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Nonisospectral and Isomonodromic
Deformations.

Self-Similar Solutions.

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Abstract

We construct self-similar reductions of various nonisospectral soliton equations and show that they may be solved in the framework of the Method of Isomonodromic Deformations. We demonstrate that new Ordinary Differential Equations (ODE) include, as a rule, variable coefficients that satisfy additional nonlinear ODE. These additional ODE may be without Painlevé property. Some new ODE may be regularly reduced to certain Painlevé equations via point transformations. The most interesting new ODE, from the physical point of view, is a self-similar reduction for the Maxwell–Bloch system, applied in nonlinear optics.

Introduction

Exact analytical solutions to nonlinear equations of mathematical physics are essential for mathematical modeling of physical wave phenomena. Soliton, shock and self-similar waves are three the most well-known types of exact special solutions. Our paper concerns (on an application level) self-similar solutions. They are significant not only because they represent simple and clear special models of certain wave physical phenomena, but also because, sometimes, one can use them to describe asymptotics of evolution of a general initial condition. To construct a self-similar substitution (if there is any for a particular equation) is rather simple mathematical problem. Much more intricately is to solve a corresponding self-similar equation, which represents an ordinary nonlinear nonautonomous differential equation. It usually excludes the possibility of applying exact analytical methods. Fortunately, there are some exceptions. The most popular ones are called Painlevé equations (e.g. see [1–5]), which are used in nonlinear optics, hydrodynamics, statistical mechanics and other fields in physics.

The Method of Isomonodromic Deformations (or Inverse Monodromy Transform (IMT)), discovered in 1979 [1–3], now is the most powerful tool to study the Painlevé equations. The connections between isospectral and isomonodromic deformations are thoroughly studied by now [1–3]. The Painlevé equations result, in particular, from self-similar substitutions into corresponding soliton equations.

In [6] was proposed a generalization of the Inverse Scattering Transform (IST), called IST with a variable spectral parameter (or a method of nonisospectral deformations). According to [6] nonisospectral soliton equations arise as a compatibility condition of the following overdetermined linear system

$$\begin{aligned}\Phi_\xi &= U\Phi \\ \Phi_\eta &= V\Phi\end{aligned}\tag{0.1}$$

Here U, V and Φ are matrix $N \times N$ complex-valued functions of ξ, η and spectral parameter λ . In a general case U and V depend on λ rationally

$$\begin{aligned}U(\lambda, \xi, \eta) &= u_0(\xi, \eta) + \sum_{n=1}^{N_1} \frac{u_n(\xi, \eta)}{\lambda - \lambda_n} \\ V(\lambda, \xi, \eta) &= v_0(\xi, \eta) + \sum_{n=1}^{N_2} \frac{v_n(\xi, \eta)}{\lambda - \mu_n}\end{aligned}\tag{0.2}$$

, where simple poles λ_n and μ_m do not coincide. In the conventional IST $\lambda, \lambda_n, \mu_m$ are supposed to be complex constants. According to [6] let poles λ_n, μ_n are some functions of ξ and η ; λ depends on ξ, η and also locally-analytically depends on "hidden" spectral parameter k (in this case simple fractions $1/(\lambda - \lambda_n), 1/(\lambda - \mu_n)$ are linear-independent). These functions are not arbitrary at all. They obey the

system of equations which is uniquely fixed by condition : compatibility condition for (0.1):

$$U_\eta - V_\xi + [U, V] = 0 \quad (0.3)$$

must be fulfilled identically over k and related system of nonlinear equations for the matrices u_n, v_n must have exactly gauge underterminacy. So the equations for λ are

$$\frac{\partial \lambda}{\partial \eta} + \sum_{m=1}^{N_2} \frac{b_m}{\lambda - \mu_m} = 0 \quad \frac{\partial \lambda}{\partial \xi} + \sum_{m=1}^{N_1} \frac{c_m}{\lambda - \lambda_m} = 0 \quad (0.4)$$

Therefore we get nonlinear equations for u_n, v_n

$$\partial_\eta u_0 - \partial_\xi v_0 + [u_0, v_0] = 0 \quad (0.5)$$

$$\begin{aligned} \frac{\partial u_n}{\partial \eta} + [u_n, \sum_{m=1}^{N_2} \frac{v_m}{\lambda_n - \mu_m}] &= \sum_{m=1}^{N_2} \frac{b_m u_n + c_n v_m}{(\lambda_n - \mu_m)^2} \\ \frac{\partial v_n}{\partial \xi} + [v_n, \sum_{m=1}^{N_1} \frac{u_m}{\mu_n - \lambda_m}] &= \sum_{m=1}^{N_1} \frac{c_m v_n + b_n u_m}{(\mu_n - \lambda_m)^2} \end{aligned}$$

System

$$\begin{aligned} \frac{\partial \lambda_n}{\partial \eta} + \sum_{m=1}^{N_2} \frac{b_m}{\lambda_n - \mu_m} &= 0 & \frac{\partial c_n}{\partial \eta} - 2c_n \sum_{m=1}^{N_2} \frac{b_m}{(\lambda_n - \mu_m)^2} &= 0 \\ \frac{\partial \mu_n}{\partial \xi} + \sum_{m=1}^{N_1} \frac{c_m}{\mu_n - \lambda_m} &= 0 & \frac{\partial b_n}{\partial \xi} - 2b_n \sum_{m=1}^{N_1} \frac{c_m}{(\mu_n - \lambda_m)^2} &= 0 \end{aligned} \quad (0.6)$$

is a compatibility condition for (0.4). In the particular case , when λ, λ_n and μ_n are constant, the right hand side of (0.5) is zero . Then equations (0.5) determine a general "nondeformed" system integrable in the framework of the IST (0.1,0.2) with "unmovable" poles λ_n, μ_n .

A lot of new integrable equations was constructed in [6]. The Maxwell-Bloch system with pumping :

$$\begin{aligned} E_\eta &= \rho \\ N_\xi + \frac{1}{2}(\rho \bar{E} + \bar{\rho} E) &= c \\ \rho_\xi &= NE \end{aligned} \quad (0.7)$$

$$x = \eta \quad t = \eta + \xi$$

is the most interesting new one, from physical point of view. The possibility of optical application of (0.7) was understood only later in [20]. Equations (0.7) describe propagation of ultra-fast electromagnetic impulse in a medium containing two-level resonant atoms and take into account additional factor – pumping of optical medium, modeled by parameter c . To get (0.7) let

$$U = \lambda u_1 + u_0 \quad V = v/\lambda \quad (0.8)$$

in (0.1), where

$$u_1 = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad u_0 = -1/2 \begin{pmatrix} 0 & E \\ -\bar{E} & 0 \end{pmatrix} \quad v = -1/4i \begin{pmatrix} N & \rho \\ \bar{\rho} & -N \end{pmatrix}$$

Spectral parameter λ must obey equations

$$\begin{aligned} \lambda_\eta &= -c/4\lambda \\ \lambda_\xi &= 0 \end{aligned} \quad (0.9)$$

If pumping is absent: $c = 0$, then one gets the well-known Maxwell–Bloch system[7]

$$\begin{aligned} E_\eta &= \rho \\ N_\xi + \frac{1}{2}(\rho \bar{E} + \bar{\rho} E) &= 0 \\ \rho_\xi &= NE \end{aligned} \quad (0.10)$$

System (0.10) is effectively applied in nonlinear optics for modeling a series of important and beautiful phenomena: self-induced transparency[7–10], superfluorescence [11], photon echo.

For each soliton equation, integrable within a framework of conventional IST, one can construct a whole class of new equations with a variable spectral parameter. The elements of this class were coined [6] as deformations of the initial equation. To construct soliton solutions to the deformations one can use the "dressing" technique developed previously for the case of a constant λ [12,13,15] (or Darboux transformation [22]). The procedure for constructing the finite-gap solutions to deformations was developed in [21]. It should be noted the method suggested in [6] is the development of ideas [13–15]. Also, the series of soliton equations with a variable spectral parameter were constructed in [16–17]. The symmetry approach to nonisospectral equations was developed in [19].

Via self-similar substitution

$$E(\xi, \eta) = \eta E(z) \quad N(\xi, \eta) = N(z) \quad \rho(\xi, \eta) = \rho(z) \quad (0.11)$$

$$z = \xi\eta$$

into (0.10) we obtain (for real valued fields ρ and E):

$$\begin{aligned} (zE)_z &= \rho \\ N_z + \rho E &= 0 \\ \rho_z &= NE \end{aligned} \tag{0.12}$$

which, in turn , can be reduced to the third Painlevé equation **PIII**. Self-similar solution (0.11–0.12) is very important , because asymptotics of superfluorescence pulse is described by it [11]. System (0.7) has a self-similar solution too:

$$\begin{aligned} E(\xi, \eta) &= \sqrt{2\eta}E(z) & N(\xi, \eta) &= \frac{1}{\sqrt{2\eta}}N(z) & \rho(\xi, \eta) &= \frac{1}{\sqrt{2\eta}}\rho(z) \\ z &= \sqrt{2\eta}\xi \end{aligned} \tag{0.13}$$

which is different from (0.11). Corresponding system of nonlinear ODE

$$\begin{aligned} (zE)_z &= \rho \\ N_z + \rho E &= c \\ \rho_z &= NE \end{aligned} \tag{0.14}$$

represents isomonodromic deformation .

Our principal aims in this paper are :

1. To identify and study relationships between nonisospectral and isomonodromic deformations.
2. To obtain new integrable nonlinear ODE via regular construction of self-similar substitutions into nonisospectral equations. To demonstrate new equations are isomonodromic deformations.

We show that each conventional integrable (by means of IMT) nonlinear ODE (e.g. Painlevé equations) has a corresponding whole class of new integrable ODE. We call the class elements the deformations of the initial equation. For instance (0.14) presents one of the possible deformations of **PIII** equation. In general case in deformations arise variable coefficients , which satisfy additional nonlinear ODE. The equations do not necessarily have Painlevé property (see particular example (2.6) in §2 and Appendix A) and finding their solution is a very important problem to be solved. In some cases the change of variables can reduce the deformed Painlevé equation to the nondeformed one. In §3 we construct a general algorithm for identifying such "hidden" transformations and present , as an example, system (0.14), which can be transformed to the fifth Painlevé equation. System (0.14) is of the third order, but due to it's isomonodromic nature it has one conservation law q

$$q^2 = [(N - cz)^2 + \rho^2 - cz^2 E^2]/16c \quad (0.15)$$

that helps to lower the order of (0.14) by one.

The results of this work were earlier presented in [23].

§1. Self-similar solutions of deformations of soliton equations

In this paragraph we construct self-similar equations and (related Lax pair) for the general deformation (0.5–0.6). We use known technique (see for example [18]), based on a one-parameter group of invariant scaling.

System (0.5–0.6) is invariant under *three-parameter* scaling group (α, β, γ) (the third parameter γ arises due to the variability of spectral parameter) :

$$\begin{aligned} \underline{\xi} &= \alpha \xi & \underline{\lambda}_n &= \gamma \lambda_n & \underline{b}_n &= \beta^{-1} \gamma^2 b_n & \underline{v}_n &= \beta^{-1} \gamma v_n \\ \underline{\eta} &= \beta \eta & \underline{\mu}_n &= \gamma \mu_n & \underline{c}_n &= \alpha^{-1} \gamma^2 c_n & \underline{u}_n &= \alpha^{-1} \gamma u_n \end{aligned} \quad (1.1)$$

To get a self-similar solution it is necessary to select one-parameter subgroup by imposing a restraint on α, β and γ . Let, e.g.

$$\gamma = 1 \quad \alpha = \beta^{-a} \quad (1.2)$$

Related family of self-similar substitutions

$$\begin{aligned} u_n(\xi, \eta) &= z u_n(z)/\xi & v_n(\xi, \eta) &= a z v_n(z)/\eta & \lambda_n(\xi, \eta) &= \lambda_n(z) \\ c_n(\xi, \eta) &= z c_n(z)/\xi & b_n(\xi, \eta) &= a z b_n(z)/\eta & \mu_n(\xi, \eta) &= \mu_n(z) \\ z &= \xi \eta^a \end{aligned} \quad (1.3)$$

generates matrix system of ODE

$$\begin{aligned} \frac{du_n}{dz} + \frac{u_n}{z} + [u_n, \sum_{m=1}^{N_2} \frac{v_m}{\lambda_n - \mu_m}] &= \sum_{m=1}^{N_2} \frac{b_m u_n + c_n v_m}{(\lambda_n - \mu_m)^2} \\ \frac{dv_n}{dz} + \frac{v_n}{z} + [v_n, \sum_{m=1}^{N_1} \frac{u_m}{\mu_n - \lambda_m}] &= \sum_{m=1}^{N_1} \frac{c_m v_n + b_n u_m}{(\mu_n - \lambda_m)^2} \end{aligned} \quad (1.4)$$

and additional independent scalar system for c_n, b_n, λ_n and μ_n :

$$\begin{aligned} \frac{d\lambda_n}{dz} + \sum_{m=1}^{N_2} \frac{b_m}{\lambda_n - \mu_m} &= 0 & \frac{dc_n}{dz} + \frac{c_n}{z} - 2c_n \sum_{m=1}^{N_2} \frac{b_m}{(\lambda_n - \mu_m)^2} &= 0 \end{aligned} \quad (1.5)$$

$$\frac{d\mu_n}{dz} + \sum_{m=1}^{N_1} \frac{c_m}{\mu_n - \lambda_m} = 0 \quad \frac{db_n}{dz} + \frac{b_n}{z} - 2b_n \sum_{m=1}^{N_1} \frac{c_m}{(\mu_n - \lambda_m)^2} = 0$$

Equations (1.5) have a trivial solution $b_n = 0 = c_n; \lambda_n(z), \mu_n(z) = \text{const.}$ In this case right-hand sides of (1.4) are zeroes and we have got general nondeformed system. Equations (1.4,1.5) present one of the possible deformations of this system. To get Lax pair for (1.4,1.5) one can use the following fact [6]. Systems (0.5–0.6) can be obtained in a quite a different way. Let now λ be a constant spectral parameter while equations (0.1) are replaced by

$$D_1 \Phi = U \Phi \quad D_2 \Phi = V \Phi \quad (1.6)$$

Here operators $D_{1,2}$ contain the derivation by λ

$$D_1 = \frac{\partial}{\partial \xi} + F \frac{\partial}{\partial \lambda} \quad D_2 = \frac{\partial}{\partial \eta} + G \frac{\partial}{\partial \lambda}$$

$$F = - \sum_{m=1}^{N_1} \frac{c_m}{\lambda - \lambda_m} \quad G = - \sum_{m=1}^{N_2} \frac{b_m}{\lambda - \mu_m}$$

Matrices U and V are defined by the previous formula (0.2). We demand from operators D_1 and D_2 to commute. Hence

$$F_\eta + GF_\lambda = G_\xi + FG_\lambda \quad (1.7)$$

One gets again the scalar system (0.6) by inserting (1.6) into (1.7). Compatibility condition for (1.6) :

$$D_2 U - D_1 V + [U, V] = 0 \quad (1.8)$$

is equivalent to (0.5). Inserting (1.3) ($\Phi(\xi, \eta, \lambda) = \Phi(z, \lambda)$) into (1.6), one extracts the Lax pair for (1.4)

$$D_1 \Phi = U \Phi \quad D_2 \Phi = V \Phi \quad (1.9)$$

$$D_1 = \frac{\partial}{\partial z} + F \frac{\partial}{\partial \lambda} \quad D_2 = \frac{\partial}{\partial z} + G \frac{\partial}{\partial \lambda}$$

$$F = - \sum_{m=1}^{N_1} \frac{c_m}{\lambda - \lambda_m} \quad G = - \sum_{m=1}^{N_2} \frac{b_m}{\lambda - \mu_m}$$

The condition for vector fields D_1 and D_2 to commute is equivalent to the system (1.5).

Equations (1.9) include derivatives both by λ and by z . For analysis (within a framework of the Inverse Monodromy Transform) we should uncouple the derivatives:

$$P(\lambda)\Phi_\lambda = A(\lambda)\Phi \quad (1.10a)$$

$$\Phi_z = B(\lambda)\Phi \quad (1.10b)$$

Here

$$P(\lambda) = -\sum_{n=1}^{N_1} \frac{c_n}{\lambda - \lambda_n} + \sum_{n=1}^{N_2} \frac{b_n}{\lambda - \mu_n}$$

$$A(\lambda) = \sum_{n=1}^{N_1} \frac{u_n}{\lambda - \lambda_n} - \sum_{n=1}^{N_2} \frac{v_n}{\lambda - \mu_n}$$

$$B(\lambda) = \sum_{n=1}^{N_1} \frac{c_n P^{-1} A + u_n}{\lambda - \lambda_n}$$

From the basic equation (1.10a) it follows that $\lambda = \lambda_n, \mu_n$ are the regular points of function $\Phi(\lambda, z)$. This time finite singular points coincide with zeroes of the function $P(\lambda)$. (1.10a) is the equation of the Fuchs type [3]. Z -variable is the deformation parameter. It parameterized all singular points of Φ (finite branching points and essential singularity $\lambda = \infty$). The most simple case: $N_1 = 1 = N_2$ provides a good illustration:

$$\Phi_\lambda = (A_\infty + \frac{A_0}{\lambda - \lambda_0})\Phi \quad (1.11a)$$

$$\Phi_z = \frac{B_0}{\lambda - \lambda_0}\Phi \quad (1.11b)$$

where

$$\begin{aligned} A_\infty &= \frac{u_1 - v_1}{b_1 - c_1} & A_0 &= \frac{b_1 u_1 - c_1 v_1}{(b_1 - c_1)^2} (\lambda_1 - \mu_1) \\ \lambda_0 &= \frac{\lambda_1 b_1 - \mu_1 c_1}{b_1 - c_1} & B_0 &= \frac{b_1 u_1 - c_1 v_1}{b_1 - c_1} = \frac{c_1 A_0}{\lambda_0 - \lambda_1} \end{aligned}$$

System (1.11a) has two singular points. The first is regular singular point $\lambda = \lambda_0$; the second is irregular $\lambda = \infty$.

§2. Deformation of the Painlevé PIII equation, system without Painlevé property.

Let the matrix functions U and V in (0.1,0.2) both have only by one simple pole. We move these poles to the points $\lambda = 0, \infty$ via a fractional - linear transformation. We suggest that spectral parameter λ obeys the system of nonlinear equations:

$$\frac{\partial \lambda}{\partial \eta} = a\lambda + b + \frac{c}{\lambda} \quad \frac{\partial}{\partial \xi} \frac{1}{\lambda} = \bar{a}\lambda + \bar{b} + \frac{\bar{c}}{\lambda} \quad (2.1)$$

Taking (2.1) into account we extract from (0.3)[6]:

$$\frac{\partial u_1}{\partial \eta} = -au_1 + \bar{a}v \quad (2.2)$$

$$\frac{\partial u_0}{\partial \eta} + [u_1, v] = -bu_1 + \bar{b}v$$

$$\frac{\partial v}{\partial \xi} + [v, u_0] = cu_1 - \bar{c}v$$

If one fixes matrices u_0, u_1, v by formula (0.8) and takes a special case (0.9) of equations (2.1), then one gets the Maxwell-Bloch system with pumping (0.7). Now we are not going to build and analyze self-similar solution of (2.2). Instead we consider an interesting special case : $a = \bar{a} = 0; \bar{b}_0 = \text{const} \neq 0; u_1 = J = \text{const}$:

$$\frac{\partial u_0}{\partial \eta} + [J, v] = -bJ + \bar{b}_0v \quad (2.3)$$

$$\frac{\partial v}{\partial \xi} + [v, u_0] = cJ - \bar{c}v$$

Compatibility condition for (2.1) fixes an additional scalar system for coefficients $c, \bar{c}$ and b :

$$c_\xi + 2c\bar{c} = 0 \quad \bar{c}_\eta + 2b\bar{b}_0 = 0 \quad b_\xi + 2b\bar{c} + 3\bar{b}_0c = 0 \quad (2.4)$$

System (2.3–2.4) is invariant under two-parameter scaling group. Omitting calculations we present a family (over parameter m) of self-similar solutions:

$$\begin{aligned} b(\xi, \eta) &= m\eta^{m-1}zb(z) & \bar{c}(\xi, \eta) &= \eta^m\bar{c}(z) \\ v(\xi, \eta) &= m\eta^{m-1}zv(z) & u_0(\xi, \eta) &= \eta^mu_0(z) \\ c(\xi, \eta) &= m\eta^{2m-1}zc(z) & z &= \xi\eta^m \end{aligned} \quad (2.5)$$

Here:

$$c_z + c/z + 2c\bar{c} = 0 \quad (2.6)$$

$$\bar{c}_z + \bar{c}/z + 2\bar{b}_0b = 0$$

$$b_z + b/z + 2b\bar{c} + 3\bar{b}_0c = 0$$

$$(u_0)_z + u_0/z + [J, v] = -bJ + \bar{b}_0v \quad (2.7)$$

$$v_z + v/z + [v, u_0] = cJ - \bar{c}v$$

System (2.6) is extraordinary, because its general (three-parameter) solution has movable branching points, so the system has not the Painlevé property. It is possible to produce special (two-parameter) solution of (2.6) without movable branching points. The detailed proof of all these facts is given in an Appendix A.

We finally construct the Lax pair for (2.6–2.7):

$$D_1 \Phi = (\zeta J + u_0) \Phi \quad D_2 \Phi = \frac{v}{\zeta} \Phi \quad (2.8)$$

$$D_1 = \partial_z + (-\bar{b}_0 \zeta^2 - \bar{c} \zeta) \partial_\zeta$$

$$D_2 = \partial_z + (-\zeta/z + b + c/\zeta) \partial_\zeta$$

The condition for the vector fields $D_{1,2}$ to commute : $[D_1, D_2] = 0$ is equivalent to (2.6). $D_{1,2}$ include, both z - and ζ - derivatives. Uncoupling the derivatives, we get:

$$P(\zeta) \Phi_\zeta = A(\zeta) \Phi \quad \Phi_z = B(\zeta) \Phi \quad (2.9)$$

$$P(\zeta) = \bar{b}_0 \zeta^2 + (\bar{c} - 1/z) \zeta + b + c/\zeta$$

$$A(\zeta) = -\zeta J - u_0 + v/\zeta$$

$$B(\zeta) = \zeta J + u_0 + (\bar{b}_0 \zeta^2 + \bar{c} \zeta) P^{-1} A$$

Finite singular points of function Φ (2.9) coincide with zeros $\zeta_{1,2,3}$ of function $\zeta P(\zeta)$. $\zeta_{1,2,3}$ are parameterized by deformation variable z .

3. " Hidden " transformations

We start with the general algorithm and then present specific formulae, linking system (0.14) with the fifth Painlevé equation PV.

Assume that we have a certain deformed (with $\underline{L}_1$ -operator, e.g. (3.1)) and nondeformed (with L_1 -operator, e.g. (3.4.1)) Painlevé equations. We suspect them of a linking transformation. To get a full check of suspects we need to make certain steps that are simple because we analyze not nonlinear ODE, but their corresponding linear $\underline{L}_1$ - and L_1 - equations (that depend rationally on ζ).

Step 1. One compares analytical structure (over spectral parameter ζ) of $\underline{L}_1$ - and L_1 - operators. If

- a) the operators contain different number of singularities, or
 - b) the number of singularities is the same, but their types are different
- then there is no desired transformation. If the number of singularities and their types are identical, then chances for transformation's existence increase and one should make

Step 2. Via fractional-linear transformation $\zeta \rightarrow (\alpha\zeta + \beta)/(\gamma\zeta + \delta)$, $\alpha\delta - \beta\gamma = 1$ one tries to get a coincidence (on the complex ζ - plane) of $\underline{L}_1$ - and L_1 - singularities.

Step 3. One performs, if necessary, a change of self-similar variable and a gauge transformation in $\underline{L}_1$ - equation.

Step 4. One demands from $\underline{L}_1$ - and L_1 - operators to have the same matrix coefficients of all their rational functions of ζ .

Steps 3,4 are necessary to make $\underline{L}_1$ - and L_1 - analytical characteristics to coincide: formal monodromy exponents and matrices related to the irregular singular points [3]. If the final Step 4 is successful, then, as a result, one gets the desired transformation. The point is that examined nonlinear ODE present the isomonodromic deformations of their own $\underline{L}_1$ - and L_1 - equations. So $\underline{L}_2$ - and L_2 - equations may be uniquely restored from $\underline{L}_1$ - and L_1 - equations. Therefore, all one has to do is to construct a substitution that transforms $\underline{L}_1$ - equation into L_1 - equation. Let us proceed to the specific example.

There are two types of substitutions (somewhat different) which transform (0.14) into the **PV** equation. The substitutions are twice degenerate with respect to the sign of conservation law q (0.15). Let us introduce Lax pair for the deformation (0.14) of the **PIII** equation

$$\underline{L}_1 \Psi = 0 \quad \underline{L}_1 = \partial_\zeta + fA \quad (3.1a)$$

$$\underline{L}_2 \Psi = 0 \quad \underline{L}_2 = \partial_z + B \quad (3.1b)$$

Here

$$f = \frac{\zeta^2}{\zeta^2 + c/4} = 1 + i\sqrt{c} \left[\frac{1}{\zeta - i\sqrt{c}/2} - \frac{1}{\zeta + i\sqrt{c}/2} \right] / 4$$

$$A = i \left(z + \frac{N}{4\zeta^2} \right) \sigma_3 + \frac{iz}{2\zeta} E \sigma_2 + \frac{i}{4\zeta^2} \rho \sigma_1$$

$$B = i\zeta \sigma_3 + iE \sigma_2 / 2$$

σ_i are the Pauli matrices. We also write down the **PV** equation [3]

$$u_{xx} = u_x^2 \frac{3u-1}{2u(u-1)} - u_x \frac{1}{x} + \frac{(u-1)^2}{x^2} (\alpha u + \beta/u) + \frac{\gamma u}{x} + \frac{\delta u(u+1)}{u-1} \quad (3.2)$$

Here $\delta = -1/2$; $\alpha, \beta, \gamma \in \mathbb{C}$ are nontrivial constants that can be parameterized with the additional constants θ_0, θ_1 and θ_∞ :

$$\alpha = (\theta_0 - \theta_1 + \theta_\infty)^2 / 8$$

$$\beta = -(-\theta_0 + \theta_1 + \theta_\infty)^2 / 8$$

$$\gamma = 1 - \theta_0 - \theta_1$$

For our purpose we should present the **PV** (3.2) equation as a pair of the first order ODE for functions w and u :

$$xw_x = uw\left[w + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right] - \frac{1}{u}(w + \theta_0)\left[w + \frac{\theta_0 + \theta_1 + \theta_\infty}{2}\right] \quad (3.3a)$$

$$xu_x = \left[-2w - \frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right](u - 1)^2 + (\theta_0 + \theta_1)(u - 1) + xu \quad (3.3b)$$

The Lax representation for the **PV** is as follows [3]:

$$L_{1,2}\Psi = 0 \quad (3.4_{1,2})$$

$$L_1 = -\partial_\lambda + \frac{x}{2}\sigma_3 + \frac{u_0}{\lambda} + \frac{u_1}{\lambda - 1} \quad (3.5_1)$$

$$L_2 = -\partial_x + \frac{\lambda}{2}\sigma_3 + \frac{1}{x}(u_0 + u_1 + \frac{\theta_\infty}{2}\sigma_3) \quad (3.5_2)$$

$$u_0 = \begin{pmatrix} w + \theta_0/2 & -v(w + \theta_0) \\ w/v & -w - \theta_0/2 \end{pmatrix}$$

$$u_1 = \begin{pmatrix} -w - (\theta_0 + \theta_\infty)/2 & vu[w + (\theta_0 - \theta_1 + \theta_\infty)/2] \\ -[w + (\theta_0 + \theta_1 + \theta_\infty)/2]/vu & w + (\theta_0 + \theta_\infty)/2 \end{pmatrix}$$

Compatibility condition for (3.4_{1,2}) is a couple of equations (3.3a, 3.3b), and also an equation (in fact, a definition) for an additional function v :

$$xv_x/v = u\left[w + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right] + \frac{1}{u}\left[w + \frac{\theta_0 + \theta_1 + \theta_\infty}{2}\right] - 2w - \theta_0 \quad (3.3c)$$

Now we formulate the main result on "hidden" transformations in terms of the **PV** written down as (3.3a-3.3c).

Transformation I :

$$x = 2\sqrt{c}z \quad (3.6\pm)$$

$$u(x) = \frac{\rho + xE/2}{\rho - xE/2}$$

$$w(x) = (N - \sqrt{cx}/2 - 2\sqrt{c}\theta_0)/4\sqrt{c}$$

$$v(x) = \frac{N - \sqrt{cx}/2 - 2\sqrt{c}\theta_0}{\rho + xE/2} \exp(x/2)$$

reduces the **PV** (3.3a-3.3c) equation to the self-similar system (0.14), if

$$\theta_\infty = 0 \quad (3.7)$$

$$\theta_0 = -\theta_1 = \pm 2q \quad (3.8)$$

Transformation II

$$x = 2\sqrt{c}z \quad (3.9\pm)$$

$$u(x) = \frac{\rho + xE/2}{\rho - xE/2} \frac{N - \sqrt{cx}/2 + 2\sqrt{c}\theta_0}{N - \sqrt{cx}/2 - 2\sqrt{c}\theta_0}$$

$$w(x) = (N - \sqrt{cx}/2 - 2\sqrt{c}\theta_0)/4\sqrt{c}$$

$$v(x) = \frac{N - \sqrt{cx}/2 - 2\sqrt{c}\theta_0}{\rho + xE/2} \exp(x/2)$$

reduces the **PV** (3.3a-3.3c) equation to the self-similar system (0.14), if

$$\theta_0 = \theta_1 = \pm 2q \quad (3.10)$$

and formula (3.7) is true.

The proof is done according to the general procedure. The results mean that there are *four* possible different substitutions, transforming (0.14) into the Painlevé **PV**. The abundance of the substitutions is due to the **PV** equation being in fact a family of equations that are parameterized by nontrivial constants θ_0, θ_1 and θ_∞ . Therefore, each of the transformations (3.6 $\pm$), (3.9 $\pm$) maps (0.14) into a corresponding element of the family. If $\theta_\infty = 0, \theta_0 = -\theta_1$, then system (3.3a-3.3c) has a conservation law: $\exp(-x)uv^2(w + \theta_0)/w = A$. It defines v for given u and w . Substituting:

a) (3.6 $\pm$) into

$$A = -1 = \exp(-x)uv^2(w + \theta_0)/w \quad (3.11)$$

one automatically gets the conservation law q (0.15), if $\theta_0 = \pm 2q$.

b) expression for w (3.6 $\pm$) into w - equation and using (3.11), one finds equation for population inversion N (see (0.14))

Equations for polarization ρ and electric field E arise, if one substitutes u - expression (3.6 $\pm$) into the related equation, differentiates (3.11) by x and solves the derived linear algebraic system with respect to the derivatives $(xE)_x$ and ρ_x .

We note that (3.6 $\pm$), (3.9 $\pm$) transforms are easily inverted.

The **PV** equation defines the isomonodromic deformation of the L_1 - equation (4.3₁). The existence of the "hidden" transformation means that (0.14) system define isomonodromic deformation too. This result is also obtained directly (we omit calculations) using technique from Chapter 2 [3]. We suggest that all possible deformations of the Painlevé equations, arising in the proposed approach, are also isomonodromic.

Just before sending this paper to the journal the author was notified about paper [24], containing the algorithm analogous to the one presented here; and work [25] devoted to reduction of self-similar equations (0.7) to the **PV**. Authors of [24] also derived asymptotics of solutions of (0.7).

Appendix A.

The proof that (2.6) system has movable critical points (i.e. without the Painlevé property) is based on a traditional method presented, e.g. in [5]. According to [5], the Painlevé test is reduced to a construction of solution for a differential system near a movable pole z_0 (which is one of integration constants) as a series $\sum_n f_n(z - z_0)$. Whether there are movable critical points or not follows from analysis of functions f_n analytical structure.

We rewrite (2.6) in a more compact form, using new notations: $\tilde{c} \rightarrow f/z; c \rightarrow c/z$:

$$zc_z + 2fc = 0 \quad (A.1)$$

$$zf_{zz} + ff_z - 6\tilde{b}_0^2 zc = 0$$

First, we determine governing orders of expansions (over $\zeta = z - z_0$) for functions f and c

$$f \sim \frac{3z_0/2}{\zeta} \quad c \sim \frac{3z_0/4}{\zeta^3} \frac{1}{6\tilde{b}_0^2} \quad (A.2)$$

We assumed that $\tilde{b}_0 \neq 0$. If the opposite is true: $\tilde{b}_0 = 0$, then (A.1) system degenerates into an equation

$$zf_{zz} + ff_z = 0 \quad (A.3)$$

which is self-similar reduction for the Liouville equation:

$$\phi_{\xi\eta} = e^\phi \quad (A.4)$$

$$\phi(\xi, \eta) = \phi(z = \xi\eta) \quad e^\phi = -f_z \quad (A.5)$$

Equation (A.3) has the Painlevé property, unlike (A.1). From now on we assume that $6\tilde{b}_0^2 = 1$ (we can also achieve this by rescaling of z). Now we should continue the f -, c - expansions (A.2) until the appearance of remaining two integration constants. To determine the corresponding powers of ζ at which the constants may appear we substitute:

$$f \sim \frac{3z_0/2}{\zeta} [1 + \alpha\zeta^p] \quad (A.6)$$

$$c \sim \frac{3z_0/4}{\zeta^3} [1 + \beta \zeta^q]$$

into (A.1). We find $p = q$; $\alpha = -\beta q/3$, as well as a cubic equation for q

$$q(q-2)(2q+1) + 3 = 0 \quad (A.7)$$

Equation (A.7) has three different real roots: $\{-1, 1, 3/2\}$. The root $q = -1$ is trivial, because it relates to redefinition of z_0 . The root $q = 3/2$ unambiguously indicates that z_0 is a movable branching point for f and c . Hence, (A.1) system does not have the Painlevé property.

We present the expansion of a general solution of (A.1):

$$f = \frac{3z_0/2}{\zeta} [1 + (1/z_0 - \beta/3)\zeta - \beta_1 \zeta^{3/2}/2 + O(\zeta^2)]$$

$$c = \frac{3z_0/4}{\zeta^3} [1 + \beta \zeta + \beta_1 \zeta^{3/2} + O(\zeta^2)]$$

β, β_1, z_0 are integration constants. In the particular case $\beta_1 = 0$ we arrive at the special solution without movable branching points.

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