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Time in the Chaotic Inflation Model and Numerical Calculations

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ABSTRACT

The time variable is considered in the quantum gravity theory and calculated explicitly in the framework of the chaotic inflationary scenario where the scalar matter field has a contribution to the time variable in addition to the gravity field. The time formulated under the semiclassical approximation is a natural extension of that in the classical orbital motion. A perturbation method is introduced in terms of the scalar mass to obtain analytically solvable expressions for the time. The Wheeler-DeWitt equation is solved numerically to ensure that the semiclassical approximation is well justified. We examine the obtained time in details and find it is reasonable to consider it as time in the region where the semiclassical approximation is well justified.

1. Introduction

One of the long standing problems in quantum cosmology is that the wave function does not have time variable owing to the constraints in the framework of the Dirac quantization.^[1] Though the problem is still controversial, one of the most promising candidates for the time variables was introduced by Banks and others under the assumption that the semiclassical approximation is well justified.^{[2]–[13]} [1] They considered the gravity coupled with matter fields and wrote down a functional Schrödinger equation for the quantum matter fields, whose time variable T was defined by the semiclassical approximation to the wave function.

In the present paper we consider the gravity coupled with a background scalar field and other quantum matter fields (massless), namely quarks, leptons and gauge fields. The gravity and the background field constitute the background space-time geometry and give rise to the chaotic inflation.^[14] It is natural to suppose that the time originates not only from the gravity but also from the background field. It was shown in Ref. [15] that, when the semiclassical methods are applied to the gravity coupled with a background scalar field and quantum matter fields, a functional Schrödinger equation for the quantum matter fields can be obtained. It is, however, very difficult to calculate the time T explicitly, since the discussions were based on a general superspace.

We think it is important to make the manifest expressions for the time variable and to judge how far the variable is justified effectively under the semiclassical calculations. We take a homogeneous minisuperspace model to simplify the discussion and apply WKB approximation for the wave function to obtain the definition of the time T . Without the background scalar field the semiclassical equations become ordinary differential equations and can be solved exactly. However, it is very difficult to solve these equations exactly in the case of the gravity with a background scalar field, because they become non-linear partial differential equations. Since the background scalar field has a small mass M_ϕ in

chaotic inflation models, the perturbative expansion will be applied in terms of the mass M_y .^[16] Since it is interesting to see explicitly how the time variable plays a role in the presence of the background scalar field, we consider that the scalar mass is small but not negligible. Then, there is a possibility to solve the equations analytically and to calculate the time T explicitly. In the present paper we perform the lowest and first order perturbations in terms of M_y and calculate T explicitly in the same approximation. Since the discussions crucially depend on the semiclassical approximation, we will check the validity of this approximation by doing numerical calculations for the Wheeler-DeWitt equation of the chaotic model. We will find that the semiclassical wave functions agree with those calculated numerically. In order to see how the perturbation method works well, the explicit form of the time obtained will be plotted against the classical orbital time. Equal time lines will be also plotted against the classical paths to get a consistency check for the time variable.

The paper is organized as follows. In §2 we apply the semiclassical methods to the gravity coupled with a background scalar field and quantum matter fields. We define the time T and calculate it in the perturbative expansion. In §3 we check the semiclassical approximation by comparing the semiclassical wave functions with the numerical wave functions. In §4 we examine the obtained T in details. We conclude in §5. Appendix A describes the detailed calculation in terms of the perturbative expansion in M_y .

2. Definition of time and quantum mechanics of matter fields

2.1 The model

We will consider a chaotic inflation model which has the gravitational field $g_{\mu\nu}$, a background scalar field ϕ_b and massless quantum matter fields.

The action is written as

$$S = \int d^4x \mathcal{L} , \quad (1)$$

$$\mathcal{L} = \sqrt{-g} \left[\frac{R}{16\pi G} - \frac{g^{\mu\nu}}{2} \partial_\mu \phi_b \partial_\nu \phi_b - V_b(\phi_b) \right] + \mathcal{L}_q ,$$

where $V_b(\phi_b) = \frac{\mu^2}{2} \phi_b^2$ and $\mathcal{L}_q$ is the Lagrangean of matter fields.

As is well known in quantum cosmology the wave function is time independent because of the constraints in the Dirac theory. Banks and others applied semiclassical methods to quantum cosmology and introduced the notion of time T . They used a model of gravitational field $g_{\mu\nu}$ coupled to matter fields and defined the time T . We have a background scalar field ϕ_b which is as important as $g_{\mu\nu}$ in defining the time variable T . In Ref. [15] the definition of T is given in the general superspace. However, it is very difficult to calculate T in the general superspace. Therefore, to simplify the discussion, we will take the homogeneous minisuperspace model:

$$ds^2 = \sigma^2 [-N^2 dt^2 + a^2(t) d\Omega_3^2] , \quad (2)$$

where $\sigma^2 = \frac{2G}{3\pi}$, $d\Omega_3^2$ is the metric on the three sphere, and $\phi_b = \phi_b(t)$.

Then the action (1) reads

$$S = \int dt L , \quad (3)$$

$$L = \frac{N}{2} \left[-\frac{a^2 \dot{a}^2}{N^2} + \frac{a^3 \dot{y}^2}{N^2} + a - a^3 V_y(y) \right] + L_q ,$$

where L_q is the Lagrangean of quantum matter fields,

$$y = \sqrt{\frac{4\pi G}{3}} \phi_b , \quad V_y(y) = \left(\frac{4\pi G}{3}\right)^2 V_b(\phi_b) = M_y^2 y^2 = \frac{m_y^2}{2} y^2 , \quad (4)$$

$$M_y = \frac{m_y}{\sqrt{2}} = \sigma \mu_b , \quad \dot{a} = \frac{da}{dt} \quad \text{and} \quad \dot{y} = \frac{dy}{dt} .$$

The momenta conjugate to a, y can be written as

$$p_a = \frac{\partial L}{\partial \dot{a}} = -\frac{a\dot{a}}{N}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = \frac{a^3 \dot{y}}{N}, \quad (5)$$

respectively. The Hamiltonian H and Hamiltonian constraint $\mathcal{H}$ are

$$H = \frac{N}{2a} \mathcal{H}, \quad (6)$$

$$\mathcal{H} = -p_a^2 + \frac{p_y^2}{a^2} - a^2 + a^4 V_y(y) + \mathcal{H}_q \approx 0.$$

Here $\mathcal{H}_q$ is the Hamiltonian constraint of the quantum matter fields.

When the quantum matter fields are small enough and can be neglected, classical equations can be written as

$$\dot{x}^2 - \dot{y}^2 = M_y^2 y^2 - e^{-2x}, \quad (7)$$

$$\ddot{y} + 3\dot{x}\dot{y} + M_y^2 y = 0,$$

where we have set

$$N = 1 \quad \text{and} \quad x = \log a. \quad (8)$$

2.2 Wheeler-DeWitt equation

The Hamiltonian constraint (6) is quantized to yield the Wheeler-DeWitt equation:

$$\left\{ \frac{\hbar^2}{a^p} \frac{\partial}{\partial a} a^p \frac{\partial}{\partial a} - \frac{\hbar^2}{a^2} \frac{\partial^2}{\partial y^2} - a^2 [1 - a^2 V_y(y)] + 2a \mathcal{H}_z \right\} \Psi(a, y, z) = 0. \quad (9)$$

Here p is a parameter of the operator-ordering, z represents the homogeneous quantum matter fields and $\mathcal{H}_z = \frac{1}{2a} \mathcal{H}_q$. If we take $\hbar = 1$ and use Eqs. (4),(8),(9),

then we have

$$\left[\frac{\partial^2}{\partial x^2} - (1-p) \frac{\partial}{\partial x} - \frac{\partial^2}{\partial y^2} - e^{4x} (1 - e^{2x} M_y^2 y^2) + 2e^{3x} \mathcal{H}_z \right] \Psi(x, y, z) = 0. \quad (10)$$

We take a small scalar mass M_y , $M_y \ll 1$, and a large initial value of y as in the inflational scenario. Let us use a new variable,

$$\hat{y} = M_y y, \quad (11)$$

and suppose that $\hat{y}$ takes a magnitude of order 1 in the region we are going to treat. The Wheeler-DeWitt equation (10) becomes

$$\left[\frac{\partial^2}{\partial x^2} - (1-p) \frac{\partial}{\partial x} - M_y^2 \frac{\partial^2}{\partial \hat{y}^2} - U(x, \hat{y}) + 2e^{3x} \mathcal{H}_z \right] \Psi(x, \hat{y}, z) = 0, \quad (12)$$

with $U(x, \hat{y}) = e^{4x} (1 - e^{2x} \hat{y}^2)$. Later we will take a perturbative expansion in this small mass M_y for the second derivative term of $\hat{y}$.

2.3 Semiclassical (WKB) approximation and the definition of time

In order to look for semiclassical solutions to the Wheeler-DeWitt equation (12), write

$$\Psi(x, \hat{y}, z) = e^{iS_0(x, \hat{y})} \Phi(x, \hat{y}, z). \quad (13)$$

If we revive $\hbar$ here, then S_0 in Eq. (13) is replaced with $\frac{S_0}{\hbar}$, and each partial derivative in Eq. (12) is multiplied by $\hbar$. We regard the energy of the quantum matter field is the order of $\hbar\nu$. When we substitute Eq. (13) into Eq. (12), we obtain

$$-\left(\frac{\partial S_0}{\partial x}\right)^2 + M_y^2 \left(\frac{\partial S_0}{\partial \hat{y}}\right)^2 - U(x, \hat{y}) = 0, \quad (14)$$

$$i \frac{\partial^2 S_0}{\partial x^2} \Phi - i(1-p) \frac{\partial S_0}{\partial x} \Phi + 2i \frac{\partial S_0}{\partial x} \frac{\partial \Phi}{\partial x} - i M_y^2 \frac{\partial^2 S_0}{\partial \hat{y}^2} \Phi - 2i M_y^2 \frac{\partial S_0}{\partial \hat{y}} \frac{\partial \Phi}{\partial \hat{y}} + 2e^{3x} \mathcal{H}_z \Phi = 0, \quad (15)$$

where $\hbar$ is suppressed again.

Eq. (14) is the Hamilton-Jacobi equation for the gravity coupled to the background scalar field $\hat{y}$, and S_0 is a solution of this equation. We denote the solution Φ of Eq. (15) as Φ_0 , when there is no quantum matter field. Namely

$$\frac{\partial^2 S_0}{\partial x^2} \Phi_0 - (1-p) \frac{\partial S_0}{\partial x} \Phi_0 + 2 \frac{\partial S_0}{\partial x} \frac{\partial \Phi_0}{\partial x} - M_y^2 \frac{\partial^2 S_0}{\partial \hat{y}^2} \Phi_0 - 2M_y^2 \frac{\partial S_0}{\partial \hat{y}} \frac{\partial \Phi_0}{\partial \hat{y}} = 0. \quad (16)$$

Let us write

$$\Psi(x, \hat{y}, z) = e^{iS_0(x, \hat{y})} \Phi_0(x, \hat{y}) \psi(x, \hat{y}, z). \quad (17)$$

Using Eqs. (15)-(17), we obtain

$$2i \frac{\partial S_0}{\partial x} \frac{\partial \psi}{\partial x} - 2i M_y^2 \frac{\partial S_0}{\partial \hat{y}} \frac{\partial \psi}{\partial \hat{y}} + 2e^{3x} \mathcal{H}_z \psi = 0. \quad (18)$$

If we define $T(x, \hat{y})$ and $X(x, \hat{y})$ as

$$\begin{aligned} -\frac{1}{e^{3x}} \frac{\partial S_0}{\partial x} \frac{\partial T}{\partial x} + \frac{M_y^2}{e^{3x}} \frac{\partial S_0}{\partial \hat{y}} \frac{\partial T}{\partial \hat{y}} &= 1, \\ -\frac{1}{e^{3x}} \frac{\partial S_0}{\partial x} \frac{\partial X}{\partial x} + \frac{M_y^2}{e^{3x}} \frac{\partial S_0}{\partial \hat{y}} \frac{\partial X}{\partial \hat{y}} &= 0, \end{aligned} \quad (19)$$

then we can write Eq. (18) as

$$i \frac{\partial \psi}{\partial T} = \mathcal{H}_z \psi, \quad (20)$$

by changing variables $x, \hat{y}$ to T, X . This is a Schrödinger equation for the quantum matter fields in terms of the time variable T .

2.4 Expansion in M_y

In order to obtain the explicit form of T , we must solve Eqs. (14) and (19). However, Eq. (14) is a nonlinear partial differential equation and is difficult to be solved exactly.

Since $M_y \ll 1$, we can make a perturbative expansion in terms of M_y :

$$\begin{aligned} S_0 &= S_0^{(0)} + S_0^{(1)} + \dots, \\ \Phi_0 &= \Phi_0^{(0)} + \Phi_0^{(1)} + \dots, \\ T &= T^{(0)} + T^{(1)} + \dots, \end{aligned} \quad (21)$$

where $S_0^{(0)}$, $\Phi_0^{(0)}$, $T^{(0)}$ are in the order of M_y^0 (0-th approximation), and $S_0^{(1)}$, $\Phi_0^{(1)}$, $T^{(1)}$ are in the order of M_y^2 (1-st approximation). The calculation of these quantities is given in the appendix A, and the results are the following.

$$\begin{aligned} S_0^{(0)} &= \frac{\epsilon}{3\hat{y}^2} (\alpha^2 - 1)^{\frac{3}{2}}, \\ S_0^{(1)} &= \frac{\epsilon M_y^2}{18\hat{y}^4} \left[\sqrt{\alpha^2 - 1} \left(\frac{\alpha^2}{3} + \frac{11}{3} - \frac{2}{\alpha^2} \right) + 2 \sin^{-1} \frac{1}{\alpha} \right], \\ \Phi_0^{(0)} &= c_\phi(\hat{y}) \left(\frac{\alpha}{\hat{y}} \right)^{-\frac{p+1}{2}} (\alpha^2 - 1)^{-\frac{1}{4}}, \\ \Phi_0^{(1)} &= c_\phi(\hat{y}) \left(\frac{\alpha}{\hat{y}} \right)^{-\frac{p+1}{2}} (\alpha^2 - 1)^{-\frac{1}{4}} \times \\ &\quad \times \left\{ -\frac{M_y^2}{\hat{y}^2} \left[\frac{1}{6} \log \alpha - \frac{7}{18\alpha^2} + \frac{1}{9\alpha^4} - \frac{\hat{y} c'_\phi(\hat{y})}{3 c_\phi(\hat{y})} \left(\log \alpha - \frac{1}{\alpha^2} \right) + c_0(\hat{y}) \right] \right\}, \\ T^{(0)} &= -\frac{\epsilon}{\hat{y}} \log |\alpha + \sqrt{\alpha^2 - 1}|, \\ T^{(1)} &= -\frac{\epsilon M_y^2}{18\hat{y}^3} \left[-\frac{2}{3\alpha} \left(1 + \frac{2}{\alpha^2} \right) \sqrt{\alpha^2 - 1} + \left(5 + \frac{6}{\alpha^2} \right) \log |\alpha + \sqrt{\alpha^2 - 1}| \right. \\ &\quad \left. - 6I(\alpha) \right]. \end{aligned} \quad (22)$$

Here

$$\begin{aligned} \epsilon &= \pm 1, \quad \alpha = e^x \hat{y}, \\ I(\alpha) &= \int d\alpha \frac{1}{\alpha} \log |\alpha + \sqrt{\alpha^2 - 1}|, \quad c'_\phi(\hat{y}) = \frac{dc_\phi(\hat{y})}{d\hat{y}}, \end{aligned} \quad (23)$$

and $c_\phi(\hat{y})$, $c_0(\hat{y})$ are arbitrary functions of $\hat{y}$. Our calculations crucially depend on the semiclassical approximation. So we will check the validity of it in the next section before discussing these results.

3. The WKB solution and a numerical solution to the Wheeler-DeWitt equation

In this section we calculate a numerical solution to the Wheeler-DeWitt equation and compare it with the WKB solution. It is expected that they agree well with each other under the good semiclassical approximation. Since the quantum matter fields are decoupled in the definition of the time T , we take the following Wheeler-DeWitt equation:

$$\left[\frac{\partial^2}{\partial x^2} - (1-p) \frac{\partial}{\partial x} - M_y^2 \frac{\partial^2}{\partial \hat{y}^2} - U(x, \hat{y}) \right] \Psi(x, \hat{y}) = 0, \quad (24)$$

where $U(x, \hat{y})$ is given in Eq. (12). In order to make the comparison evident, we take the same initial condition for the numerical solution as that for the WKB solution as follows.

3.1 Initial condition

From Eqs. (17) and (22), the 0-th approximation of the WKB solution to Eq. (24) can be derived as

$$\begin{aligned} \Psi &= \Phi_0^{(0)} e^{i s_0^{(0)}}, \\ &= c_\phi(\hat{y}) \left(\frac{\alpha}{\hat{y}} \right)^{-\frac{p+1}{2}} (\alpha^2 - 1)^{-\frac{1}{4}} \text{Exp} \left[\frac{\epsilon i}{3 \hat{y}^2} (\alpha^2 - 1)^{\frac{3}{2}} \right]. \end{aligned} \quad (25)$$

Let us take the initial condition with Gaussian distribution:

$$\begin{aligned} \Psi \Big|_{x=0} &= A(\hat{y}), \\ A(\hat{y}) &= \frac{1}{\sqrt{2\pi}\sigma_A} \text{Exp} \left[-\frac{(\hat{y} - \hat{y}_0)^2}{2\sigma_A^2} \right]. \end{aligned} \quad (26)$$

This means

$$c_\phi(\hat{y}) = A(\hat{y}) (\hat{y}^2 - 1)^{\frac{1}{4}} \text{Exp} \left[-\frac{\epsilon i}{3 \hat{y}^2} (\hat{y}^2 - 1)^{\frac{3}{2}} \right], \quad (27)$$

and

$$\begin{aligned} \Psi(x, \hat{y}) &= A(\hat{y}) (\alpha^2 - 1)^{-\frac{1}{4}} (\hat{y}^2 - 1)^{\frac{1}{4}} \left(\frac{\alpha}{\hat{y}} \right)^{-\frac{p+1}{2}} \times \\ &\times \text{Exp} \left\{ \frac{\epsilon i}{3 \hat{y}^2} \left[(\alpha^2 - 1)^{\frac{3}{2}} - (\hat{y}^2 - 1)^{\frac{3}{2}} \right] \right\}. \end{aligned} \quad (28)$$

Therefore,

$$\frac{\partial \Psi}{\partial x} \Big|_{x=0} = A(\hat{y}) \left[-\frac{\hat{y}^2}{2} (\hat{y}^2 - 1)^{-1} - \frac{p+1}{2} + \epsilon i (\hat{y}^2 - 1)^{\frac{1}{2}} \right]. \quad (29)$$

Eqs. (26) and (29) are the initial condition for the WKB solution (25). We adopt the same initial condition for the numerical solution to the Wheeler-DeWitt equation (24).

3.2 Numerical calculation

We numerically solve the Wheeler-DeWitt equation (24) under the initial conditions (26) and (29), where we set $p = -1$ and $\epsilon = 1$. The Wheeler-DeWitt equation (24) is a hyperbolic second-order partial differential equation. The method of integrating along characteristics, which is called the method of characteristics, is most accurate for a numerical calculation using computer. We use this method for the present numerical calculation.

The characteristics of the Wheeler-DeWitt equation (24)

$$y = \pm x + \text{constant}$$

gives a natural rectangular grid on which a numerical solution will be computed. We take the grid with constant interval Δ . The values of $\frac{\partial \Psi}{\partial x}$ and $\frac{\partial \Psi}{\partial y}$ are given by integrating the ordinary differential equations along the characteristic curves,

starting from the initial values (29). The value of Ψ on a grid is also given by integration of the differentials $\frac{\partial\Psi}{\partial x}$ and $\frac{\partial\Psi}{\partial y}$ along the characteristics,

$$d\Psi = \frac{\partial\Psi}{\partial x}dx + \frac{\partial\Psi}{\partial y}dy ,$$

starting from the initial values (26). A solution of the Wheeler-DeWitt equation is uniquely determined in the triangular region bounded by the initial curve $x = 0$ and the characteristic curves $y = -x + N_{min}\Delta$ and $y = x + N_{max}\Delta$. The direction of propagation of concern is upward in the x axis. The region corresponds to the future light-cone starting from the interval on which the initial conditions are set.

The numerical results depend on the grid size Δ , which should be chosen sufficiently small in order to get an enough accuracy. So, it is very difficult to carry out a numerical calculation far away from the initial curve, which needs huge memory size and long CPU time of a computer. Acceptable grid size depends on the initial conditions, i.e. the values of M_y, σ_A and $\hat{y}_0$. We estimate the reasonable grid size by recalculating the whole process by varying grid sizes. We think that the present numerical results shown in Figs. 1,2 have an accuracy sufficient for comparing with the WKB solutions. As are shown in Figs. 1,2, the numerical and WKB solutions agree well with each other. It is mentioned here that to get the solution for $x > 1.5$ is very difficult, because the wave function oscillates rapidly as x increases and its wave length becomes smaller than the grid size.

Fig. 1

Fig. 2

4. Discussions on the time T

In this section we want to give a consistency check for the time T obtained in the previous sections. If the semiclassical approximation is well justified, it is supposed that the wave function is peaked around the region where the action is stationary. In this region, the momenta and coordinates have a correlation of classical Hamilton-Jacobi relations,

$$p_a = \frac{\partial S_0}{\partial a} = \frac{1}{e^x} \frac{\partial S_0}{\partial x} , \quad p_y = \frac{\partial S_0}{\partial y} = M_y \frac{\partial S_0}{\partial \hat{y}} . \quad (30)$$

These relations were examined in the actual example in Refs. [3], [4]. If we assume these relations, we obtain classical equations,

$$\frac{\partial S_0}{\partial x} = -\frac{e^{3x}}{N} \dot{x} , \quad \frac{\partial S_0}{\partial \hat{y}} = \frac{e^{3x}}{M_y^2 N} \dot{\hat{y}} , \quad (31)$$

by using the definition of momenta (5). These equations and the definition (19) of T yield

$$\frac{1}{N} \frac{dT}{dt} = \frac{\dot{x}}{N} \frac{\partial T}{\partial x} + \frac{\dot{\hat{y}}}{N} \frac{\partial T}{\partial \hat{y}} = -\frac{1}{e^{3x}} \frac{\partial S_0}{\partial x} \frac{\partial T}{\partial x} + \frac{M_y^2}{e^{3x}} \frac{\partial S_0}{\partial \hat{y}} \frac{\partial T}{\partial \hat{y}} = 1 . \quad (32)$$

This means that

$$T = Nt + constant , \quad (33)$$

which is nothing but the proper time Nt . It is interesting to show how far Eq. (33) is satisfied, if we use the time T in the lowest and next order perturbations in M_y . In this section we choose

$$\epsilon = -1 , \quad (34)$$

because the time T should be increasing function of $x = \log a$. This choice corresponds to the out-going mode in Vilenkin's initial condition.^[17] (We can see this from the form of $S_0^{(0)}$ in Eq. (22) and p_a in Eq. (5).)

Let us express $T^{(0)}(x, y)$ and $T^{(1)}(x, y)$ in terms of t by inserting the classical orbit $x(t), y(t)$ which are solved numerically in Eq.(7) with (8). In Fig. 3, we compare $T^{(0)}$ and $T^{(0)} + T^{(1)}$ to t in a classical orbit and find $T^{(0)} + T^{(1)}$ is closer to t than $T^{(0)}$. This suggests that our perturbative expansion in M_y is reasonable.

Fig. 3

In order for us to be able to regard T as time, classical orbits must cross the equal-time line, $T = \text{constant}$, once and only once.^[8] Fig. 4 shows that there is no problem to regard T as time in the region where the semiclassical approximation is well justified.

Fig. 4

5. Conclusions

A manifest expression for time variable is given in the framework of a homogeneous minisuperspace model with a background scalar field. The time variable is defined in the semiclassical approximation and expressed by a contribution from the background scalar field in addition to the gravity.

The analytic form of the time variable is derived by making a perturbative expansion in terms of the mass of the background scalar field. To justify the formalism, the following three items are examined.

(1) The wave function in the semiclassical approximation is compared with the results solved numerically for the Wheeler-DeWitt equation. It is found that both are close enough with each other in the case of small M_y .

(2) The time variable T is compared with the time of the classical orbit under the assumption that the orbit has a dominant contribution to the wave function in the semiclassical system. It is found that the T is close to the classical time and the perturbative expansion by M_y works well.

(3) Equal time curve $T(x, y) = \text{constant}$ is plotted against the classical orbits in the x, y plane. It is found that the classical orbits cross an equal time curve once and only once. This fact will ensure the consistency condition for the time variables.

The future task will be to see what a role the time variable T has in the exact wave function of the Wheeler-DeWitt equation. To study it, first we have to solve the variables T, X as accurate as possible. Second, it is desirable to obtain the wave function in a more extended region than that given in the present paper.

APPENDIX A Perturbative expansion in M_y

Substituting Eqs. (21) into Eqs. (14), (16), (19) and equating powers of M_y , we obtain

$$-\left(\frac{\partial S_0^{(0)}}{\partial x}\right)^2 - U(x, \hat{y}) = 0, \quad (A1)$$

$$-2\frac{\partial S_0^{(0)}}{\partial x}\frac{\partial S_0^{(1)}}{\partial x} + M_y^2\left(\frac{\partial S_0^{(0)}}{\partial \hat{y}}\right)^2 = 0, \quad (A2)$$

$$\frac{\partial^2 S_0^{(0)}}{\partial x^2}\Phi_0^{(0)} - (1-p)\frac{\partial S_0^{(0)}}{\partial x}\Phi_0^{(0)} + 2\frac{\partial S_0^{(0)}}{\partial x}\frac{\partial \Phi_0^{(0)}}{\partial x} = 0, \quad (A3)$$

$$\begin{aligned} 2\frac{\partial S_0^{(0)}}{\partial x}\frac{\partial \Phi_0^{(1)}}{\partial x} + \left[\frac{\partial^2 S_0^{(0)}}{\partial x^2} - (1-p)\frac{\partial S_0^{(0)}}{\partial x}\right]\Phi_0^{(1)} \\ + \frac{\partial^2 S_0^{(1)}}{\partial x^2}\Phi_0^{(0)} + \frac{\partial S_0^{(1)}}{\partial x}\left[-(1-p)\Phi_0^{(0)} + 2\frac{\partial \Phi_0^{(0)}}{\partial x}\right] \\ - M_y^2\frac{\partial^2 S_0^{(0)}}{\partial \hat{y}^2}\Phi_0^{(0)} - 2M_y^2\frac{\partial S_0^{(0)}}{\partial \hat{y}}\frac{\partial \Phi_0^{(0)}}{\partial \hat{y}} = 0, \end{aligned} \quad (A4)$$

$$-\frac{1}{e^{3x}}\frac{\partial S_0^{(0)}}{\partial x}\frac{\partial T^{(0)}}{\partial x} = 1, \quad (A5)$$

$$-\frac{1}{e^{3x}}\frac{\partial S_0^{(0)}}{\partial x}\frac{\partial T^{(1)}}{\partial x} - \frac{1}{e^{3x}}\frac{\partial S_0^{(1)}}{\partial x}\frac{\partial T^{(0)}}{\partial x} + \frac{M_y^2}{e^{3x}}\frac{\partial S_0^{(0)}}{\partial \hat{y}}\frac{\partial T^{(0)}}{\partial \hat{y}} = 0. \quad (A6)$$

A.1 0-th approximation

From Eq. (A1) we obtain

$$\frac{\partial S_0^{(0)}}{\partial x} = \epsilon \frac{\alpha^2}{\hat{y}^2} \sqrt{\alpha^2 - 1}, \quad (A7)$$

with $\epsilon = \pm 1$, $\alpha = e^x \hat{y}$. Integration of (A7) with respect to x gives

$$S_0^{(0)} = \frac{\epsilon}{3\hat{y}^2}(\alpha^2 - 1)^{\frac{3}{2}} + c_s(\hat{y}), \quad (A8)$$

where $c_s(\hat{y})$ is a "constant" of integration. As the phase of Φ_0 has an ambiguity

in the definition (17), we renormalize $c_s(\hat{y})$ into $\Phi_0^{(0)}$. Then we have

$$S_0^{(0)} = \frac{\epsilon}{3\hat{y}^2}(\alpha^2 - 1)^{\frac{3}{2}}. \quad (A9)$$

Differentiation of (A9) with respect to $\hat{y}$ yields

$$\frac{\partial S_0^{(0)}}{\partial \hat{y}} = \frac{\epsilon \alpha^2}{3\hat{y}^3} \sqrt{\alpha^2 - 1} \left(1 + \frac{2}{\alpha^2}\right). \quad (A10)$$

Eq. (A3) can be rewritten as

$$\frac{1}{e^{(p-1)x}\Phi_0^{(0)}} \frac{\partial}{\partial x} \left[e^{(p-1)x} \frac{\partial S_0^{(0)}}{\partial x} (\Phi_0^{(0)})^2 \right] = 0.$$

This equation has a solution

$$\Phi_0^{(0)} = c_\phi(\hat{y}) \left[\epsilon e^{(p-1)x} \frac{\partial S_0^{(0)}}{\partial x} \right]^{-\frac{1}{2}},$$

where $c_\phi(\hat{y})$ is an arbitrary function of $\hat{y}$. This and Eq. (A7) give

$$\Phi_0^{(0)} = c_\phi(\hat{y}) \left(\frac{\alpha}{\hat{y}} \right)^{-\frac{p+1}{2}x} (\alpha^2 - 1)^{-\frac{1}{4}}. \quad (A11)$$

Differentiating (A11) by x and $\hat{y}$, we obtain

$$\begin{aligned} \frac{\partial \Phi_0^{(0)}}{\partial x} &= \Phi_0^{(0)} \left[-\frac{\alpha^2}{2(\alpha^2 - 1)} - \frac{p+1}{2} \right], \\ \frac{\partial \Phi_0^{(0)}}{\partial \hat{y}} &= \Phi_0^{(0)} \left[-\frac{\alpha^2}{2\hat{y}(\alpha^2 - 1)} + \frac{c'_\phi(\hat{y})}{c_\phi(\hat{y})} \right], \end{aligned} \quad (A12)$$

with $c'_\phi(\hat{y}) = \frac{dc_\phi(\hat{y})}{d\hat{y}}$.

Eqs. (A5) and (A7) mean

$$\frac{\partial T^{(0)}}{\partial x} = -\frac{e^{3x}}{\frac{\partial S_0^{(0)}}{\partial x}} = -\epsilon \frac{\alpha}{\hat{y}\sqrt{\alpha^2-1}}. \quad (A13)$$

Integrating (A13) with respect to x , we get

$$T^{(0)} = -\frac{\epsilon}{\hat{y}} \log|\alpha + \sqrt{\alpha^2-1}| + c_T(\hat{y}), \quad (A14)$$

where $c_T(\hat{y})$ is any function of $\hat{y}$. Let us adjust $T^{(0)}$ such that $T^{(0)} = 0$ at $\alpha = 1$, and this means that $c_T(\hat{y}) = 0$. Then we have

$$T^{(0)} = -\frac{\epsilon}{\hat{y}} \log|\alpha + \sqrt{\alpha^2-1}|. \quad (A15)$$

Differentiation of (A15) by $\hat{y}$ gives

$$\frac{\partial T^{(0)}}{\partial \hat{y}} = -\frac{\epsilon}{\hat{y}^2} \left(\frac{\alpha}{\sqrt{\alpha^2-1}} - \log|\alpha + \sqrt{\alpha^2-1}| \right). \quad (A16)$$

A.2 1-st approximation

Substituting Eqs. (A7) and (A10) into Eq. (A2), we obtain

$$\frac{\partial S_0^{(1)}}{\partial x} = \frac{\epsilon M_y^2 \alpha^2}{18 \hat{y}^4} \sqrt{\alpha^2-1} \left(1 + \frac{2}{\alpha^2} \right). \quad (A17)$$

Integration of Eq. (A17) with respect to x yields

$$S_0^{(1)} = \frac{\epsilon M_y^2}{18 \hat{y}^4} \left[\sqrt{\alpha^2-1} \left(\frac{\alpha^2}{3} + \frac{11}{3} - \frac{2}{\alpha^2} \right) + 2 \sin^{-1} \frac{1}{\alpha} \right]. \quad (A18)$$

So as to rearrange Eq. (A4) for $\Phi_0^{(1)}$, we need following equations:

$$\begin{aligned} \frac{\partial^2 S_0^{(0)}}{\partial x^2} &= \frac{\epsilon \alpha^2 (3\alpha^2 - 2)}{\hat{y}^2 \sqrt{\alpha^2-1}}, \\ \frac{\partial^2 S_0^{(0)}}{\partial \hat{y}^2} &= -\frac{\epsilon (\alpha^2 - 2)}{\hat{y}^4 \sqrt{\alpha^2-1}}, \\ \frac{\partial^2 S_0^{(1)}}{\partial x^2} &= \frac{\epsilon M_y^2 (\alpha^2 + 2)}{18 \hat{y}^4 \sqrt{\alpha^2-1}} \left(3\alpha^2 - 4 + \frac{4}{\alpha^2} \right), \end{aligned} \quad (A19)$$

which can be calculated from Eqs. (A7), (A10) and (A17). With Eqs. (A4), (A7), (A10)-(A12), (A17) and (A19), we can derive easily

$$\begin{aligned} \frac{\partial \Phi_0^{(1)}}{\partial x} + P \Phi_0^{(1)} + Q &= 0, \\ P &= \frac{3\alpha^2 - 2}{2(\alpha^2 - 1)} - \frac{1-p}{2}, \\ Q &= \frac{M_y^2}{\hat{y}^2} \Phi_0^{(0)} \left[\frac{3\alpha^4 + 14\alpha^2 - 8}{18\alpha^4} - \frac{\hat{y}}{3} \left(1 + \frac{2}{\alpha^2} \right) \frac{c'_\phi(\hat{y})}{c_\phi(\hat{y})} \right]. \end{aligned} \quad (A20)$$

Since the general solution of $\frac{df(x)}{dx} + P(x)f(x) + Q(x) = 0$ is $f(x) = e^{-\int dx P} \left(-\int dx Q e^{\int dx P} + C \right)$, where C is an integration constant, we can obtain a solution of Eq. (A20) after a short calculation:

$$\begin{aligned} \Phi_0^{(1)} &= c_\phi(\hat{y}) \left(\frac{\alpha}{\hat{y}} \right)^{-\frac{p+1}{2}} (\alpha^2 - 1)^{-\frac{1}{4}} \times \\ &\times \left\{ -\frac{M_y^2}{\hat{y}^2} \left[\frac{1}{6} \log \alpha - \frac{7}{18\alpha^2} + \frac{1}{9\alpha^4} - \frac{\hat{y}}{3} \frac{c'_\phi(\hat{y})}{c_\phi(\hat{y})} \left(\log \alpha - \frac{1}{\alpha^2} \right) + c_0(\hat{y}) \right] \right\}, \end{aligned} \quad (A21)$$

where $c_0(\hat{y})$ is an arbitrary function of $\hat{y}$.

Substituting Eqs. (A7), (A10), (A13), (A16) and (A17) into Eq. (A6), we have

$$\frac{\partial T^{(1)}}{\partial x} = -\frac{\epsilon M_y^2}{18 \hat{y}^3} \left(1 + \frac{2}{\alpha^2} \right) \left[\frac{1}{\sqrt{\alpha^2-1}} \left(5\alpha - \frac{2}{\alpha} \right) - 6 \log|\alpha + \sqrt{\alpha^2-1}| \right]. \quad (A22)$$

When we integrate Eq. (A22) with respect to x , we can easily calculate the following equation.

$$\begin{aligned} T^{(1)} &= -\frac{\epsilon M_y^2}{18 \hat{y}^3} \left[-\frac{2}{3\alpha} \left(1 + \frac{2}{\alpha^2} \right) \sqrt{\alpha^2-1} + \left(5 + \frac{6}{\alpha^2} \right) \log|\alpha + \sqrt{\alpha^2-1}| \right. \\ &\quad \left. - 6I(\alpha) \right]. \end{aligned} \quad (A23)$$

Here $I(\alpha) = \int d\alpha \frac{1}{\alpha} \log|\alpha + \sqrt{\alpha^2-1}|$ and this can not be expressed by elementary functions.

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FIGURE CAPTIONS

1. Comparison between the WKB solution and a numerical solution. Values of the wave function Ψ multiplied by M_y are shown along $y = 5.0$ with an initial condition:

$$m_y = \sqrt{2}M_y = 0.5, \quad y_0 = \frac{\hat{y}_0}{M_y} = 5.0, \quad \sigma_A = 0.5M_y.$$

We have taken $\Delta = 0.001$. Solid lines denote the WKB solutions, and dashed lines denote numerical ones.

2. Comparison between the WKB solution and a numerical solution. Values of the wave function Ψ multiplied by M_y are shown along $y = 5.5$ with the same initial condition as Fig. 1.
3. Comparison between the classical time t and T in a classical orbit. The orbit is a solution of Eqs. (7) with an initial condition: $x(0) = 0$, $y(0) = 2\sqrt{2}$, $\dot{y}(0) = 0$ and $m_y = 0.5$. In this figure the line represents the classical time t , $\diamond$ represents $T^{(0)}$ and $+$ represents $T^{(0)} + T^{(1)}$. The calculation error of T is less than 10^{-2} .
4. Solid lines represent equal-time lines, $T^{(0)} = \text{constant}$, and dashed lines represent classical orbits. These orbits are obtained by solving Eqs. (7) with an initial condition: $x(0) = 0$, $\hat{y}(0) = \hat{y}_0$, $\dot{\hat{y}}(0) = -\frac{M_y^2}{3}$ and $M_y = \frac{\sqrt{2}}{4}$. Three dashed lines correspond to $\hat{y}_0 = 4M_y, 5M_y, 6M_y$, respectively. We can see that classical orbits intersect each equal-time line only once.

Fig. 1

Fig. 2

Fig. 3

Fig. 4