}^N \frac{c_j}{k - k_j} - \sum_{j=1}^M \frac{c_j^+}{k - k_j^+} \right\}, \quad \arg k = \frac{\pi}{3}, \frac{2\pi}{3}.$$

Having $\{\alpha(k); k_j, k_j^+, c_j, c_j^+\}$ one can reconstruct $c(k)$ in a **closed form**. Also, to reconstruct $s_1^+(k)$ one needs to know $\beta(k) \doteq \operatorname{Res}_1^+(k)$. Thus the set of independent parameters for our RH problem is

$$\{\alpha(k); \beta(k); k_j, k_j^+, c_j, c_j^+\}$$

where $\alpha(k), \beta(k)$ are real-valued decreasing functions defined on the whole line and satisfying the symmetry condition $\alpha(k) = \alpha(-\bar{k})$, $\beta(k) = \beta(-k)$; $k_j, k_j^+ \in II$, $c_j, c_j^+ \in \mathbb{C} \setminus \{0\}$ (k_j, k_j^+, c_j, c_j^+ should be pure imaginary). In particular, this means that this set can be represented by two real valued functions defined on the half-line.

2.4 Solitons

It turns out that the zeros of $\rho(k)$ give rise to solitons.

Proposition 2.5 Assume that $\rho(k)$ has a finite number of simple zeros for $k \in II$. Let these zeros be denoted by $\{k_j\}_1^N$, $k_j \in i\mathbb{R}_+^+$, $j = 1, \dots, N$. Let $\{c_j\}_1^N$ be the corresponding residues of $c(k)$. Then:

(i) The constants $c_j, j = 1, \dots, N$ are pure imaginary.

(ii) The first column $\hat{Z}_p^{(1)}(x, t, k)$ of $\hat{Z}_p(x, t, k)$ has a simple pole at $k_j, j = 1, \dots, N$ and the second column $\hat{Z}_p^{(2)}(x, t, k)$ of $\hat{Z}_p(x, t, k)$ has a simple pole at $\bar{k}_j, j = 1, \dots, N$; the corresponding residues satisfy the equations

$$\text{res}_{k_j} \hat{Z}_p^{(1)}(x, t, k) = c_j e^{\theta(k_j)} \hat{Z}_p^{(2)}(x, t, k_j), \quad \theta(k) = 2i(kx + 4k^3t)$$

$$\text{res}_{\bar{k}_j} \hat{Z}_p^{(2)}(x, t, k) = \bar{c}_j e^{-\theta(\bar{k}_j)} \hat{Z}_p^{(1)}(x, t, \bar{k}_j). \quad (2.4.1)$$

(iii) Equations (2.4.1) together with the equations (1.23), (1.24) characterize the function $\hat{Z}_p(x, t, k)$ uniquely.

(iv) The RH problem (1.23), (1.24), (2.4.1) can be solved as follows: Let $\tilde{Z}(x, t, k)$ be the solution of a RH problem satisfying the same jump conditions as $\hat{Z}_p(x, t, k)$, but with $c(k)$ and $b(k)$ replaced by $c_0(k) \doteq c(k) \prod_{j=1}^N (k - k_j)/(k - \bar{k}_j)$ and $b_0(k) \doteq b(k) \prod_{j=1}^N (k - \bar{k}_j)/(k - k_j)$. This RH problem is regular, i.e. $\tilde{Z}(x, t, k)$ is a holomorphic function in the complex k -plane. Then $\tilde{Z}_p(x, t, k)$ can be found from $\tilde{Z}$ through equation (2.3.10) where

$$\tilde{Z}_p(x, t, k) = (kI + B_N)kI + B_{N-1}) \dots (kI + B_1) \tilde{Z}(x, t, k) \begin{pmatrix} \frac{1}{\prod_{j=1}^N (k - k_j)} & 0 \\ 0 & \frac{1}{\prod_{j=1}^N (k - \bar{k}_j)} \end{pmatrix}, \quad (2.4.2)$$

where the 2×2 matrices $B_1, \dots, B_N$ are independent of k . These matrices can be determined recursively by solving the algebraic equations

$$(k_j I + B_j) \tilde{Z}_{j-1}(x, t, k_j) \begin{pmatrix} 1 \\ -d_j(x, t) \end{pmatrix} = 0, \quad (\bar{k}_j I + B_j) \tilde{Z}_{j-1}(x, t, \bar{k}_j) \begin{pmatrix} -\bar{d}_j(x, t) \\ 1 \end{pmatrix} = 0, \quad j = 1, \dots, N \quad (2.4.3)$$

where

$$\tilde{Z}_j(x, t, k) = (kI + B_j) \tilde{Z}_{j-1}(x, t, k), \quad j = 1, \dots, N-1, \quad \tilde{Z}_0 = \tilde{Z},$$

and

$$d_j(x, t) = c_j \frac{\prod_{l=1, l \neq j}^N (k_j - k_l)}{\prod_{l=1}^N (k_j - \bar{k}_l)} \exp(2ik_j x + 8ik_j^3 t). \quad (2.4.4)$$

Proof Statement (i) is a direct consequence of the underline symmetries:

$$c(k) = \overline{c(-\bar{k})}$$

(ii) $\hat{Z}_p(x, t, k)$ in $I \cup III$ and in II is given by equations (2.3.3) and (2.3.7) respectively. Since $s_2^+(k)$ has no zeros in $I \cup III$, $\hat{Z}_p$ is analytic for $k \in I \cup III$. The first column of $\hat{Z}_p$ has poles for $k \in II$ at the zeros of $\rho(k)$, i.e. at the points $k_1, \dots, k_N$, which are simultaneously the poles of $c(k)$. For $\arg k = \pi/3, 2\pi/3$,

$$\hat{Z}_p^-(x, t, k) \begin{pmatrix} 1 & 0 \\ -c(k)e^\theta & 1 \end{pmatrix} = \hat{Z}_p^+(x, t, k). \quad (2.4.5)$$

Because $\hat{\Phi}(t, k)$ is an entire function in the punctured complex k -plane, both sides of equation (2.4.5) have an analytic continuation into the region II of the k -complex plane; moreover, $\hat{Z}_p^+(x, t, k)$ has no singularities at $k_1, \dots, k_N$. This shows that the l.h.s. of equation (2.4.5) has no singularities at $k_1, \dots, k_N$ as well. Similar considerations apply for $k \in V$, thus

$$\hat{Z}_p(x, t, k) \begin{pmatrix} 1 & \\ -c(k)e^\theta & \end{pmatrix}; \quad \hat{Z}_p(x, t, k) \begin{pmatrix} -\bar{c}(\bar{k})e^{-\theta} & \\ & 1 \end{pmatrix}$$

have no singularities at k_j and $\bar{k}_j$ respectively. This leads directly to the equations (2.4.1).

In order to prove statement (iii) we suppose that $\hat{Z}'_p(x, t, k)$ is another solution of the same RH problem (1.23), (1.24), (2.4.1). The product $\hat{Z}'_p(\hat{Z}_p)^{-1}$ has no jumps at $k^3 \in \mathbb{R}$; its only possible singularities are at the points k_j , $\bar{k}_j$, and at the $k = 0$ (recall that $\det \hat{Z}_p = 1$). For $k \in II$ one can rewrite $\hat{Z}'_p(\hat{Z}_p)^{-1}$ as

$$\hat{Z}'_p(\hat{Z}_p)^{-1} = \hat{Z}'_p(k) \begin{pmatrix} 1 & 0 \\ -c(k)e^\theta & 1 \end{pmatrix} \left[\hat{Z}_p(k) \begin{pmatrix} 1 & 0 \\ -c(k)e^\theta & 1 \end{pmatrix} \right]^{-1},$$

which shows that $\hat{Z}'_p(\hat{Z}_p)^{-1}$ has actually no singularities at k_j . Similar considerations for $k \in V$ imply that $\hat{Z}'_p(\hat{Z}_p)^{-1}$ has no singularities at $\bar{k}_j$ either. To complete the proof of the statement (iii) it remains to refer to the end of the proof of the Proposition 2.3.

The essence of deriving statement (iv) is the usual idea [9] of using Darboux transformations to solve a RH problem with singularities. We need to show that the function $\tilde{Z}_p$ defined by (2.4.2)-(2.4.4) satisfies the RH problem (1.23), (2.4.1) (conditions (1.23), (2.4.1) are preserved by the gauge transformation (2.3.10)). Indeed, by construction the matrix $\tilde{Z}_p$ satisfies the correct jump condition, and $\tilde{Z}_p \rightarrow I$ as $k \rightarrow \infty$. Also,

$$res_{k_j} \tilde{Z}_p^{(1)}(x, t, k) = P_j(k_j)(k_j I + B_j) \tilde{Z}_{j-1}(x, t, k_j) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\prod_{l=1, l \neq j}^N (k_j - k_l)},$$

$$\tilde{Z}_p^{(2)}(x, t, k) = P_j(k_j)(k_j I + B_j) \tilde{Z}_{j-1}(x, t, k_j) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{1}{\prod_{l=1}^N (k_j - \bar{k}_l)},$$

where

$$P_j(k) = (kI + B_N)(kI + B_{N-1}) \dots (kI + B_{j+1}).$$

This yields equation (2.4.1a) as a direct consequence of equation (2.4.3a); actually, since $\det P_j(k_j) \neq 0$ these two equations are equivalent. Similarly, equation (2.4.1b) follows from (2.4.3b). This completes the proof of the proposition 2.5.

Because of the underline symmetry, the regular RH problem for $\tilde{Z}(x, t, k)$ is always solvable (i.e. there exists a vanishing lemma [16]). To discuss the solvability of the algebraic equations (2.4.3) let us rewrite them in the form

$$B_j = -W_j \begin{pmatrix} k_j & 0 \\ 0 & \bar{k}_j \end{pmatrix} W_j^{-1},$$

$$W_j \doteq \left(\tilde{Z}_{j-1}(x, t, k_j) \begin{pmatrix} 1 \\ -d_j(x, t) \end{pmatrix}, \tilde{Z}_{j-1}(x, t, \bar{k}_j) \begin{pmatrix} -\bar{d}_j(x, t) \\ 1 \end{pmatrix} \right).$$

This implies that the system of the algebraic equations for the matrices $B_1, \dots, B_N$ is solvable iff

$$\det W_j \neq 0, \quad j = 1, \dots, N, \quad \forall x, t > 0. \quad (2.4.6)$$

However equation (2.4.6) is not valid. This corresponds to the well known fact that the MKdV on the whole line has no solitons. This means that the corresponding solution $y(x, t)$ of the MKdV equation is always singular. The solution $u(x, t)$ of the initial-boundary value problem (1.1), is related to $y(x, t)$ through the Miura transformation $u = -y_x - y^2$. Taking into account what happens in the full-line problem for KdV equation, one can expect that since all k_j are pure imaginary, a singular y yields a smooth u . More precisely, one can formulate the conjecture. Let $k_j = ih_j$, $0 < h_1 < h_2 < \dots < h_N$ and suppose that $\text{sign}(-ic_j) = (-1)^{N-j+1}$. Then the RH problem (1.23), (1.24), (2.4.1) is solvable for all $x, t > 0$, and $u(x, t)$ has no singularities.

3 Asymptotic Analysis

In order to determine the large t behavior of the solution $u(x, t)$ one needs to study the large t asymptotic behavior of the oscillatory RH problem (1.23). The corresponding problem for integrable equations on the full line was first studied in [10] (see also [11], [12] and the review [13]). The first rigorous result concerning the asymptotics in the soliton sector for the full line problem for KdV equation was obtained in [15]. A rigorous and elegant approach to studying the asymptotic behavior of RH problems has been recently developed in [14]. In what follows we shall use this new approach.

We first study the solution $\tilde{Z}(x, t, k)$ of the regular RH problem corresponding to the RH problem (1.23) (see Proposition 2.5 (iv)).

Theorem 3.1 Under the assumption of Proposition 2.5, the solution $\tilde{Z}(x, t, k)$ of the regular RH problem corresponding to (1.23) satisfies the asymptotic equations,

$$\tilde{Z}(x, t, k) = I + o(1), \quad t \rightarrow \infty, \quad 0 < A \leq x/t \leq B < \infty \quad (3.1)$$

uniformly for $|Imk| \geq \varepsilon > 0$.

Proof The method of [14] can be thought of as a nonlinear steepest descent method. The stationary point, and the directions of the steepest descent associated with $\exp 2i(kx + 4k^3t)$, are given by

$$k_{\pm} = \pm i \sqrt{\frac{x}{12t}}, \quad Imi(k^3 - 3kk_+^2) = 0.$$

This implies that we must deform the original RH problem to one defined on the above steepest descent contours (see the solid lines of Figure 3.1). The fact that the stationary phase points are complex, simplifies the corresponding deformation procedure dramatically (compare with [14], [7])

Figure 3.1

The jump matrix along the positive real axis can be factorized into two triangular matrices. Using this factorization we find

$$\tilde{Z}^- \begin{pmatrix} 1 & b_0(k)e^{-\theta} \\ 0 & 1 \end{pmatrix} = \tilde{Z}^+ \begin{pmatrix} 1 & 0 \\ \overline{b_0(k)}e^\theta & 1 \end{pmatrix}, \quad k \in \mathbb{R}^+. \quad (3.2)$$

Suppose that $b_0(k)$ is a rational function with appropriately chosen poles such that $b_0(k)$ is analytic in regions 6 and 10. Since $e^{-\theta}$ and e^θ are analytic and decreasing in regions 6, 10 and 1, 5 respectively, it follows that the relevant jump matrices can be absorbed into $\tilde{Z}^-$ and $\tilde{Z}^+$. Also, the jumps along the broken lines separating regions 1-11, 11-5, 6-12, and 12-10 can be eliminated since $c_0(k)$ has analytic continuation into the bounded domain 11 (notice also that e^θ is decreasing as $t \rightarrow \infty$, $k \in 11$). The rest of the broken lines in Figure 3.1 disappear if we assume $c_0(k)$ be, like $b_0(k)$, an appropriate rational function. Introducing the sectionally holomorphic function

$$X(x, t, k) = \tilde{Z}(x, t, k)K(x, t, k) \quad (3.3)$$

where $K(x, t, k)$ is given by

$$\begin{pmatrix} 1 & 0 \\ \overline{b_0(k)}e^\theta & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ c_0(k)e^\theta & 1 \end{pmatrix}, \quad I, \quad \begin{pmatrix} 1 & 0 \\ c_0(k)e^\theta & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ \overline{b_0(k)}e^\theta & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & b_0(k)e^{-\theta} \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \overline{c_0(k)}e^{-\theta} \\ 0 & 1 \end{pmatrix}, \quad I, \quad \begin{pmatrix} 1 & \overline{c_0(k)}e^{-\theta} \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & b_0(k)e^{-\theta} \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ [\bar{b}_0(\bar{k}) - c_0(k)]e^\theta & 1 \end{pmatrix}, \begin{pmatrix} 1 & [b_0(k) - \bar{c}_0(\bar{k})]e^{-\theta} \\ 0 & 1 \end{pmatrix},$$

for $k \in 1, 2, \dots, 11, 12$ respectively we obtain that X satisfies the RH problem depicted in Figure 3.2 (also $X \rightarrow I$ as $k \rightarrow \infty$)

Figure 3.2

The deformed RH problem arising in the asymptotic analysis of the initial-boundary value problem of KdV. The x, t dependence enters only through $\theta(x, t) = 2i(kx + 4k^3t)$.

The RH problem for $X(x, t, k)$ has the crucial property that its jump matrices decay exponentially to the identity, as $t \rightarrow +\infty$, uniformly with respect of k . This, just like the classical steepest descent method, implies

$$X(x, t, k) = I + O(t^{-m}), \quad \forall m \in \mathbb{N}, \quad (3.4)$$

for all k away from the conjugation contours. Indeed, if Γ and $\tilde{G}_0$ denote the contours and the jump matrices of Figure 3.2, then

$$X(x, t, k) = I + \frac{1}{2\pi i} \int_{\Gamma} \frac{X^+(x, t, k')(I - \tilde{G}_0(x, t, k'))}{k' - k} dk' \quad (3.5)$$

for k not on Γ . Taking into account the apriori estimate $|X^+(x, t, k)| \leq \text{const}$, and using the classical Laplace's method to the integral of the r.h.s. of equation (3.5), one finds equation (3.4).

This concludes the proof of Theorem 3.1 in the case that b_0 and c_0 are rational. The general case can be reduced to this case following the construction of [14]. Of course, in the general case we can't guarantee the estimate $O(t^{-m})$, $\forall m$ in (3.1). In fact, one can prove that indeed in (3.1)

we have that $o(1) = o(t^{-m})$ where exponent m depends on the character of the decreasing $b(k)$ and $c(k)$ as $k \rightarrow \infty$.

As an immediate consequence of Theorem 3.1 we obtain:

Corollary 3.1 Under the assumption of Proposition 2.5, the solution $\hat{Z}_p(x, t, k)$ of the RH problem (1.23), (1.24), (2.4.1) satisfies the asymptotic equation

$$\hat{Z}_p(x, t, k) = (I + o(1))\hat{Z}_s(x, t, k), \quad t \rightarrow \infty, \quad 0 < A \leq x/t \leq B < \infty, \quad (3.6)$$

uniformly for $|Imk| \geq \varepsilon > 0$. The matrix function $\hat{Z}_s(x, t, k)$ is given by the equation (2.3.10) where $\tilde{Z}_p$ is defined in (2.4.2) - (2.4.4) with $\tilde{Z}$ replaced by I .

The function $\hat{Z}_s$ is nothing but the matrix eigenfunction corresponding to the pure N -soliton solution $u_s(x, t)$ of the KdV equation with parameters $\{k_j\}_1^N$ and $\{c_j\}_1^N$. This implies the final asymptotic formula for the solution of the initial-boundary value problem (1.1):

$$u(x, t) = u_s(x, t) + o(1), \quad t \rightarrow \infty, \quad 0 < A \leq x/t \leq B < \infty. \quad (3.7)$$

Corollary 3.2 The function $\rho(k)$, $k \in II$, may have zeros only on $i\mathbb{R}^+$. In fact, if there were $k_{j_0} \in II$, $\rho(k_{j_0}) = 0$, $Rek_{j_0} \neq 0$, the corresponding N -soliton solution $u_s(x, t)$ of the KdV equation would have a singularity. Since $k_{j_0} \in II$, it follows that

$$3(Rek_{j_0})^2 < (Imk_{j_0})^2,$$

and the singularity moves to the right. Because of the asymptotic formula (3.7) (that is obviously valid for general case $k_j \in II$), this would mean the nonexistence of the solution of the initial-boundary problem under consideration, which contradict the general results of [3].

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