NEW CORRECTIONS OF ORDER $\alpha^6$ TO $S$-LEVELS OF TWO-BODY SYSTEMS

Michael I. Eides$^1$
Petersburg Nuclear Physics Institute,
Gatchina, St.Petersburg 188350, Russia

and

Howard Grotch$^2$
Department of Physics, Pennsylvania State University,
University Park, Pennsylvania 16802, USA

November, 1994

$^1$E-mail address: eides@lnpi.spb.su
$^2$E-mail address: hlg@psuvm.psu.edu
Abstract

New corrections to the energy of $S$-levels of positronium of order $m_0^6$ which are as large as several hundred kilohertz are obtained. New recoil correction of order $\alpha(Z\alpha)^5(m/M)m$ to the Lamb shift in hydrogen is calculated. This correction is equal to $-6.59$ kHz for 1S-level and to $-0.82$ kHz for 2S-level.
1. Recent progress in the spectroscopy of positronium [1, 2, 3, 4, 5] triggered theoretical work on the corrections of order $\alpha^6 m$ to the positronium energy levels. All logarithmic corrections of this order to $S$-levels were calculated recently in [6, 7]. Complete results for the corrections of order $\alpha^6 m$ to $P$-levels were obtained in [8]. As emphasized in this last work the large magnitude of the nonlogarithmic corrections to $P$-levels suggest that calculation of corresponding nonlogarithmic corrections to $S$-levels is also important. Some of these corrections are already known, e.g., contributions induced by the two- and three-photon annihilation kernels [9, 10, 11]. We present below results of the calculation of nonlogarithmic contributions of order $\alpha^6 m$ to the $S$-levels of positronium induced by radiative corrections to the Breit potential and by the polarization insertions in the graphs with two-photon exchange.

A new radiative-recoil correction of order $\alpha(Z\alpha)^3(m/M)m$ to the Lamb shift in hydrogen induced by a polarization operator insertion in the two-photon exchange graph is also calculated in this note. Recent experimental achievements in measuring $1S - 2S$ splitting in hydrogen [12] and the well-known results on the $2S$ Lamb shift [13, 14, 15] clearly demonstrate that theoretical calculation of all corrections to the Lamb shift of the order of several kHz for the $1S$-state and about $1$ kHz for the $2S$-state is necessary. Several such contributions were obtained quite recently [16, 17, 18] and the result presented below is one more contribution of comparable magnitude (for more detailed description of the current theoretical status of the Lamb shift calculations see, e.g. [19]).

2. Let us consider first corrections of order $\alpha^6 m$ to the $S$-levels of positronium connected with radiative insertions in the graph with one-photon exchange in Fig 1. As is well known this graph leads to the Breit potential. One may easily obtain the radiatively corrected expression for the Breit potential in the form\(^1\) (see, e.g. [20] and paper in preparation)

$$U(p, r) = -\frac{1}{r} - \pi \left( \frac{1 + 8 f_1' + 2 f_2}{m c^2} \right) \delta^3(r) + \frac{4 \alpha p}{m c} \delta^3(r) + \frac{r( rp)}{2 m^2 c^2 r^3} + \frac{p^2}{2 m^2 c^2 r} - (3 + 4 f_2) \frac{sl}{2 m^2 c^2 r^3}$$

\(^1\)The annihilation diagram contribution is missing in this expression since we do not consider annihilation contributions in this paper.
where \( m \) is the electron mass, \( p \) is the relative momentum of the electron and positron, \( r \) is their relative position, \( f'_1 \) is the slope of the Dirac formfactor, \( f_2 \) is the Pauli formfactor at zero momentum transfer and \( p \) is the polarization operator contribution. With two-loop accuracy we have

\[
f'_1 = \frac{\epsilon_1 \alpha}{m^2 \pi} + \frac{\epsilon_2}{m^2} \left( \frac{\alpha}{\pi} \right)^2,
\]

\[
f_2 = g_1 \frac{\alpha}{\pi} + g_2 \left( \frac{\alpha}{\pi} \right)^2,
\]

\[p = p_1 \frac{\alpha}{\pi} + p_2 \left( \frac{\alpha}{\pi} \right)^2.\]

It is an easy task now to obtain corrections of order \( \alpha^6 m \) to the positronium energy levels

\[
\Delta E_{F_1} = \epsilon_2 \frac{\alpha^6}{\pi^2 n^3} m\delta_{\text{lo}} = 0.46994 \frac{\alpha^6}{\pi^2 n^3} m\delta_{\text{lo}},
\]

\[
\Delta E_{F_2, \mu=0} = g_2 \frac{\alpha^4 m}{4n^3} = -0.082 \frac{\alpha^6 m}{\pi^2 n^3},
\]

\[
\Delta E_{p2} = -p_1 \frac{\alpha^6}{2\pi^2 n^3} m\delta_{\text{lo}} = -\frac{41}{324} \frac{\alpha^6}{\pi^2 n^3} m\delta_{\text{lo}}.
\]

where we used the value of the two-loop contribution \( \epsilon_2 \) to the slope of the Dirac formfactor obtained numerically in [21] and analytically in [22], the explicit results for the two-loop electron magnetic moment \( g_2 \) [23, 24], and the two-loop irreducible vacuum polarization operator [25] obtained a long time ago.

With the help of the effective potential in eq.(1) we may also easily calculate radiative corrections to the levels of positronium which have nonvanishing angular momentum. Our results in this case reproduce and confirm the respective results in [8, 18].

3. Consider now corrections of relative order \( \alpha^6 \) to the energy levels of two-body systems which are generated by the diagrams with intermediate momenta which are high on the scale of the typical atomic momenta. It is well known that all such corrections are generated by the diagrams with two exchanged photons containing also either a polarization operator insertion in
one of the exchanged photons or radiative photon insertions in the electron line (see, e.g. [26]). To sufficient accuracy external electron lines in the diagrams under consideration may be safely taken to be on-mass shell. It is not difficult to obtain an explicit expression for the infrared divergent skeleton integral corresponding to the sum of ladder and crossed diagrams in Fig.2. Direct integration over loop momentum advocated in [27] leads to the following expression for the skeleton integral

\[\Delta E_{skel} = -32mM(Z\alpha)^2|\psi(0)|^2\]

\[\int_0^\infty \frac{dk}{\pi k} \int_0^\pi \frac{d\theta}{2} \frac{\sin^2 \theta(1 + 2\cos^4 \theta)}{k^2 + 4m^2\cos^2 \theta(k^2 + 4M^2\cos^2 \theta)}\]

\[= -16mM(Z\alpha)^2|\psi(0)|^2\int_0^\infty \frac{dk}{k^3} \frac{1}{m^2 - M^2} \left[1 + \frac{k^2}{4m^2} \left(1 + \frac{k^3}{8m^2}\right)\right.\]

\[\left. - M\sqrt{1 + \frac{k^2}{4M^2} \left(1 + \frac{k^3}{8M^2}\right)} - \frac{k^2}{8m^2} \left(1 + \frac{k^2}{2m^2}\right) + \frac{k^2}{8M^2} \left(1 + \frac{k^2}{2M^2}\right)\right],\]

where \(m\) and \(M\) are the masses of the negatively and positively charged particles, respectively, \(Z\) is the charge of the positive particle in terms of the proton charge and \(\psi(0)\) is the value of the reduced mass Schrödinger-Coulomb wave function at the origin.

All contributions to hydrogen Lamb shift of order \(\alpha(Z\alpha)^3 m\), both recoil and nonrecoil, calculated over years by different methods [28, 29, 30], may be obtained from the expression for the skeleton integral in eq.(4) by insertion of radiative corrections.

Consider first recoil contributions of order \(\alpha(Z\alpha)^3 m\) to the Lamb shift in hydrogen. The contribution induced by the radiative photon insertions in the electron line was obtained in [30]. With the help of explicit expression in eq.(4) above, it is easy to confirm the result of [31] that correction induced by the radiative photon insertions in the heavy line is suppressed by the factor \((m/M)^2\) relative to the contribution induced by the radiative photon insertion in the electron line. It is also easy to see that the recoil correction corresponding to the polarization operator insertion in the exchanged photon is suppressed by the factor \(m/M\) relative to respective nonrecoil correction. Let us calculate this last correction. The general expression in eq.(4) contains the skeleton integral both for recoil and nonrecoil corrections. The skeleton
integral for the recoil corrections may be obtained by subtracting the heavy pole residue in eq.(4) and has the form

\[
\Delta E_{\text{skei-rec}} = \frac{16(Z\alpha)^2|\psi(0)|^2}{m^2(1-\mu^2)} \int_0^\infty \frac{kd\bar{k}}{(k^2 + \lambda^2)^2} \left\{ \mu \sqrt{1 + \frac{k^2}{4}(\frac{1}{\bar{k}} + \frac{k^3}{8})} \right\}
\]

(5)

\[- \sqrt{1 + \frac{\mu^2 k^2}{4}(\frac{1}{k} + \frac{\mu^2 k^3}{8})} - \frac{\mu k^2}{8}(1 + \frac{k^2}{2}) + \frac{\mu^3 k^2}{8}(1 + \frac{\mu^2 k^2}{2}) + \frac{1}{k}, \]

where \( \mu = m/M \) and we transferred to a dimensionless integration momentum measured in units of the electron mass.

For calculation of the radiative-recoil contribution to the Lamb shift induced by the polarization operator insertions one has to make a substitution in the integrand in eq.(5)

\[
\frac{1}{k^2} \to \frac{\alpha}{\pi} I_1(k),
\]

(6)

where

\[
I_1(k) = \int_0^1 dv \frac{v^2(1 - v^2/3)}{4m^2 + (1 - v^2)k^2}.
\]

(7)

However, the skeleton integrand in eq.(5) behaves as \( \mu/k^4 \) at small momenta and naive substitution in eq.(6) leads to divergence. This divergence \( dk/k^2 \) actually diminishes the power of the \( Z\alpha \) factor and the respective contribution turns out to be of order \( \alpha(Z\alpha)^4 \). In order to get the recoil correction of order \( \alpha(Z\alpha)^5m \) we have to perform a subtraction of the leading low frequency asymptote in the polarization operator insertion

\[
\tilde{I}_1(k) \equiv I_1(k) - I_1(0) = -\frac{k^2}{4m^2} \int_0^1 dv \frac{v^2(1 - v^2)(1 - v^2/3)}{4m^2 + (1 - v^2)k^2} \]

(8)

and substitute this subtracted expression in eq.(5) (one has to insert an additional factor of 2 which takes into account possible insertions of the polarization in both photon lines)

\[
\Delta E_r = \frac{32(Z\alpha)^2|\psi(0)|^2}{m^2(1-\mu^2)} \left( \frac{\alpha}{\pi} \right) \int_0^\infty \frac{dk}{k} \tilde{I}_1(k) \left\{ \mu \sqrt{1 + \frac{k^2}{4}(\frac{1}{k} + \frac{k^3}{8})} \right\}
\]

(9)
\[ -\sqrt{1 + \frac{\mu^2 k^2}{4} \left( 1 + \frac{\mu^4 k^3}{8} \right) - \frac{\mu k^2}{8} \left( 1 + \frac{k^2}{2} \right) + \frac{\mu^3 k^2}{8} \left( 1 + \frac{\mu^2 k^2}{2} + \frac{1}{k} \right)}. \]

Integrals of this kind also emerge in the investigation of other recoil corrections, e.g., for the hyperfine splitting in \([26]\), and all methods developed for the respective calculations in \([26]\) are also applicable in the case under investigation. Following closely \([26]\) we obtain

\[ \Delta E_r = \left( \frac{4}{5} \ln \mu + \frac{2\pi^2}{9} - \frac{70}{27} \right) \frac{\mu}{1 - \mu^2} \frac{\alpha(Z\alpha)^5}{\pi^2 n^3} \frac{m_r}{m}. \]  

(10)

4. Consider now the contribution of order \(\alpha^6\) induced by insertion of the one-loop polarization operator for the positronium case. Calculation is similar to the one for hydrogen. The analog of the skeleton integral in eq.(4), for the case of equal masses, has the form

\[ \Delta E = -16m^2(Z\alpha)^2|\psi(0)|^2 \int_0^\infty \frac{dk}{k^4} \frac{1}{\sqrt{k^2 + 4m^2}} \]

\[ + \frac{k^3}{8m^4} - \frac{k^4(k^2 + 3m^2)}{8m^6\sqrt{k^2 + 4m^2}} + \frac{k^5}{8m^6} \]

\[ = -\frac{2\alpha^5 m^5}{\pi n^3} \int_0^\infty \frac{dk}{k^4} \frac{1}{\sqrt{k^2 + 4m^2}} \]

\[ + \frac{k^3}{8m^4} - \frac{k^4(k^2 + 3m^2)}{8m^6\sqrt{k^2 + 4m^2}} + \frac{k^5}{8m^6}. \]

Consideration of the hydrogen case above teaches us that we have to substitute into the skeleton integrand in eq.(11) the subtracted expression for the vacuum polarization operator. This procedure corresponds to subtraction of the polarization operator contribution to the energy level shift of the previous order. Hence, contribution to the energy levels of positronium of order \(\alpha^6\) is given by the expression (remember combinatorial factor 2)

\[ \Delta E_{\alpha^6} = -\frac{4\alpha^6 m^5}{\pi^2 n^3} \int_0^\infty \frac{dk}{k^2} \tilde{f}(k) \left\{ \frac{1}{\sqrt{k^2 + 4m^2}} + \frac{k^3}{8m^4} - \frac{k^4(k^2 + 3m^2)}{8m^6\sqrt{k^2 + 4m^2}} + \frac{k^5}{8m^6} \right\} \]

\[ = \left( \frac{\pi^2}{36} - \frac{127}{540} \right) \frac{m\alpha^6}{\pi^2 n^3}. \]

(12)
5. Numerical values of the corrections obtained above are presented in the Table.

<table>
<thead>
<tr>
<th>$\Delta E$</th>
<th>$2S$ kHz</th>
<th>$1S$ kHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positronium, $\Delta E_{F_1}$</td>
<td>0.469 $\frac{a_0^6}{\pi^2} m$</td>
<td>111.05</td>
</tr>
<tr>
<td>Positronium, $\Delta E_{F_2}$</td>
<td>$-0.082 \frac{a_0^6}{\pi^2} m$</td>
<td>$-19.38$</td>
</tr>
<tr>
<td>Positronium, $\Delta E_{p2}$</td>
<td>$-\frac{41}{514} \frac{a_0^6}{\pi^2} m$</td>
<td>$-29.90$</td>
</tr>
<tr>
<td>Positronium, $\Delta E_{p1}$</td>
<td>$\left( \frac{\pi^2}{36} - \frac{127}{30} \right) \frac{a_0^6}{\pi^2} m$</td>
<td>9.21</td>
</tr>
<tr>
<td>Hydrogen, $\Delta E_r$</td>
<td>$\left( \frac{4}{5} \ln m.M + \frac{2\alpha^2}{9} - \frac{307}{37} \frac{a_0^4 Z^2}{\Delta \pi^2} \frac{m}{M} \right) m$</td>
<td>$-0.82$</td>
</tr>
</tbody>
</table>

These corrections turn out to be of the same order of magnitude as other corrections to the energy levels calculated recently [6, 7, 8, 9, 10, 11, 18] and are significant for comparison of the theory with the current experimental results both in the case of hydrogen and positronium. Detailed derivation of the results of this paper will be presented elsewhere. In the case of positronium there remain some other yet unknown contributions of order $a_0^6 m$ to the energy shift of $S$-levels. Work on their calculation is in progress now.

We are deeply grateful to I. B. Khriplovich for useful remark on the radiative corrections to the Breit potential. This work was done during the visit of M.E. to the Penn State University. He is deeply grateful to the colleagues at the Physics Department of the Penn State University for their kind hospitality. This research was supported by the National Science Foundation under grant #NSF-PHY-9120102. Work of M.E. was also supported in part by grant #R2E000 from the International Science Foundation and by the Russian Foundation for Fundamental Research under grant #93-02-3853.
References


Figure Captions

Fig. 1. One-photon exchange skeleton graph.

Fig. 2. Two-photon exchange skeleton graphs.