EQUIVALENCE THEOREM AND PROBING THE
ELECTROWEAK SYMMETRY BREAKING SECTOR

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Abstract

We examine the Lorentz non-invariance ambiguity in longitudinal weak-boson scatterings and the precise conditions for the validity of the Equivalence Theorem (ET). Safe Lorentz frames for applying the ET are defined, and the intrinsic connection between the longitudinal weak-boson scatterings and probing the symmetry breaking sector is analyzed. A universal precise formulation of the ET is presented for both the Standard Model and the Chiral Lagrangian formulated Electro-Weak Theories. It is shown that in electroweak theories with strongly interacting symmetry breaking sector, the longitudinal weak-boson scattering amplitude under proper conditions can be replaced by the corresponding Goldstone-boson scattering amplitude in which all the internal weak-boson lines and fermion loops are ignored.

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1. Introduction

The electroweak gauge symmetry is spontaneously broken. As a consequence of absorbing the corresponding spin-0 would-be Goldstone bosons (GB), the spin-1 weak-bosons acquire masses and their longitudinal components \( V_L^\alpha (= \omega^0_L, \omega^0_T ) \) become physical degrees of freedom. While the transverse components \( V_T^\alpha (= \omega^\pm_L, \omega^\pm_T ) \) are irrelevant to the symmetry breaking (SB) mechanism, the interactions of the longitudinal weak-bosons ( \( V_L^\alpha \)'s ) are sensitive to probing the SB sector.

Technically, the electroweak Equivalence Theorem (ET) is used to give a quantitative relation between the \( V_L \)-amplitude and the corresponding GB-amplitude in the high energy region ( \( E \gg M_W \) ), as shown in Refs.[1]-[7]. The most rigorous relation between these two amplitudes (including all the possible multiplicative and additive factors) is given by a general identity, eq.(1) or (2) in this paper, derived at the level of the LSZ reduced \( S \)-matrix elements [5]. Based upon this identity we derive the precise formulation of the ET which is given in this paper as the ensemble of our equations (10) and (10a,b). By this formulation we show that the ET is not just a technical tool in calculating a \( V_L \)-amplitude using a GB-amplitude, it has an even more profound physical content for being able to discriminate processes which are insensitive to probing the SB sector.

We know that the physical \( V_L \)-amplitude can be measured by experiments and the GB-amplitude, though not directly measurable, carries information about the SB sector. Hence, physically, the ET as a bridge tells us how the \( V_L \)-scattering experiments probe the SB sector; while technically, it replaces the calculation of the \( V_L \)-amplitude by a much simpler calculation of the scalar GB-amplitude\(^2\) in certain energy regime where their difference can be safely ignored. The formulation of the ET in the Standard Model (SM) and in the Chiral Lagrangian formulated Electro-Weak Theories (CLEW\( T \)) have been recently given in Refs.[4]-[6], where the quantization effects and problems related to the renormalization-scheme and the gauge-parameter dependence have been systematically studied. However, there is another important problem in this subject which has not yet been carefully examined. It is about the Lorentz non-invariance ambiguity in the \( V_L \)-scattering amplitudes. We noticed that the spin-0 GB’s (and thus the GB-amplitudes) are invariant under the proper Lorentz transformation, but both the longitudinal and the transverse components of the \( \text{spin-1 massive weak-bosons} \) (and thus their scattering amplitudes) are Lorentz non-invariant (LNI). After a Lorentz transformation, the longitudinal component may mix with the transverse components, and hence the

\(^2\)This is an essential simplification since the \( V_L \)-amplitude is even much more involved than the \( V_T \)-amplitude due to the non-trivial cancellations of large \( E \)-power factors from the longitudinal polarization vectors in the high energy region. This fact was first pointed out by Chanowitz and Gaillard [2].
original $V_L$-amplitude will become a mixture of longitudinal and transverse amplitudes. Undoubtedly, one can even Lorentz-transform a longitudinal component into a pure transverse one\(^3\). Thus a conceptual and fundamental question arises: How can we use the LNI $V_L$-amplitudes to probe the electroweak SB sector of which the physical mechanism should clearly be independent of the choices of the Lorentz frames? In this paper, starting from a careful examination of this problem, we construct a universal precise formulation of the ET which shows that the $V_L$-amplitudes can probe the SB sector unambiguously as long as certain general conditions, as in eqs.\((10a,b)\), are satisfied.

Generally speaking, the replacement between the $V_L$-amplitude and the GB-amplitude (with possible multiplicative factors) is LNI unless the LNI-part in the $V_L$-amplitude can be ignored. This LNI-part has the same origin as the transverse amplitudes because they can mix or turn into each other under Lorentz transformations. Hence, the physically important and interesting object is the Lorentz invariant (LI) part of the $V_L$-amplitude. When we use the GB-amplitude to predict the physical $V_L$-amplitude measured by experiments, it does not distinguish the difference between the experimental results from different Lorentz frames. Thus, by estimating the LNI-part in the $V_L$-amplitude we can determine the accuracy and the validity region of our quantitative predictions for the physical $V_L$-amplitudes based on the ET. We emphasize that the content of our precise formulation of the ET is more than just a technical tool for simplifying the calculations of the $V_L$-amplitude. The importance of the ET is firstly because it provides a conceptual connection between the would-be Goldstone-boson amplitudes directly related to the SB mechanism and the experimentally measurable longitudinal weak-boson amplitudes. Secondly, as a technical tool, it may simplify the calculation of the $V_L$-amplitude which however can always be directly calculated in spite of its complexity. Hence the most important task is to find out the conditions under which the LNI-part of the $V_L$-amplitude can be safely ignored and the LI-part becomes dominant in the experimentally measured $V_L$-amplitudes so that the physical $V_L$-scatterings can be used to sensitively and unambiguously probe the SB sector.

\(^3\)This can be done by, for example, first boosting $V_L$ to its rest frame and then boosting it in a direction perpendicular to the first boost.
2. Avoidance of Lorentz non-invariance ambiguity and the universal precise formulation of the ET

Let us start from the Ward-Takahashi identity derived in Refs.\cite{2}-\cite{5}:

\[ < 0 | F_0^a(k_1) F_0^a(k_2) \cdots F_0^a(k_n) \Phi_a | 0 > = 0 \]

in which $F_0^a$ is the bare gauge fixing function and $\Phi_a$ denotes other possible physical in/out states. After a rigorous LSZ reduction for the external $F^a$-lines, we derived in Ref.\cite{5} the following general identity for the renormalized $S$-matrix elements:

\[ T[V_L^{a_1}, \cdots, V_L^{a_n}; \Phi_a] = T[\bar{Q}^{a_1}, \cdots, \bar{Q}^{a_n}; \Phi_a] \]

\[ \bar{Q}^{a} \equiv -i C_{mod}^{a} \pi^{a} + v^{a}, \quad v^{a} \equiv v^{a} V^{a}, \quad v^{a} \equiv \epsilon^{a} - k^{a}/M_{W} = O(M_{W}/E), \]

where $\pi^{a}$'s are GB fields\footnote{The subscript $a$ denotes possible Lorentz indices.}. (In this paper, we use $W$ to denote either $W^{\pm}$ or $Z$, and $E$ is the energy of the $W$-boson, unless specified otherwise.) The finite constant modification factor $C_{mod}^{a}$ has been systematically studied in the literatures \cite{3}-\cite{7} and is proved to be renormalization-scheme and gauge-parameter dependent. In General, $C_{mod}^{a}$ is not unity and the difference $C_{mod}^{a} - 1$ comes from loop contributions \cite{3}-\cite{7}. A convenient renormalization scheme, scheme-II, was constructed in Refs.\cite{4}-\cite{6} so that the modification factors $C_{mod}^{a}$ in both the SM and the CLEW are exactly unity, and the application of the ET is greatly simplified. It has also been shown that $C_{mod}^{a} - 1 = O((g^2, \lambda) / 16 \pi^2)$ for the SM with a light Higgs boson \cite{3}-\cite{5}, and $C_{mod}^{a} - 1 = O(g^2 / 16 \pi^2)$ for both the heavy Higgs SM \cite{3, 5, 10} and the CLEW \cite{6, 10}, provided that the GB wavefunction renormalization constant $Z_{\pi^{a}}$ is subtracted at a scale $\mu \sim O(M_{W})$ and the physical mass pole of weak-boson propagator is determined by the on-shell scheme.

The identity in (1) can be re-written as

\[ T[V_L^{a_1}, \cdots, V_L^{a_n}; \Phi_a] = C \cdot T[-i \pi^{a_1}, \cdots, -i \pi^{a_n}; \Phi_a] + B \]

(2)

where

\[ C \equiv C_{mod}^{a_1} \cdots C_{mod}^{a_n} \]

\[ B \equiv B[v, -i \pi^{a}; \Phi_a] \equiv \sum_{\pi^{a}^0} (C_{mod}^{a_1} \cdots C_{mod}^{a_n} T[v^{a_1}, \cdots, v^{a_n}, -i \pi^{a_1}, \cdots, -i \pi^{a_n}; \Phi_a] + \text{permutations of } v^{a} s \text{ and } \pi^{a} s ). \]

(2a, b)

Hereafter we shall use the shorthand notations $T[V_L; \Phi_a]$ and $T[-i \pi; \Phi_a]$ for $T[V_L^{a_1}, \cdots, V_L^{a_n}; \Phi_a]$ and $T[-i \pi^{a_1}, \cdots, -i \pi^{a_n}; \Phi_a]$, respectively. Under Lorentz transformations, the amplitude of spin-0 scalar

\footnote{The $\pi^{a}$-field by definition has a sign difference compared with that in Ref.\cite{5}. Correspondingly, in $Q^{a}$ we have $-i$ instead of $+i$ for the coefficient of $\pi^{a}$.}

\footnote{The $C$-factor has also been examined for the $U(1)$ Higgs theory in Refs.\cite{4, 5} and \cite{8}.}
particles is invariant. If \( \Phi_n \), in (2), contains no field or only external physical scalar field(s) and/or photons, then from (2) the Lorentz non-invariant \( V_L \)-amplitude can be decomposed into two parts. The first part is \( C \cdot T[-i \pi; \Phi_n] \) which is Lorentz invariant (LI), and the second part is the \( v_\mu \)-suppressed \( B \)-term which is Lorentz non-invariant (LNI) because of the external spin-1 massive vector field(s). Such a decomposition clearly shows the essential difference between the \( V_L \)-amplitude and the \( V_T \)-amplitude since the former contains a Lorentz-invariant GB-amplitude which is the intrinsic source causing a large \( V_L \)-amplitude in the case of strongly coupled SB sector. We note that only the LI part of the \( V_L \)-amplitude is sensitive to probing the SB sector, while its LNI part contains a significant Lorentz-frame-dependent \( B \)-term and therefore can not be sensitive to the SB mechanism.

Strictly speaking, when \( \Phi_n \) contains field(s) such as \( V_T \)'s and fermions, the GB-amplitude is not exactly LI due to non-trivial Lorentz transformations of \( \Phi_n \). The change of the GB-amplitude due to Lorentz transformations of \( \Phi_n \) may not be small when compared with the GB-amplitude itself. For instance, if \( \Phi_n \) contains a \( V_T \)-field, this change can be of the same order of magnitude as the GB-amplitude itself because after a Lorentz transformation the mixed GB-amplitude (with one external \( V_T \) replaced by \( V_L \) ) is only suppressed by \( \mathcal{O}(M_W/E) \) (see the 2nd relation in eq.(7) ), and this suppression factor is largely compensated by the enhancement factor \( \mathcal{O}(E/M_W) \) arising from the polarization vector of the resulting \( V_L \). For a fermion field in \( \Phi_n \), it is easy to see that this change is always \( \mathcal{O}(m_f/E) \)-suppressed [10]. (Here, \( m_f \) and \( E \) are the mass and energy of the fermion, respectively.) Since the basic properties of the physical mechanism of the electroweak SB sector are clearly independent of Lorentz frames, this LNI GB-amplitude (due to the LNI \( \Phi_n \) field(s) ) would be less sensitive to probing the SB mechanism. In the case of strongly coupled SB sector, the extra \( V_T \)'s and/or fermion field(s) in \( \Phi_n \) make the leading contribution of the GB-amplitude contain more pure gauge couplings and/or Yukawa couplings (of the SM fermions ) and lower \( E \)-power dependence [10, 14]. Therefore, the corresponding amplitude in each order of perturbative expansion is at least \( \mathcal{O}(M_W/E) \)- or \( \mathcal{O}(m_f/E) \)-suppressed relative to the pure GB-amplitude (containing no external \( V_T \) and/or fermion fields).

\(^7\)One exception is the top-condensate SM [11] in which the top quark Yukawa coupling is related to the Higgs boson self-couplings. For \( m_t \approx \mathcal{O}(M_W) \), this model must predict a light Higgs boson which can be detected through processes other than the \( V_L \)-scatterings.

\(^8\)Taking the CLEWT as an example, we easily see that only the pure scalar vertices contain the largest \( E \)-power dependence, while all other vertices containing gauge bosons and/or fermions involve less derivatives and more gauge and/or Yukawa couplings.

\(^9\)The heaviest known external fermions are (anti-)top quarks. Thus \( \mathcal{O}(m_f/E) \leq \mathcal{O}(m_t/E) \approx \mathcal{O}(M_W/E) \).
Despite that $\Phi_n$ might contain some LNI contributions, it will not cause the longitudinal-transverse ambiguity in replacing a longitudinal weak-boson line in the $V_L$-amplitude by a corresponding Goldstone-boson line in the GB-amplitude as long as the LNI $B$-term can be safely ignored. Thus, we have to find the conditions under which the $B$-term in (2) is negligibly small compared with the $C \cdot T[-i\pi; \Phi_n]$ term. Such conditions can be conveniently found from (2) by estimating the magnitude of the $B$-term from the analysis of the Lorentz transformation of the $V_L$-amplitude. To estimate the $B$-term, we first examine how the $V_L$-amplitude transforms under Lorentz transformations\textsuperscript{10}. Without loss of generality, let us consider a Lorentz boost with velocity $\beta_0$ along the $\hat{x}$-direction (from $\alpha x y z$-frame to $\alpha' x' y' z'$-frame) for an external longitudinal boson $V_L$ (and also an external transverse boson) with momentum $k^\mu = (E, 0, 0, k)$ in $\alpha x y z$-frame\textsuperscript{11}:

\[\text{in } \alpha x y z \text{ frame } : \quad \text{in } \alpha' x' y' z' \text{ frame :}\]
\[k^\mu = (E, 0, 0, k), \quad k'^\mu = (\gamma_0 E, -\beta_0 \gamma_0 E, 0, k),\]
\[\epsilon_L^{\mu} = \frac{1}{M_W} (k, 0, 0, E), \quad \epsilon_L'^{\mu} = \frac{1}{M_W} (\gamma_0 k, -\beta_0 \gamma_0 k, 0, E),\]
\[\epsilon_T^\mu = (0, 1, 0, 0), \quad \epsilon_T'^\mu = (-\beta_0, 0, 0, 0),\]
\[\epsilon_T^\mu = (0, 0, 1, 0), \quad \epsilon_T'^\mu = (0, 0, 1, 0).\]

The three new polarization vectors in $\alpha' x' y' z'$-frame are defined as
\[\epsilon_L'^{\mu} = \frac{1}{M_W} (a, -\beta_0 \gamma_0 E / a, 0, \gamma_0 E / a), \quad \epsilon_T'^{\mu} = (0, k / a, 0, \beta_0 \gamma_0 E / a), \quad \epsilon_T'^{\mu} = (0, 0, 1, 0),\] where $a \equiv \sqrt{k^2 + \beta_0^2 \gamma_0^2 E^2}$, $\gamma_0 = 1 / \sqrt{1 - \beta_0^2}$, and $k' \cdot \epsilon^{\lambda} = 0$, for $\lambda = L, T_1, T_2$. After a little algebra, we get
\[\epsilon_L'^{\mu} - \epsilon_L^{\mu} = b_L \epsilon_L'^{\mu} + \sum_{j=1}^{2} h_{1,T_1} \epsilon_{T_j}'^{\mu} + h_{1,L} \epsilon_L'^{\mu}, \quad b_L = \gamma_0 k / a - 1, \quad b_{T_1} = \beta_0 \gamma_0 M_W / a, \quad b_{T_2} = 0;\]
\[h_{1,T_1} = \gamma_0 k / a - 1, \quad h_{1,T_2} = 0, \quad h_{i,L} = -\beta_0 \gamma_0 M_W / a, \quad h_{2,T_j} = h_{2,L} = 0.\]

Hence, for high energy scattering $E \sim k \gg M_W$, we generally have
\[b_L \approx O(M_W^2 / E^2), \quad b_{T_1} \leq O(M_W / E), \quad h_{i,T_1} \leq O(M_W^2 / E^2), \quad h_{i,L} \approx O(M_W / E),\] where we have taken $\gamma_0 \geq O(1)$. Thus, for a boosted external weak-boson field,

\textsuperscript{10} We thank Lay Nam Chang for enlightening discussions on this point.

\textsuperscript{11} Equivalently, one can study the Lorentz transformation relation of the spin-1 helicity amplitudes by using the spin-rotation matrices as shown in Ref.\textsuperscript{9}. But, here as an illustration, we equivalently study the Lorentz transformations of the longitudinal polarization vector $\epsilon_L^{\mu}$ and the transverse polarization vector $\epsilon_T^{\mu}$.
\[ V_{(L)}^{a'} = \epsilon_\nu^\mu \vec{V}_{\rho}^{a'} \approx \left[ 1 + \mathcal{O} \left( \frac{M_W}{E_j} \right) \right] V_L^{a'} + \sum_{j=1}^{2} \mathcal{O} \left( \frac{M_W}{E_j} \right) V_{j}^{a'} , \]
\[ V_{(T)}^{a'} = \epsilon_\nu^\mu \vec{V}_{\rho}^{a'} \approx \sum_{j=1}^{2} \left[ 1 + \mathcal{O} \left( \frac{M_W}{E_j} \right) \right] V_{j}^{a'} + \mathcal{O} \left( \frac{M_W}{E_j} \right) V_L^{a'} . \]  

(7)

Now, consider the variation \( \Delta B \equiv B[(v'), -i\pi; \Phi_{(a)}'] - B[v', -i\pi; \Phi_{a}' \right] \), which is the difference between the boosted amplitude \( B[(v'), -i\pi; \Phi_{(a)}'] \) and the corresponding amplitude \( B[v', -i\pi; \Phi_{a}'] \) defined in the \( dx'y'z't' \)-frame. Since the LNI \( B \)-term does not contain L1 spin-0 scalar sub-set which is the only intrinsic source that might cause the \( V_L \)-amplitude to be large, the variation \( \Delta B \) should be of the same order of magnitude as \( B \)-term itself, i.e. \( \mathcal{O}(\Delta B) \approx \mathcal{O}(B[(v'), -i\pi; \Phi_{(a)}']) \approx \mathcal{O}(B[v', -i\pi; \Phi_{a}']) \approx \mathcal{O}(B[v, -i\pi; \Phi_{a}']) \). Thus we can estimate \( B \) by estimating \( \Delta B \). From (2) and (7),

\[ \Delta B = B[(v'), -i\pi; \Phi_{(a)}'] - B[v', -i\pi; \Phi_{a}'] = T[V_L^{(a)}; \Phi_{(a)}'] - T[V_L^{(a)}; \Phi_{(a)}] - C \cdot T[-i\pi; \Phi_{(a)}] - \Phi_{a}' \]
\[ \equiv T[V_L^{(a)} + \Delta V_L^{(a)}; \Phi_{(a)} + \Delta \Phi_{(a)}'] - T[V_L^{(a)}; \Phi_{(a)}] - C \cdot T[-i\pi; \Delta \Phi_{(a)}'] \]
\[ = T[\Delta V_L^{(a)}; \Phi_{(a)}'] + (T[\Delta V_L^{(a)}; \Phi_{(a)}'] + B[v', -i\pi; \Delta \Phi_{(a)}'] \quad ( \text{c.f. (2)}) \]
\[ \approx \mathcal{O}(T[\Delta V_L^{(a)}; \Phi_{(a)}']) \]
\[ \approx \mathcal{O}(\frac{M_W}{E_j}) T[V_L^{(a)}; \Phi_{(a)}'] \mathcal{O}(\frac{M_W}{E_j}) T[V_{j}^{(a)}; \Phi_{(a)}'] + \mathcal{O}(\frac{M_W}{E_j}) C' \cdot T[V_{j}^{(a)}; -i\pi; \Phi_{(a)}] \quad ( \text{c.f. (7)}) \]
\[ \approx \mathcal{O}(\frac{M_W}{E_j}) C \cdot T[-i\pi; \Phi_{(a)}] + \mathcal{O}(\frac{M_W}{E_j}) C' \cdot T[V_{j}^{(a)}; -i\pi; \Phi_{(a)}] \quad ( \text{c.f. (2)}) \]
\[ C = C_{mod}^{a}, \quad C' = C_{mod}^{a'} \cdot C_{mod}^{a} . \] 

(8)

Here, in estimating the order of magnitude of \( \Delta B \), we have ignored \( T[\Delta V_L^{(a)}; \Phi_{(a)}'] \) and \( B[v', -i\pi; \Delta \Phi_{(a)}'] \), which vanish when \( \Phi_{(a)} \) contains no field or only scalar(s) and/or photon(s), and can be at most of the same order of magnitude as \( B \)-term itself. For the same reason, we have also neglected the LNI-parts generated from replacing \( V_{j}^{(a)} \) and \( \Phi_{(a)}' \) by \( V_{j}^{(a)} \) and \( \Phi_{(a)} \) in the last step of (8). Let \( E_j \) be the energy of the \( j \)-th external longitudinal weak-boson. We can thus estimate the order of magnitude of \( B \) from (8) by making the \( M_W/E_j \)-expansion when \( E_j \approx k_j \gg M_W \). Then,

\[ B \approx \sum_{a=1}^{n} C_{mod}^{a} \cdots C_{mod}^{a} \cdot T[v^{a}, \cdots, v^{a}, \pi^{a}, \cdots, \pi^{a}; \Phi_{a}] \]
\[ \approx \mathcal{O}(\frac{M_W}{E_j}) C \cdot T[-i\pi^{a}, \cdots, -i\pi^{a}; \Phi_{a}] + \mathcal{O}(\frac{M_W}{E_j}) C' \cdot T[V_{j}^{(a)}, -i\pi^{a}, \cdots, -i\pi^{a}; \Phi_{a}] . \]

(9)

We emphasize that the condition \( E_j \approx k_j \gg M_W, \quad (j = 1, 2, \cdots, n) \) for each external longitudinal weak-boson is necessary in making the \( M_W/E_j \)-expansion and ensuring the \( B \)-term (and its Lorentz variation) to be much smaller than \( C \cdot T[-i\pi; \Phi_{a}] \), as shown in (2). If the energy of one of \( V_{L}^{(a)} \)'s is low, say \( E_j \approx k_j \approx \mathcal{O}(M_W) \), then a Lorentz transformation may cause large variations in the \( V_L \)-amplitude and the Lorentz-frame-dependent \( B \)-term can be as large as \( C \cdot T[-i\pi; \Phi_{a}] \).
In conclusion, our general and precise formulation of the ET is \(^{12}\)

\[
T[V_L^{\pi^0}, \cdots, V_L^{\pi^n}; \Phi_n] = C \cdot T[-i\pi^0, \cdots, -i\pi^n; \Phi_n] + O(\frac{M_W}{E_j}-suppressed),
\]
and, from eqs.\((2b)\) and \((9)\), the conditions for ignoring the LNI and \(M_W/E_j\)-suppressed \(B\)-term on the RHS of \((10)\) are:

\[
E_j \sim k_j \gg M_W, \quad (j = 1, 2, \cdots, n) \; ; \\
B \ll C \cdot T[-i\pi^0, \cdots, -i\pi^n; \Phi_n].
\]**(10a,b)**

Some discussions on the important implications of eqs.\((10a)\) and \((10b)\) are in order. First, we note that condition \((10a)\) defines the safe Lorentz frames for the precise formulation and the application of the ET. As we pointed out, a longitudinal weak-boson can turn into a mixture of longitudinal and transverse state under Lorentz transformations while the scalar Goldstone boson is invariant. This implies that \((10)\) cannot hold in all Lorentz frames. To resolve this longitudinal-transverse ambiguity, a set of safe Lorentz frame has to be defined such that for each external \(V_L\) particle \(E_j \gg M_W\) \(^{13}\). This means that \(V_L\) is sensitive to probing the SB sector only in the sufficiently high energy region where the \(V_L\), originally coming from “eating” the GB, mainly behaves like the GB, and the effects of its mixing with the transverse components are always \(M_W/E_j\)- or \((M_W/E_j)^2\)-suppressed and negligibly small. If we change this high energy property by making Lorentz transformations such that \(M_W/E_j \approx O(1)\), this longitudinal-transverse ambiguity can no longer be ignored and the LNI-part of \(T[V_L; \Phi_n]\) will be of the same order of magnitude as the LL-part of \(T[-i\pi; \Phi_n]\) (cf. \((9)\)).

The condition \((10a)\) is actually quite strong. Naively one may expect that requiring the total center-of-mass (CM) energy \(E_{CM} \gg M_W\) can always guarantee the equivalence of the \(V_L\)-amplitude and the GB-amplitude. However, we shall show as follows that even in the SM, there are counter examples to this weaker condition in which only \(E_{CM} \gg M_W\) is satisfied but \((10a)\) is violated. Subsequently, eq.\((10)\) does not hold. To illustrate this point, we consider the scattering process \(Z_L + H \rightarrow Z_L + H\), where \(H\) is the

\(^{12}\)Here we still generally keep the modification \(C\)-factor in the ET. The exact simplification of the \(C\)-factor as unity has been given before for both the SM \([4, 5]\) and the CLEWIT \([6]\).

\(^{13}\)Here we do not take the unphysical limit as \(M_W(= g_f z/2) \rightarrow 0\), which requires either the gauge coupling \(g = 0\), implying no Higgs mechanism and the disappearance of physical longitudinal component of the \(W\)-boson, or the vacuum expectation value \(f_z = 0\), in contradiction with the non-vanishing physical Fermi-scale and the presence of the electroweak symmetry breaking. Such limits are actually unnecessary for the precise formulation of the ET.
SM Higgs particle. In the CM frame of $Z_L H$, the exact tree-level $Z_L$- and GB-amplitudes are:

$$T[Z_L H \rightarrow Z_L H] = i g^2 \left[ \frac{p^2 (1 - \cos \theta) - M_Z^2 \cos \theta}{2M_Z^2} \frac{t + 2m_h^2}{t - m_h^2} + [p^2 (1 - \cos \theta) - M_Z^2 \cos \theta] \cdot \left( \frac{1}{u - M_Z^2} + \frac{1}{s - M_Z^2} \right) - \frac{p^2}{M_Z^2} \left( \frac{\cos \theta \sqrt{p^2 + M_Z^2} + \sqrt{p^2 + m_h^2}}{u - M_Z^2} + \frac{s}{s - M_Z^2} \right) \right],$$

$$T[\pi^0 H \rightarrow \pi^0 H] = i \left[ -\frac{m_h^2 t + 2m_h^2}{f_s^2 t - m_h^2} - \frac{m_h^4}{f_s^2} \left( \frac{1}{u - M_Z^2} + \frac{1}{s - M_Z^2} \right) + \frac{g^2}{4} \left( \frac{s - t}{u - M_Z^2} + \frac{u - t}{s - M_Z^2} \right) \right],$$

$$T[Z_L H \rightarrow Z_L H] = T[i\pi^0 H \rightarrow -i\pi^0 H] + O(g^2 M_Z^2/p^2, \lambda m_h^2/p^2).$$

where $p$ is the CM momentum, $\theta$ is the scattering angle and $s, t, u$ are the Mandelstam variables. We consider two typical high energy limits: $E_{CM} \gg m_H \sim M_Z$ and $E_{CM} > m_H \gg M_Z$, where $E_{CM} = \sqrt{s}$ is the total energy. In the first case, the energy of the $Z$-boson $E_Z \sim p \gg M_Z$ so that our new condition (10a) is satisfied; while in the second case, $E_Z \sim p \sim O(M_Z)$ which violates (10a). In both cases the conventional condition $E_{CM} \gg M_Z$ is satisfied.

(i). For the first case $E_{CM} \gg m_H \sim M_Z$, which implies $E_Z \sim p \gg M_Z$, (11) gives

$$T[Z_L H \rightarrow Z_L H] = -i \left[ \frac{m_h^2}{f_s^2} + \frac{g^2}{4} \frac{3 + \cos^2 \theta}{1 + \cos \theta} + O(g^2 M_Z^2/p^2, \lambda m_h^2/p^2) \right],$$

$$T[\pi^0 H \rightarrow \pi^0 H] = -i \left[ \frac{m_h^2}{f_s^2} + \frac{g^2}{4} \frac{3 + \cos^2 \theta}{1 + \cos \theta} + O(g^2 M_Z^2/p^2, \lambda m_h^2/p^2) \right],$$

$$T[Z_L H \rightarrow Z_L H] = T[i\pi^0 H \rightarrow -i\pi^0 H] + O(g^2 M_Z^2/p^2, \lambda m_h^2/p^2) .$$

Thus, the $V_L$-amplitude is equivalent to the GB-amplitude, and can be used to probe the SB sector. In this case, the CM frame is a safe frame in applying the ET.

(ii). For the second case $E_{CM} > m_H \gg M_Z$, \footnote{For example, $E_{CM} = 1$ TeV, $m_H = 800$ GeV.}, which implies $E_Z \sim p \sim O(M_Z)$, (11) gives

$$T[Z_L H \rightarrow Z_L H] = i4 \left\{ \frac{(p^2 + M_Z^2) \cos \theta - 3p^2}{f_s^2} + O(p/m_H, M_Z/m_H) \right\},$$

$$T[\pi^0 H \rightarrow \pi^0 H] = i2 \left\{ -2p^2 (1 - \cos \theta) + M_Z^2 \right\} \frac{f_s^2}{f_s^2} + O(p/m_H, M_Z/m_H),$$

$$T[Z_L H \rightarrow Z_L H] = T[i\pi^0 H \rightarrow -i\pi^0 H] = i2 \left\{ -4p^2 + M_Z^2 (2 \cos \theta - 1) \right\} \frac{f_s^2}{f_s^2} + O(p/m_H, M_Z/m_H) .$$

As shown in the above equation, the difference between the $V_L$-amplitude and the GB-amplitude has the same size as the $V_L$-amplitude itself. Thus, the $V_L$-amplitude is not equivalent to the GB-amplitude. The CM frame in this case is therefore not a safe frame for applying the ET because in this frame our condition (10a) is violated.
Next, we examine our condition (10b) for ignoring the LNI $B$-term, which is the sum of all the $v_{\mu}$-suppressed terms in (2). Based upon the order of magnitude estimate of the $B$-term given in eq.(9), we can further express (10b) as

$$O\left(\frac{M_{\mu}}{E_j}\right) T[-i\pi^{a_1}, \cdots, -i\pi^{a_{n}}; \Phi_\alpha] + O\left(\frac{M_{\mu}}{E_j}\right) T[V_{T_j}^{a_{n+1}}, -i\pi^{a_{n+2}}, \cdots, -i\pi^{a_{r_2}}; \Phi_\alpha] \ll T[-i\pi^{a_1}, \cdots, -i\pi^{a_{n}}; \Phi_\alpha].$$

(14)

Here we have dropped the factor $1/C_{mod}^a$ in the second term on the LHS since we can always adopt the Scheme-II of Refs. [4]-[6] to make $C_{mod}^a \equiv 1$. Even in some other schemes as described in the paragraph just below eq.(1), $C_{mod} - 1$ is of $O((g^2, \lambda)/16\pi^2)$ and $O(g^2/16\pi^2)$ for the light Higgs SM and the heavy Higgs SM (or the CLEWIT), respectively, so that $1/C_{mod}^a$ will not affect the order of magnitude estimate on the LHS of (14) since only the leading terms are relevant. The condition (14) shows that after ignoring the $B$-term, we only need to keep in the GB-amplitude the contributions that satisfy the condition in (14). If we further make a perturbative expansion on the GB-amplitude, (14) would then constrain the smallest term to be included in the GB-amplitude for a fixed energy, or the lowest energy required to calculate the GB-amplitude to a desired accuracy.

In perturbative calculations, we may make loop expansion with the expansion parameter $\hat{h}$, or the momentum expansion with the expansion parameter $E/\Lambda$, or the large $N$ expansion with the expansion parameter $1/N$, etc. Practically we can only calculate the amplitude $T$ to a finite order in the perturbation expansion, i.e. $T = \sum_{\ell=0}^{N} T_\ell = \sum_{\ell=0}^{N} T_\ell \alpha^\ell$, where $\alpha$ denotes the expansion parameter. In perturbative expansion, we have $T_0 > T_1, T_2, \cdots, T_N$. Let $T_{min}$ be the smallest one in the set $\{T_0, T_1, \cdots, T_N\}$. The condition (14) then implies

$$O\left(\frac{M_{\mu}}{E_j}\right) T_0[-i\pi^{a_1}, \cdots, -i\pi^{a_{n}}; \Phi_\alpha] + O\left(\frac{M_{\mu}}{E_j}\right) T_0[V_{T_j}^{a_{n+1}}, -i\pi^{a_{n+2}}, \cdots, -i\pi^{a_{r_2}}; \Phi_\alpha] \ll T_{min}[-i\pi^{a_1}, \cdots, -i\pi^{a_{n}}; \Phi_\alpha].$$

(15)

When $N = 0$, i.e. only the leading order in the expansion is kept, (15) reduces to (10a). Hence, to leading order in any perturbative expansion, the condition (10a) is always sufficient to ensure the smallness of the $B$-term. The extra condition (15) is non-trivial only if higher order contributions are included.

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15 For special cases with both $T_0$-amplitudes on the LHS of (15) vanishing, the non-trivial condition is given via replacing the two $T_0$-amplitudes by corresponding higher order amplitudes of maximum values among $T_1, \cdots, T_N$. In this case, (15) simply reduces to (10a) up to next-to-leading order. Explicit examples of such kind are discussed in detail elsewhere [10].

16 For example, in the $1/N$-expansion formalism, some previous studies [12] applied the ET only to leading order so that condition (15) is unnecessary there. The specific form of (15) in the $1/N$-expansion beyond leading order will be given elsewhere [13].
Actually, when applying the ET to any perturbation theory, two kinds of expansions have to be considered:
one is the expansion in $\alpha$, the intrinsic expansion parameter of the theory itself; another is the expansion in power of $M_W/E_j$, as required by the ET (cf. eq. (10)). In the first expansion we usually try to include contributions beyond the leading order, while in the second expansion we always keep only the leading order term for both the physical and the technical reasons explained above. The condition (15) is required to ensure the $M_W/E_j$-suppressed $B$-terms from the leading order in $\alpha$ to be much smaller than the smallest term $T_{\min}[-i\pi; \Phi_n]$ kept in the GB-amplitude. If (15) is satisfied, i.e. (10b) is satisfied, the $V_L$-amplitude is equivalent to the GB-amplitude. Thus in this case, the $V_L$-amplitude can be given by a much simpler calculation of the GB-amplitude. This is the technical aspect of (10). Physically, the applicability of (10) implies that this $V_L$-amplitude is sensitive to probing the SB sector to the accuracy of $T_{\min}[-i\pi; \Phi_n]$. If (15) is not satisfied, i.e. the smallest term kept in the GB-amplitude does not dominate the LNI and $M_W/E_j$-suppressed $B$-term, then (10b) is not satisfied, therefore (10) is not true. Hence, the $V_L$-amplitude and the GB-amplitude are not equivalent, and this $V_L$-scattering process cannot be sensitive to probing the SB sector to the accuracy of $T_{\min}[-i\pi; \Phi_n]$. In addition to its technique content as a tool in simplifying the $V_L$-amplitude calculations, our formulation of the ET, eqs. (10) and (10a,b), has a profound physical content in discriminating processes which are insensitive to probing the SB sector to certain required precision.

To illustrate the condition (15), we consider two typical examples with $N = 1$, i.e. up to the next-to-leading order. They are the high energy $2 \rightarrow 2$ pure $V_L$-scatterings predicted in the CLEWT, and in the SM with a light Higgs boson $(m_H \ll E)$. We shall work in the CM frame of $V_L$-$V_L$ which is a safe Lorentz frame for $M_W \ll E$.

First, we examine (15) in the CLEWT, where the SB sector is non-linearly realized and strongly interacting. Now $T_0$ and $T_1$ are the $E^2$-level and the $E^4$-level contributions, respectively. From our precise power counting rule [14], these scattering amplitudes behave as\footnote{The counting results in (16) are also in accord with our explicit calculations [10].}

\[ T_0[\pi^+ \pi^- \rightarrow \pi^+ \pi^-] = \mathcal{O}\left(\frac{E^3}{f_s^2}\right), \quad T_0[V_0 \pi^+ \pi^- \rightarrow \pi^+ \pi^-] = \mathcal{O}\left(\frac{g^2 E^3}{f_s^2}\right), \]
\[ T_1[\pi^+ \pi^- \rightarrow \pi^+ \pi^-] = \mathcal{O}\left(\frac{E^4}{f_s^2 \Lambda^2}\right), \quad \]

where $\Lambda \approx 4\pi f_s \approx 3$ TeV is the cut-off of the CLEWT according to the usual dimensional analysis [15]. Substituting (16) into (15), we have $\mathcal{O}(g^2) \ll \frac{E^4}{f_s^2 \Lambda^2}$. After replacing $g^2$ by $(2M_W/f_s)^2$, we obtain

\[ \frac{M_W^2}{E^2} \ll \frac{1}{4 \Lambda^2}, \quad \text{or} \quad (0.7 \text{TeV}/E)^2 \ll 1, \quad (17) \]
which coincides with what we derived by explicit calculations [10] where the $v_{\mu}$-suppressed $B$-term, as defined in (2b), is precisely shown to be of $O(g^2)$. From (17), we see that the higher the energy $E$ is, the better the condition (17) is satisfied. For examples, for $E = 800$ GeV, 1 TeV, and 1.3 TeV, (17) gives $0.56 \ll 1$, $0.23 \ll 1$ and $0.081 \ll 1$, respectively. These numerical results indicate that the ET technically works well if $E \geq 1 \text{ TeV}$. Most importantly, it also tells us that in order to sensitively probe the strongly interacting SB sector, up to the order of $E^4$, we must raise the collider energy far beyond the TeV region so that there will be enough $V_L$-$V_L$ luminosities in the TeV region for $V_L V_L \to V_L V_L$ scatterings. In this example, we assume that there is no light resonance ($\xi$ defined as a resonance with mass much less than 1 TeV ) involved in the pure $V_L$-scattering. Next, let's examine what if there is a resonance, such as a SM Higgs boson, far below TeV.

In the case of the SM with $m_H, M_W \ll E$, our precise power counting rule [14] shows that the one-loop contribution $T_i$ is of the order:\footnote{Since the $U(1)_{em}$ gauge coupling $e$ is suppressed by $\sin \theta_W \approx 0.48$ relative to $g$, it is sufficient to take $g$ for the order of magnitude estimate.}

$$T_i[\pi^{a_1}, \cdots, \pi^{a_4}] \approx \mathcal{O} \left( \frac{g^2 \lambda}{16\pi^2} \right) T_0[\pi^{a_1}, \cdots, \pi^{a_4}] ,$$

$$T_0[V_{T_i}^{a_1}, \pi^{a_2}, \cdots, \pi^{a_4}] \approx \mathcal{O} \left( \frac{M_W}{E} \right) T_0[\pi^{a_1}, \cdots, \pi^{a_4}] ,$$

where the factor $1/16\pi^2 \ (= \pi^2/(2\pi)^4)$ is the characteristic of each loop correction\footnote{(18a) also coincides with previous explicit 1-loop calculations.}. Thus (15) and (18a,b) give

$$\mathcal{O} \left( \frac{M_W}{E^2} \right) \ll \mathcal{O} \left( \frac{g^2 \lambda}{2 \cdot 16\pi^2} \right) , \quad \text{or} \quad \left( \frac{1.4 \text{ TeV}}{\mathcal{O}(g, \sqrt{\lambda}) \cdot E} \right)^2 \ll 1 ,$$

which is a rather strong condition. For $\lambda \approx 10g^2$, i.e. $m_H = \sqrt{2\lambda} f_t \approx 700$ GeV, the condition (19) requires $(0.7 \text{ TeV}/E)^2 \ll 1$. For $E = 1 \text{ TeV}, 1.3 \text{ TeV},$ and $2 \text{ TeV}$, (19) gives $0.49 \ll 1$, $0.29 \ll 1$ and $0.12 \ll 1$, respectively. For $\lambda \approx g^2$, i.e. $m_H \approx 225$ GeV, (19) means $(2.2 \text{ TeV}/E)^2 \ll 1$, which requires $E$ be at least a few TeV to probe the SB sector of the SM with a light Higgs boson to the accuracy of including loop corrections in the GB-amplitude. This is however not a disaster because to probe the SB sector of the SM with a light Higgs boson we would have to search for a light resonance in the region $E_{CM} \sim m_H$. It has been extensively studied in the literature how to detect such a SM Higgs boson resonance through other production mechanisms other than the $V_L$-$V_L$ fusion process at the LHC ( Large Hadron Collider, pp ), the NLC ( Next Linear Collider, $e^- e^+$ ), and some photon-photon linear colliders [16, 17]. Because the $V_L$-$V_L$ scattering amplitude in the SM is unitary, if the SM Higgs boson is
not heavy, the $V_LV_L$ scattering amplitude in the vicinity of 1 TeV can never be large enough to be useful for probing the SB sector of the SM with a light Higgs resonance. Our condition (19) sets the lower limit of the energy range in which the ET can be used to calculate $T[V_L; \Phi_n]$ in terms of $T[-i\pi; \Phi_n]$ to the accuracy of including one loop corrections in the SM with $m_H \ll E$.

Before we conclude this section, we would like to note that the above numerical results are merely order of magnitude estimates, so that they cannot be regarded as accurate numbers. Moreover, the actual experimentally measured quantities are cross sections rather than amplitudes. The effect of the $B$-term in the cross section is about twice of that in the amplitude, depending on the phase space integration [10]. Therefore, similar conditions like (17), (19) and (22), but for cross sections, can be stronger by a factor of 2 than what has been given in the above discussions.

3. The ET for pure longitudinal scatterings in probing strongly coupled SB sector

Here we give a further discussion on the precise formulation of the ET for pure longitudinal weak-boson scatterings in the strongly interacting SB sector case. We first estimate the largest contribution in the $B$-term, as defined in (2), based on (15) and our precise power counting rule [14]. For both the SM with a heavy Higgs boson, $m_H \gg E$, and the general CLEW, we find that $B$ is of $O(g^2)f_T^{D_T}$ [14], where $D_T$ is the dimension of the scattering amplitude $T$, and $D_T = 4 - n_e$, for $n_e$ external $V_L$ or GB-lines. This result is in accord with our explicit calculations [10]. It is easy to see that in the GB-amplitudes all tree level Feynman graphs with internal gauge boson line(s) are at most of $O(g^2)f_T^{D_T}$ [10, 14], i.e. of the same order as the largest contribution in $B$. For higher loops or higher dimensional operators, the graphs with internal gauge-boson line(s) will be suppressed by higher powers of $E/\Lambda$. For instance, beyond the tree level, all graphs in the GB-amplitudes with internal gauge boson line(s) are at most of $O(g^2\frac{f_T^{D_T}}{\lambda^2})f_T^{D_T}$ [20].

Therefore, once we ignore the largest $v_\mu$-terms according to the condition (10b) or (15), we should also correspondingly ignore all the GB-graphs with internal gauge-boson lines to all orders in the heavy Higgs mass expansion or the momentum expansion. Furthermore, fermion fields can only appear in loops in the GB-amplitudes, their contributions are at most of $O(y_f^2\frac{f_T^{D_T}}{\lambda^2})f_T^{D_T}$ [10, 14], where $y_f \leq y_t \approx O(g)$ and $y_f$ is the Yukawa coupling of fermion $f$. (Here we assume all possible non-SM heavy fermions have been integrated out in the CLEW.) Thus, their contributions should also be ignored once the $B$-term, of $O(g^2)f_T^{D_T}$, is ignored.

In conclusion, for pure longitudinal weak-boson scatterings in theories with the strongly interacting SB

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20 For heavy Higgs SM, $\Lambda$ is replaced by $m_H$. For CLEW, $\Lambda$ is taken to be about $4\pi f_T$. 

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sector, the ET (eqs. (10) and (10a,b)) can be further simplified as

$$T[V_L^a,\cdots,V_L^a] = \tilde{C} \cdot T[-i\pi^a,\cdots,-i\pi^a]|_{g,\gamma j=0} + \mathcal{O}(g^2)f_{T}^{D_T}$$

$$E_j \sim k_j \gg M_W, \quad (j = 1, 2, \cdots, n)$$,

$$\mathcal{O}(g^2)f_{T}^{D_T} \ll \tilde{C} \cdot T[-i\pi^a,\cdots,-i\pi^a]|_{g,\gamma j=0}$$,

$$C = \tilde{C}^a_{mod} \cdots \tilde{C}^a_{mod}, \quad \tilde{C}^a_{mod} = C^a_{mod}|_{g,\gamma j=0} = \left( \frac{M_a}{M_a^{hgs}} \sqrt{Z_{V^a} Z_{M^a}} \right) |_{g,\gamma j=0}$$,

where $\pi^a_0 = \sqrt{Z_{V^a} \pi^a}$, $V^a_0 = \sqrt{Z_{V^a} V^a}$ and $M_a = Z_M M_a \pi^a_0$ and $V^a_0$ are bare fields, and $M_a = M_W$ or $M_Z$. $M_a^{hgs}$ denotes the physical mass of the $W^\pm$ or $Z^0$ boson and is equal to $M_a$ only in the on-shell renormalization scheme [4]-[6].

We note that in the above equations, the condition $g, \gamma, y_j = 0$ is meant to ignore all the gauge coupling or Yukawa coupling dependent contributions in the GB-amplitudes after replacing $M_W$ and $M_Z$ (or $m_J$) by the products of $g$ (or $y_j$) and $f_x$, because they are at most of the same order as $B$-term. The $g^2$- and $y^2_j$-dependent terms in the modification factor $(C^a_{mod} - 1)$ come from loop corrections and are at most of $\mathcal{O}(E^a_{\Lambda}^{3/2,y^2_j}) \leq \mathcal{O}(g^2/E^a_{\Lambda})$ [3]-[6]. (Recall that $y_j \leq \mathcal{O}(g)$.)

This modification factor times the largest term in the GB-amplitude, of $O(E^a_{\Lambda}/f_x)^{D_T}$, can only be of $O(g^2 E^a_{\Lambda}/f_x)$, which is again $E^a_{\Lambda}$-suppressed relative to the $B$-term and should be ignored. Then we find that those complicated $\Delta_y$-quantities inside of $C^a_{mod}$, as defined in [4]-[6], disappear after ignoring all $g^2$- and $y^2_j$-dependent terms.

So we can make the finite modification $C$-factor exactly unity by simply choosing the unphysical wavefunction renormalization constant $Z_{\tilde{C}}$ as

$$Z_{\tilde{C}} = \left( \frac{M_a}{M_a^{hgs}} \right)^2 Z_{V^a} \cdot Z_{M^a} |_{g,\gamma j=0} \quad (\text{Scheme - III})$$

$$\tilde{C}^a_{mod} = C^a_{mod}|_{g,\gamma j=0} = 1$$.

We call the above renormalization prescription as Scheme-III in which all other renormalization conditions can be freely chosen as in any of the standard renormalization schemes. In the general CLEWT, up to the $E^a_{\Lambda}$-level, the pure GB-amplitude without internal gauge boson lines can be counted as of the form

$$O(1)f_x^{D_T} E^a_{\Lambda}^{3/2} E^a_{\Lambda}$$ [14]. Only the one-loop graphs from the $E^2$-level operator $(f_x^2/\Lambda) \text{Tr}[(D_u U)(D_u U)]$ and the tree graphs from the $E^3$-level operators [18], such as $\alpha_1(f_x/\Lambda)^2 [\text{Tr}(D_u U)]^2$ and $\alpha_2(f_x/\Lambda)^2 [\text{Tr}(D_u U)]^3 \text{Tr}(D_u U)]^2$, can contribute to this leading energy behaviour. The Feynman diagrams from the other $E^4$-level operators such as $-ig \alpha_{0L}(f_x/\Lambda)^2 \text{Tr}[(W^\mu (D_u U)(D_u U))]$, $-ig \alpha_{0R}(f_x/\Lambda)^2 \text{Tr}[B^{\mu \nu} (D_u U)(D_u U)]$, and $gg' \alpha_{1}(f_x/\Lambda)^2 \text{Tr}[U B^{\mu \nu} U^\dagger W_{\mu \nu}]$, etc [21], must contain gauge

\footnote{The custodial SU(2)-symmetry violating operator $(1/\beta) \Delta \rho f_x^2 \text{Tr}(r^7 U^\dagger D_u U)]^2$ can contribute to some pure GB-graphs without internal gauge boson lines, whose contributions however are at most of $O(E^a_{\Lambda}/f_x)^{D_T}$ $\approx O(E^a_{\Lambda}/f_x)^{D_T}$ $\approx 0(g^2 E^a_{\Lambda}/f_x)^{D_T} \approx O(\Delta \rho E^a_{\Lambda}/f_x)^{D_T}$, where $y_i$ is the top quark Yukawa coupling.}
boson lines and are therefore not sensitive to probing the SB sector. Thus up to $E^4$-level the condition (20b) gives

$$O(g^2) \ll \frac{E^2}{f^2} \Lambda^2 , \quad \text{or} \quad \frac{M_W^2}{E^2} \ll \frac{1}{4} \Lambda^2 .$$  \hspace{1cm} (22)

We note that the result of (22) holds independent of the number of external lines involved in pure $V_L$-scattering processes. Our condition (17) for a pure $2 \to 2$ $V_L$-scattering is only a special case of (22).

As $E \geq O(1) \text{ TeV}$, eq.(22) is satisfied. Our precise formulation of the ET therefore provides a rigorous theoretical reasoning for justifying many previous applications of the ET in the literature to study the strongly coupled SB sector by ignoring all the internal gauge boson lines in the GB-amplitudes. Most importantly, our result of (22) shows that in order to probe strongly coupled SB sector from pure longitudinal weak-boson scattering processes with any number of external lines, we must experimentally measure their production rates in the energy region above 1 TeV.

5. Summary

We have examined the Lorentz non-invariance ambiguity for longitudinal weak-boson scatterings and derived the precise conditions, eqs.(10a) and (10b) ( or (15) ), for the equivalence of the $V_L$-amplitude and the GB-amplitude, as shown in (10). After analyzing the intrinsic connection between the ET and the problem of probing the SB sector, we presented the universal formulation of the ET in eqs.(10) and (10a,b) for both the SM and the general CLEWT. We have also defined the safe Lorentz frames in which the condition (10a) holds. We gave an explicit example, $Z_L H \to Z_L H$, to show that the center-of-mass frame of this scattering process for a heavy Higgs boson ($M_W \ll m_H < E_{CM}$ ) is not a safe frame because (10a) in this case is not satisfied. Therefore, in the CM frame the $Z_L H \to Z_L H$ amplitude cannot be estimated by using the corresponding GB-amplitude $\pi^0 H \to \pi^0 H$, as shown in (13). We note that our formulation of the ET not only serves as a technique tool in simplifying the $V_L$-amplitude calculation using the GB-amplitude when the conditions (10a,b) are satisfied, but, most importantly, this formulation also discriminates processes which are not sensitive to probing the SB sector when (10a) or (10b) fails. Furthermore, the condition in eq.(15) determines whether the $V_L$-scattering process of interest is sensitive to probing the SB sector to the desired precision in perturbative calculations. The minimum energy scale required for testing the SB sector ( assuming no light resonance present ) of the SM and the CLEWT beyond the leading order ( up to $E^4$-level ) were given in (17) or (22). We found that longitudinal weak-boson scatterings can only be sensitive to probing strongly coupled SB sector in the TeV region, i.e. $E \geq O(1) \text{ TeV}$. In this case, for pure longitudinal weak-boson scatterings, the ET takes a very simple form in which the GB-amplitude is
calculated by ignoring all the internal gauge-boson lines and fermion loops (cf. \((20,20a,b,c)\)). Here the multiplicative modification factors can be exactly simplified as unity in a very simple renormalization scheme, Scheme-III (cf. (21)).

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References


[18] For these non-linear operators, see, for example,