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COMPLEX INSTABILITY: A TYPICAL INSTABILITY IN MULTIDIMENSIONAL HAMILTONIAN SYSTEMS

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1. Introduction

Complex instability is a type of *generic* instability occurring only in multidimensional Hamiltonian systems, as important as the well known parametric instability of 1-dimensional oscillators. Its characteristics derive only from the symplectic geometry of such systems, not from a particular form of a Hamiltonian. It appears around equilibrium points or periodic orbits, the neighborhood of which cannot, due to a rotational component, be decomposed as a combination of 1-dimensional harmonic oscillators. A typical famous example of fixed point subject to complex instability is the L_4 Lagrangian point above the Routh critical reduced mass in the restricted 3-body problem.

Complex instability was described and named so by Broucke^[1] in a study of periodic orbits in the elliptic restricted 3-body problem, in which he classified all the possible instability types of periodic orbits in 3-degree of freedom Hamiltonian systems. Complex instability was further studied numerically in celestial mechanics and galactic dynamics (e.g. [2]-[9]), and also in particle accelerators problems (e.g. [10]). The author is convinced that the fundamental nature of this phenomenon implies that it has already been encountered many times in other fields.

On the mathematical side, the existence of complex instability was considered probably as trivial, so the necessity was weak to coin a particular name avoiding the circumlocution "instability resulting from the collision of eigenvalues of opposite Krein signature". However as shown by empirical numerical evidences in the previously quoted references, there are many intricacies in phase space associated with this instability, in particular near the transition to it, that needed to be also rigorously investigated. This subject has been studied by mathematical techniques only during about the last decade (e.g. [11]-[14], [9]).

2. A prototypical case: a pair of rotating harmonic oscillators

The simplest physical example where complex instability occurs is the case of two orthogonal harmonic oscillators with different frequencies ω_x and ω_y , rotating uniformly around the origin with the frequency Ω . In the case where ω_x and ω_y are equal, rotation can be formally absorbed in the oscillation frequency by a change to polar coordinates. But when $\omega_x \neq \omega_y$ this is not reducible to the 1-dimensional case.

The Hamiltonian corresponding to this system is

$$H(p_x, p_y, x, y) = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} (\omega_x^2 x^2 + \omega_y^2 y^2) + \Omega (p_x y - p_y x). \quad (1)$$

The equations of motion of this quadratic Hamiltonian are linear, so obviously they can be fully integrated^[17], yielding

$$\begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix} (t) = A(t - t_0, \omega_x^2, \omega_y^2, \Omega) \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix} (t_0), \quad (2)$$

where the 4×4 matrix A can be expressed in terms of trigonometric and algebraic functions of the arguments (Eq. [19] in [7]). The solution extends naturally when ω_x^2 or ω_y^2 are allowed to be negative.

Matrix A has the symplectic properties expected from a Hamiltonian system, i.e., it has the property

$$A^T \Gamma A = A \Gamma A^T = \Gamma, \quad \text{where} \quad \Gamma = \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix}, \quad (3)$$

and where 0_2 and 1_2 are the 2×2 null and identity matrices, respectively. Γ defines a skew-symmetric product $\vec{x}_1^T \Gamma \vec{x}_2$, where $\vec{x}_i = \{x, y, p_x, p_y\}_i$. This skew-symmetric product is invariant by the motion:

$$(A\vec{x}_1)^T \Gamma (A\vec{x}_2) = \vec{x}_1^T A^T \Gamma A \vec{x}_2 = \vec{x}_1^T \Gamma \vec{x}_2. \quad (4)$$

Choosing $\vec{x}_2 = A\vec{x}_1$ leads to the interesting property that the quadratic form

$$Q = \vec{x}^T(t) \Gamma A(t) \vec{x}(t) \quad (5)$$

is invariant by the motion (see e.g. [18]), i.e. Q is an integral of the motion expressed in the original coordinates x, y, p_x , and p_y . From this it is clear that stability or instability depends on whether Q is positive definite (i.e. sign definite) or not. When Q is positive definite, motion is confined on an ellipsoid in phase space, while otherwise it occurs on one of the various types of hyperboloids in four dimensions, so is typically unbounded.

As for any real matrix (A is real because motion is described by real numbers), the four eigenvalues (multipliers) of A , if complex, are restricted to occur in conjugate pairs. Furthermore, since the system obeys to the symplectic geometry, to every eigenvalue another one exists which is the inverse. The seven possible combinations of eigenvalues in the complex plane around the unit circle are shown in Fig. 1. They correspond to the categories of instabilities listed by Broucke^[1]. Stability corresponds to the case in the middle of Fig. 1 where all the eigenvalues are on the unit circle. Complex instability corresponds to the case at the top where all the eigenvalues are complex and outside the unit circle. Note that in the complex instability case, and only in this case, the four eigenvalues must move together as a quadruplet in the complex plane when a system parameter is changed, while in all the other cases the eigenvalues must only move as pairs to respect the symplecticity condition.

In generic higher dimensional Hamiltonian systems the eigenvalues are just a combination of these 7 cases, they occur in pairs or quadruplets. Any other restriction (say, linking 6 or more eigenvalues to a single parameter) must come from additional symmetries in the Hamiltonian.

From stability there are only three routes to instability as a system parameter is changed:

1. A pair of eigenvalues $\lambda, \bar{\lambda}$ on the unit circle ($|\lambda| = 1$) in the complex plane may bifurcate onto the real axis at $(1, 0)$ to positive $\lambda, 1/\lambda$, which corresponds to the well studied case of tangent bifurcation.
2. The same occurring at $(-1, 0)$ in the complex plane to negative $\lambda, 1/\lambda$ corresponds to the other well studied case of period doubling bifurcation.
3. The four eigenvalues collide simultaneously by conjugate pairs on the unit circle and leave it into the complex plane; this is the transition to complex instability, also called Krein collision. Then one has to distinguish the two cases when the bifurcation position on the unit circle corresponds to an angle that is rational or

irrational with respect to 2π . Among the rational cases, the angles $\pi/3$ and $\pi/4$ are particular in the sense that they correspond to strong resonances^[16].

Returning now to the problem of the pair of rotating harmonic oscillators, we can compute for which values of the coefficients ω_x^2, ω_y^2 , and Ω^2 we have stability or any other type of instability. The result is summarized in Fig. 2, which shows, among others, that complex instability occurs naturally provided at least one of the two coefficients ω_x^2 and ω_y^2 is negative. Such a case occurs at the L_4 and L_5 Lagrangian points of the restricted 3-body problem, or in rotating triaxial galaxy potentials^{[8],[15]}.

3. Parametric resonances

However, around periodic orbits in arbitrary systems it is rare that the coefficients of the variational equations are constant as in Equ. (1). Instead they are periodically modulated. A starting point to extend the results of the previous Section to periodically modulated coefficients is to examine the effect of an *infinitesimally small but periodic* perturbation of the parameters.

A nice theorem by Krein^[19] and Moser^[20] leads to the following rules for the occurrence of parametric instability in the generic case of small but finite amplitude periodic modulation of the parameters:

The solutions are marginally unstable and the system is unstable towards parametric perturbations of frequency ω either

1. when $\lambda_i = \pm 1$ (generalized Mathieu's parametric instability), or
2. when two distinct pairs of eigenvalues $(\lambda_i, \bar{\lambda}_i), (\lambda_j, \bar{\lambda}_j)$, $i \neq j$, on the unit circle collide (Krein's collision) and furthermore, by ordering the λ_i properly^[20], $\lambda_i = \bar{\lambda}_j$ (parametric complex instability). To order the eigenvalues one needs to evaluate the Lagrange's brackets of the complex valued eigenvectors^[20]. Alternatively in a more geometrical way one can also evaluate the real quadratic form Q and determine whether it is sign definite or not^[7].

The strength of the theorem stated by Krein and Moser is to allow the prediction of instability for *arbitrary* parametric perturbations by inspecting the *unperturbed* system only. When condition 1. or 2. are fulfilled, one needs very peculiar perturbations for avoiding instability.

The application of this theory to the pair of rotating harmonic oscillators (Eq. (1)) leads to the rules of parametric resonances for a parameter modulation frequency $\omega = 1$:

1. The extension of Mathieu's instability to the rotating case leads to an implicit quartic equation for integer k :

$$\left(\Omega^2 - \omega_x^2 + \frac{k^2}{4} \right) \left(\Omega^2 - \omega_y^2 + \frac{k^2}{4} \right) = k^2 \Omega^2. \quad (6)$$

2. Noting

$$\xi^2 = \frac{1}{2}(\omega_x^2 + \omega_y^2), \quad R = \sqrt{(\Omega^2 + \xi^2)^2 - (\Omega^2 - \omega_x^2)(\Omega^2 - \omega_y^2)}, \quad (7)$$

$$\omega_{\pm} = \sqrt{\Omega^2 + \xi^2 \pm R},$$

complex instability occurs by parametric resonance when either

$$\begin{aligned} \omega_+ + \omega_- = k, \quad \text{when } \omega_x^2 > \Omega^2, \quad \text{and } \omega_y^2 > \Omega^2, \quad \text{or} \\ \omega_+ - \omega_- = k, \quad \text{when } \omega_x^2 < \Omega^2, \quad \text{and } \omega_y^2 < \Omega^2, \end{aligned} \quad (8)$$

with k integer.

Typical diagrams of the location of marginal parametric resonances are shown in Fig. 3a for a weakly rotating case ($\Omega^2 = 0.01$). The classical parametric resonance curves (dashed) are intermingled with complex instability curves (solid). At higher rotation (Fig. 3b, $\Omega^2 = 1.69$) new parametric curves appear in the stable unshaded region at $\omega_x^2 < \Omega^2, \omega_y^2 < \Omega^2$.

It is straightforward to experiment numerically that an arbitrary periodic modulation with frequency $\omega = 1$ of the frequencies ω_x, ω_y or Ω widen the parametric resonance curves to finite sized regions of instability having the corresponding instability type^[7]. As in parametric resonances of 1-dimensional oscillators, at large modulation amplitudes the instability regions occupy most of the diagram: in most of the examined cases stability becomes exceptional for large amplitude parameter modulations.

4. Transition stability–complex instability

Striking phase space structures appear at the transition from stability to complex instability. Since bifurcations have universal properties, it suffices to study a few sufficiently generic cases to unveil the general rules. We have used particular 4-dimensional symplectic maps generalizing the “standard map” to study the neighborhood of the complex instability bifurcation, searching numerically the periodic orbits and invariant curves^[6].

An example of orbits near a central periodic orbit is shown in Fig. 4, where projections of orbits starting close to a periodic orbit (a point in the map) is shown shortly before, at, and after the transition from stability to complex instability. Clearly motion around a periodic orbit degenerates into planar motion at the transition.

The main results are that, as in classical pitchfork bifurcations, the bifurcating structures may be “direct” (the bifurcating structures unfold on the unstable side), or

it may be “inverse” (the bifurcating structures unfold on the stable side) as a system parameter is varied.

Contrary to the classical routes to instability (tangent or period doubling bifurcation), this transition may have, in addition to bifurcations of new families of periodic orbits, bifurcations of families of invariant curves in a map, or 2-dimensional tori in a 3D Hamiltonian system. These invariant curves correspond to the limit cycles generated by a Hopf bifurcation in the dissipative case.

If we note “rot” the rotation number corresponding to the ratio between the position angle of the eigenvalues with 2π , we can summarize the situation as follow:

1. When $\text{rot} = p/q$ is a rational number (p, q irreducible integers with $q > 2$) we find q -periodic orbits as main bifurcating structures with the following stability properties:

TABLE 1.

2. When $\text{rot} \neq p/q$ is an irrational number, we find invariant curves as main bifurcating structures with the following stability properties:

TABLE 2.

As in usual bifurcations, the stability property of the bifurcating objects may change as the control parameter is changed, and further bifurcations may occur. This has also been observed to be the case for the bifurcating invariant curves, these invariant curves may also be subject to stability change and may bifurcate, for instance by period doubling. An example of invariant curve bifurcation is shown in Fig. 5.

In the direct case the chaotic orbits originating around the central complex-unstable orbit are *confined* for a finite time. In the inverse case, no confinement occurs. Since Arnold’s diffusion in principle connects all the stochastic sea into a single web, we expect, and indeed find that the confinement is partial: the closer the bifurcation point, the longer the confinement.

To summarize, in the neighborhood of a transition from stability to complex instability, one may have coexistence of invariant curves, and confined chaotic orbits. In the direct bifurcation case the transition to chaos is limited, while in the inverse case, it is immediate. In Fig. 6 a case of coexistence of a stable invariant curve and a confined chaotic orbit is shown^[6].

5. Application: symplectic integrators

As soon as one considers multi-dimensional Hamiltonian systems one can expect complex instability to occur as often as other types of instability. Therefore the number of problems that may be related to the above discussion are extremely numerous. For instance during investigations of periodic orbits in 3-dimensional models of rotating galaxies all the ingredients are met to have complex instability, and we indeed found it frequently^{[5]–[8],[15]}.

To illustrate a different kind of application, we give an example of a problem related with the stability of symplectic integrators. Indeed it is often believed that symplectic integrators are automatically superior to other types of integrators, just because they keep the Hamiltonian character of the modelled system. However, as for any other integrator types, it is necessary to verify that a particular symplectic integrator is stable, at least in the von Neumann sense.

In some problems one may wish to integrate motion of a particle in a potential which rotates uniformly around some axis. The generalization of the leap-frog algorithm, a second-order symplectic integrator of the equations of motion, to coordinates rotating uniformly can clearly find many applications, if shown to be stable.

5.1 Non-canonical equations of motion

In the plane (x, y) , the algorithm mentioned by Hockney & Eastwood ([21], Eqs. [4-96] and [4-97], p. 113) for charges in an electromagnetic field can be translated to the one of a particle in a rotating frame of reference. The magnetic field is then replaced by the Coriolis force and the electric field by the centrifugal force plus any other scalar potential. Hockney & Eastwood use as variables the space and velocity coordinates x , y , v_x , and v_y in the rotating frame of reference. These variables are not canonical. The equations of motion for a particle in the potential Φ are

$$\begin{aligned} \dot{x} &= v_x, & \dot{v}_x &= -\Phi_x + \Omega^2 x + 2\Omega v_y, \\ \dot{y} &= v_y, & \dot{v}_y &= -\Phi_y + \Omega^2 y - 2\Omega v_x, \end{aligned} \quad (9)$$

where Φ_x and Φ_y denote the derivatives of the potential. The corresponding discretized and time symmetric second order algorithm for a time-step τ is then

$$\begin{aligned} x(t+\tau) - x(t) &= \tau v_x(t + \tfrac{1}{2}\tau), \\ y(t+\tau) - y(t) &= \tau v_y(t + \tfrac{1}{2}\tau), \\ v_x(t + \tfrac{1}{2}\tau) - v_x(t - \tfrac{1}{2}\tau) &= \tau \left\{ -\Phi_x(t) + \Omega^2 x(t) + \Omega [v_y(t + \tfrac{1}{2}\tau) + v_y(t - \tfrac{1}{2}\tau)] \right\}, \\ v_y(t + \tfrac{1}{2}\tau) - v_y(t - \tfrac{1}{2}\tau) &= \tau \left\{ -\Phi_y(t) + \Omega^2 y(t) - \Omega [v_x(t + \tfrac{1}{2}\tau) + v_x(t - \tfrac{1}{2}\tau)] \right\}. \end{aligned} \quad (10)$$

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The von Neumann stability criterion consists to linearize the difference scheme, assuming that during the time-step τ the forces are constant, and to analyse the stability of the resulting linear system (cf. [21], §4-4-2). In a space with constant forces, it is indeed reasonable to require that neighboring orbits should not separate exponentially solely due the algorithm. The calculation of the linear variations $\delta x(t + \tau)$, $\delta y(t + \tau)$, $\delta v_x(t + \tfrac{1}{2}\tau)$, and $\delta v_y(t + \tfrac{1}{2}\tau)$ of the system yields

$$\mathcal{L} \begin{pmatrix} \delta x(t+\tau) \\ \delta y(t+\tau) \\ \delta v_x(t+\tfrac{1}{2}\tau) \\ \delta v_y(t+\tfrac{1}{2}\tau) \end{pmatrix} = \mathcal{R} \begin{pmatrix} \delta x(t) \\ \delta y(t) \\ \delta v_x(t-\tfrac{1}{2}\tau) \\ \delta v_y(t-\tfrac{1}{2}\tau) \end{pmatrix}, \quad (11)$$

where $\mathcal{L}$ and $\mathcal{R}$ are simple 4×4 matrices. Inverting $\mathcal{L}$, we get the matrix $\mathcal{A} \equiv \mathcal{L}^{-1}\mathcal{R}$ which represents the linear map of the old coordinates into the new ones, after a time-step τ :

$$\mathcal{A} = \frac{1}{1 + \zeta^2} \begin{pmatrix} 1 + 2\zeta^2 & \zeta^3 & (1 - \zeta^2)\tau & 2\zeta\tau \\ -\zeta^3 & 1 + 2\zeta^2 & -2\zeta\tau & (1 - \zeta^2)\tau \\ \Omega\zeta & \Omega\zeta^2 & 1 - \zeta^2 & 2\zeta \\ -\Omega\zeta^2 & \Omega\zeta & -2\zeta & 1 - \zeta^2 \end{pmatrix}, \quad (12)$$

where $\zeta = \Omega\tau$. It can be verified that $\mathcal{A}$ has the structure of a symplectic map, so might be taken as a symplectic integrator of second order. In 4D symplectic linear maps it is not necessary to calculate the eigenvalues explicitly to determine stability or the type of instability. Instead the following easy-to-calculate quantities characterize the map stability (see e.g. [1]),

$$\alpha = -\text{Tr}(\mathcal{A}), \quad \beta = \tfrac{1}{2} [\alpha^2 - \text{Tr}(\mathcal{A}^2)], \quad \Delta = \alpha^2 - 4\beta + 8, \quad b_{\pm} = \tfrac{1}{2} (\alpha \pm \sqrt{\Delta}). \quad (13)$$

To have stability we need $|b_{\pm}| < 2$, and $\Delta > 0$. In particular when $\Delta < 0$ the map is complex unstable (and the $b_{\pm}$ are complex). Computing α , β , and Δ explicitly yields

$$\alpha = -2 \frac{2 + \zeta^2}{1 + \zeta^2}, \quad \beta = \frac{6 + 2\zeta^2 + \zeta^4}{1 + \zeta^2}, \quad \Delta = -\frac{4\zeta^6}{(1 + \zeta^2)^2}. \quad (14)$$

Since Δ is always negative, this implies that the four eigenvalues depart from the unit circle for non-zero τ and Ω . Therefore, the above algorithm is guarantied to amplify numerical errors exponentially, and is complex-unstable because the eigenvalues are all complex with a modulus different from one. Obviously, this symplectic integrator is then unsuited to be a faithful discrete representation of a system of differential equations.

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5.2 Canonical equations of motion

The use of canonical coordinates x, y, p_x , and p_y , where the momenta $p_x = v_x - \Omega y$, and $p_y = v_y + \Omega x$ are the velocities in the instantaneous inertial reference frame coaxial to the rotating one, leads to a stable algorithm. The equations of motion are:

$$\begin{aligned} \dot{x} &= p_x + \Omega y, & \dot{p}_x &= -\Phi_x + \Omega p_y, \\ \dot{y} &= p_y - \Omega x, & \dot{p}_y &= -\Phi_y - \Omega p_x. \end{aligned} \quad (15)$$

The corresponding time symmetric and second order algorithm in implicit form reads:

$$\begin{aligned} x(t+\tau) - x(t) &= \tau \left\{ p_x(t+\tfrac{1}{2}\tau) + \tfrac{1}{2}\Omega [y(t+\tau) + y(t)] \right\}, \\ y(t+\tau) - y(t) &= \tau \left\{ p_y(t+\tfrac{1}{2}\tau) - \tfrac{1}{2}\Omega [x(t+\tau) + x(t)] \right\}, \\ p_x(t+\tfrac{1}{2}\tau) - p_x(t-\tfrac{1}{2}\tau) &= \tau \left\{ -\Phi_x(t) + \tfrac{1}{2}\Omega [p_y(t+\tfrac{1}{2}\tau) + p_y(t-\tfrac{1}{2}\tau)] \right\}, \\ p_y(t+\tfrac{1}{2}\tau) - p_y(t-\tfrac{1}{2}\tau) &= \tau \left\{ -\Phi_y(t) - \tfrac{1}{2}\Omega [p_x(t+\tfrac{1}{2}\tau) + p_x(t-\tfrac{1}{2}\tau)] \right\}. \end{aligned} \quad (16)$$

Then the corresponding matrix $\mathcal{A}$ derived as above is

$$\mathcal{A} = \frac{1}{1+\xi^2} \begin{pmatrix} 1-\xi^2 & 2\xi & \frac{1-3\xi^2}{1+\xi^2}\tau & \frac{3\xi-\xi^3}{1+\xi^2}\tau \\ -2\xi & 1-\xi^2 & -\frac{3\xi-\xi^3}{1+\xi^2}\tau & \frac{1-3\xi^2}{1+\xi^2}\tau \\ 0 & 0 & 1-\xi^2 & 2\xi \\ 0 & 0 & -2\xi & 1-\xi^2 \end{pmatrix}, \quad (17)$$

where $\xi = \frac{1}{2}\Omega\tau$. The matrix $\mathcal{A}$ has also a symplectic form, and the corresponding parameters α, β, Δ , and $b_{\pm}$ are:

$$\alpha = -4 \frac{1-\xi^2}{1+\xi^2}, \quad \beta = 2 \frac{3-2\xi^2+3\xi^4}{(1+\xi^2)^2}, \quad \Delta = 0, \quad b_+ = b_- = -2 \frac{1-\xi^2}{1+\xi^2}. \quad (18)$$

For any real ξ , $-2 \leq b_{\pm} \leq 2$. Therefore, the algorithm is stable in the sense of von Neumann. Since the eigenvalues are always located on the unit circle, neighboring orbits do not separate exponentially. Since $\Delta = 0$, the algorithm is actually only marginally stable with respect to complex instability. (The same is true when we have an uncoupled pair of non-rotating leap-frog integrators, since in this case $\xi=0$ and $\tau \neq 0$.)

When the forces are not assumed to be exactly constant, but the first terms Φ_{xx} , Φ_{yy} , and Φ_{xy} are retained and assumed to be constant during τ , the above parameters are slightly modified. A more detailed analysis^[22] shows that if the potential corresponds to a locally unstable orbit in the continuous case, i.e., when $\tau \rightarrow 0$, the

algorithm produces also locally diverging neighboring trajectories, and vice versa in the stable case.

5.3 Explicit leap-frog algorithm in rotating coordinates

Instead of using $\mathcal{A}$ to find the new position as a function of the old position, we can rewrite the algorithm in a computationally more efficient form very similar to the usual leap-frog algorithm,

$$\begin{aligned} \begin{pmatrix} p_x(t+\tfrac{1}{2}\tau) \\ p_y(t+\tfrac{1}{2}\tau) \end{pmatrix} &= B \begin{pmatrix} p_x(t-\tfrac{1}{2}\tau) \\ p_y(t-\tfrac{1}{2}\tau) \end{pmatrix} + C \begin{pmatrix} -\Phi_x(t) \\ -\Phi_y(t) \end{pmatrix}, \\ \begin{pmatrix} x(t+\tau) \\ y(t+\tau) \end{pmatrix} &= B \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + C \begin{pmatrix} p_x(t+\tfrac{1}{2}\tau) \\ p_y(t+\tfrac{1}{2}\tau) \end{pmatrix}, \end{aligned} \quad (19)$$

where

$$B = \frac{1}{1+\xi^2} \begin{pmatrix} 1-\xi^2 & 2\xi \\ -2\xi & 1-\xi^2 \end{pmatrix}, \quad C = \frac{\tau}{1+\xi^2} \begin{pmatrix} 1 & \xi \\ -\xi & 1 \end{pmatrix}. \quad (20)$$

In the limit $\Omega \rightarrow 0$ ($\xi \rightarrow 0, \tau \neq 0$), this algorithm tends towards a pair of usual leap-frog integrators.

6. Conclusions

Complex instability is typical in multidimensional Hamiltonian systems. We have presently good, although probably not totally accurate understanding about the phenomenons associated to it. The transition to complex instability, corresponding to the Hopf bifurcation in the Hamiltonian case, offers fascinating and unique curiosities about the delicate parcelling of phase space between regular and chaotic motion.

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TABLES

Table 1.

q	number of families	direct bifurcation	inverse bifurcation
odd > 1	2	stable	semi-unstable
	2	semi-unstable	doubly unstable
4	1	semi-unstable	semi-unstable
	1	stable	complex unstable
even > 4	1	semi-unstable	semi-unstable
	1	stable	doubly unstable

Table 2.

	direct bifurcation	inverse bifurcation
invariant curve	stable	unstable
partial confinement	yes	no

FIGURE CAPTIONS

1. Possible types of stability-instability of the four eigenvalues in the complex plane of the linearized map around an equilibrium point (or periodic orbit) in a 2-dimensional Hamiltonian system. Only in the complex instability case all the four eigenvalues are linked together by the constraints of real motion and symplectic geometry.
2. Stability diagram ($\omega_x^2/\Omega^2, \omega_y^2/\Omega^2$) of the pair of rotating harmonic oscillators. The regions are marked by S (stability), Δ (complex instability), U (even semi-instability), and DU (even-even instability).
3. Parametric resonance diagram for a periodic perturbation of frequency $\omega = 1$. Left: $\Omega^2 = 0.01$, Right: $\Omega^2 = 1.69$. The regions of natural instability are shaded as in Fig. 2. The dashed curves indicate classical parametric resonance for $\lambda = \pm 1$. The solid curves indicate complex instability parametric resonance.
4. Transition to complex instability in a 4D map^[6] by varying slightly a single parameter. In the top frames the shown orbit is quasi-periodic, it starts and remains close to the stable central periodic orbit. In the middle frames the central periodic orbit is marginally complex-unstable and the shown orbit is nearly confined to a 2D disk. In the bottom frames the central periodic orbit is complex-unstable, and the shown orbit is chaotic, yet well confined for 20000 consequents. In this case Arnold's diffusion into the general stochastic sea is very low. Note that the size of the frames with respect to the top ones increases by a factor 25 in the middle frames, and by a factor 500 in the bottom frames.
5. Example of "period doubling" in a family of invariant curves bifurcating at the transition stability-complex instability in a particular 4D symplectic map^[6]. In the top frame 5 members of the family are shown in projection as the parameter L is changed. When the parameter L is increased above about 0.40, two invariant curves exist for each value of L , as shown in the bottom frame.
6. Example of coexistence of confined chaotic orbit and a stable invariant curve bifurcating from the transition to complex instability in a 4D symplectic map^[6].

Pfenniger Fig. 2.

Pfenniger Fig. 3 left

Pfenniger Fig. 3 (right)

Pfenniger Fig. 4

Pfenniger Fig. 5

Pfenniger Fig. 6