Propagation of the 50 Hz Perturbations in the Large Hadron Collider Sector Dipoles and Equivalent Model Circuit

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Dipoles and Equivalent Model Circuit

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“The true scientific history of our legacy is richer than all fairy tales and, in its reality, is more mysterious and bizarre than all myths.”
- Leon Lederman
Abstract

Observations of harmonics of the mains power frequency in the beam spectrum have been observed in the past in several accelerators. Indeed, since 1986 this phenomenon was already known in the SPS at CERN. In particular, it was observed that these excitations can lead to emittance blow-up and shortening of the beam lifetime. For these reasons, in the Summary of the 220th LMC Meeting held on 3rd June 2015, it was decided to invest more energy to study the origin of this excitation and to understand the tolerance that we would have for the next upgrade of the LHC, that is the High Luminosity LHC (HL-LHC) that will enter service after 2025.

Several observations were made during the 2018 proton run in order to understand the origin of this issue; in particular, all the observations suggest that the harmonics are the result of a real beam excitation, rather than an instrumental feature. Based on these considerations, potential sources have been found leading to plausible correlation to the magnets power converters.

In this context, this work of thesis aims to provide the bases of analysis for a particle accelerator under the effect of perturbations and how their propagation occurs in the Large Hadron Collider sector dipoles at CERN. In particular, after an analytical approach at the issue, an equivalent circuit model of the LHC dipoles chain will be used in order to give a first estimation of excitation given by the ripple, at the mains frequency, due to the power converters of the main bending magnets.
Acknowledgments

Every journey has an end.
After fourteen months, my technical studentship is coming to an end. Many are the people that contributed to increase my luggage of knowledge. Firstly, I would like to express my gratitude to my supervisor at CERN, Guido Sterbini, for his daily support, for his teachings and for his useful suggestions throughout the work described in this thesis. I would like to thank Yannis Papaphilippou, my section leader, for welcoming me warmly in his team and for allowing me to improve my knowledge by following various courses and schools around the world. Last but not least I want to thank my Professors Mauro Migliorati and Andrea Mostacci for the preparation and the defence of this thesis. The list of people that I should mention at CERN is definitely too long for just one page, therefore I will limit myself to expressing my gratitude to all those who have endured and supported me during this period.

A special acknowledgement should be addressed to my family. I thank my father Pancrazio and my mother Iris who guided me with love during my education and my life encouraging me to pursue my interests. A special thanks goes to Massimo, my brother and point of reference. I also want to remember my uncle Christian who inspired me and believed in me till the end.
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Chapter 1

Introduction

The aim of this chapter is to introduce the reader at the issue of the 50 Hz excitations contaminating the proton beam spectrum. We will first present the research center where this work of thesis was performed, with particular emphasis on its goals. The chapter will proceed describing briefly the accelerator related to this study (the Large Hadron Collider) and, afterwards, will give some information and historical context about the 50 Hz perturbation that excites the transverse motion of the protons circulating in the LHC.

1.1 CERN

The European Organisation for Nuclear Research, known as CERN (Conseil Européen pour la Recherche Nucléaire), is the research organization in charge of largest accelerator complex for High Energy Physics (HEP) in the world. At the end of the Second World War, European science was no longer world-class but a group of visionary scientists imagined creating a European nuclear physics laboratory. Founded in 1954 by 12 founding states it currently counts 23 member states and its laboratory sits astride the Franco-Swiss border near Geneva. CERN employs about 2500 people from nationalities all around the world, representing a prime example of international collaboration, operating in every area of the Organization, from physicists to computer scientists, from mechanical engineers to firemen. CERN’s mission is to uncover what the universe is made of and how it works by providing a unique range of particle accelerator facilities to researchers, to push the limits of human knowledge. One of the most famous discoveries that CERN made, is the first observation of the Higgs boson, theorized in 1964 but observed only in 2012 [1][12]. Since fundamental physics needs the ultimate frontier in performance, CERN laboratory also plays a key role in developing cutting-edge technologies in various fields of application spanning from materials science to computational science. In this case, it is impossible not to mention the World Wide Web (WWW), commonly known as the Web, invented by Tim Berners-Lee in 1989 when working at CERN [3][11].

The privileged instruments of discovery used at CERN are the accelerators, machine capable to increase the kinetic energy of particles (electrons, protons,
ions...) and transform it in new particles.

1.1.1 The CERN Accelerator Complex

The CERN accelerator chain includes seven main accelerators, built in various periods starting from the foundation of the Organisation. From the beginning, it was expected that every new and more powerful machine would use the previous ones as injectors, creating a chain of accelerators that gradually brings a particle beam to increasingly higher energies. To allow the operation of this complex, all the functions of the accelerators are coordinated by a single reference signal, generated by a system of atomic clocks and distributed throughout the installation, with an accuracy of the order of nanoseconds.

The CERN accelerator complex
Complexe des accélérateurs du CERN

Figure 1.1. CERN Accelerator Complex

The main accelerators available to CERN are, from the initial source of low-energy particles up to the main collision double-ring (see Figure 1.1):

- Two linear accelerators, or LINACs, which generate low-energy particles, which are subsequently injected into the PS Booster. They are known as LINAC 2\(^1\).

\(^1\)LINAC 2 has been replaced by LINAC 4 in December 2018.
and LINAC 3 and reach 50 $MeV$ — protons, or 4.2 $MeV$ heavy ions per nucleon respectively. The whole subsequent accelerator chain depends on these sources.

- The Low Energy Ion Ring (LEIR), which increases the ion beams energy up to 72 $MeV$ per nucleon, began to run in 2010 in the LHC pre-acceleration chain.

- The PS Booster, consisting of 4 superimposed synchrotrons with a radius of 25 m, increases the energy of the particles generated by LINAC before injecting them into the PS. It is also used for separate experiments, such as ISOLDE, which studies unstable nuclei of very heavy isotopes.

- The Proton Synchrotron (PS), built in 1959, is a synchrotron with a circumference of 628.3 m able to accelerate protons up to 28 GeV, as well as to a whole series of particles for different experiments. In particular, it receives respectively protons from the Proton Synchrotron Booster and ions from the Low Energy Ion Ring.

- The Super Proton Synchrotron (SPS), is a circular accelerator of 2 km in diameter that feeds the LHC. It was built in a tunnel, which started functioning in 1976 accelerating lead ions. It currently leads to an energy equivalent to that of a 450 GeV proton. Moreover, it also has a dedicated straight beam line for fixed-target experiments, it worked as a proton-antiproton collider and as a final acceleration stage for electrons and positrons to be injected into the Large Electron-Positron Collider (LEP).

- The Large Hadron Collider (LHC), which entered into operation on 10 September 2008 after the dismantling of LEP. It extends over a circumference of 27 kilometers and was designed to accelerate protons up to a maximum of 7 $TeV$.

This thesis focuses entirely on this last machine, CERN flagship, which represents the syntheses of the cutting-edge technologies of the last decade. In the following section, a brief description of this machine is given.

### 1.1.2 The Large Hadron Collider

As mentioned previously, LHC is the world’s most powerful and largest particles accelerator. The idea of this machine was born with the commitment to achieve an energy that allows us to study elementary particles in experimental conditions comparable to those of the first moments of life in the Universe, immediately after the Big Bang. Without going into too much detail, in Figure 1.2 an artistic version of the evolution of the history of the Universe afterwards the singularity is depicted. The first stages of the universe evolution had an energy density equivalent to the energy of the LHC collisions, therefore this machine and its detectors allows us to probe them in a quantitative and reproducible way.

\[\text{LEP was dismantled in 2000 in order to replace it with LHC in 2008.}\]
Inside the accelerator, two high energy particle beams travel in opposite directions and (for most of the LHC circumference) in separate beam pipes, at a speed close to the speed of light. Beams are guided around the LHC ring through superconducting magnets, in particular, particles are forced to stay in orbit by the dipoles, which give the right curvature to proton beams, presently, up to 6.5 \( TeV \). As it is shown in Figure 1.3, the LHC is divided in 8 identical (ideally) arcs. Each arc is powered by its own power supply. Furthermore, there are four interaction points in the tunnel, the experiments, in which the particles are forced to collide and where the four detectors are placed:

- **A Toroidal LHC AparatuS (ATLAS)**: In principle it’s used to look for signs of new physics, including the origins of mass and extra dimensions but also employed for general purpose observations.

- **Compact Muon Solenoid (CMS)**: Its aim is to hunt for the Higgs boson and look for clues to the nature of dark matter.

- **A Large Ion Collider Experiment (ALICE)**: ALICE is studying a ‘fluid’ form of matter called quark–gluon plasma that existed shortly after the Big Bang.
Large Hadron Collider beauty (LHCb): Equal amounts of matter and antimatter were created in the Big Bang. LHCb tries to investigate what happened to the missing antimatter.

In the following chapters we will go into more details on the LHC lattice and its dipoles chain, in particular, we will concentrate on implementing our analysis for one arc of the machine. Before concluding this section, it’s appropriate to mention that this study was made during the second run of LHC. In Table 1.1 the reader can find the main LHC parameters.

1.2 50 Hz spaced lines in the beam spectrum

Let’s introduce now the issue faced in this thesis, giving a description and some historical considerations.

Starting from the observations that harmonics of 50 Hz (called also tones) excite the transverse spectrum of the beam, we will try to describe this mechanism in a quantitative way assuming that the main source of the excitation is the LHC main dipoles power converters and modelling the beam-dipoles interaction.

Since 1986 this phenomenon has been observed in the SPS at CERN while two years before (1984), Carlo Rubbia and Simon van der Meer had won the Nobel prize. In those years, since the first p-pbar collider runs, the Schottky signals were contaminated by families of lines spaced at 50 Hz. Figure 1.4 shows an old picture concerning the first approach at the phenomena in the SPS.

During the following years, this phenomenon has been observed also in many other machines as Tevatron (at FNAL), RHIC (at BNL), SPS (CERN).

---

3Run 1 lasted from 2011 until 2013 and Run 2 from 2015 until the end of 2018.
4Nobel Prize for work leading to the discovery of the W and Z particles at CERN.
Table 1.1. Quantity involved in LHC Run 2.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Number</th>
</tr>
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<tbody>
<tr>
<td>Circumference</td>
<td>26 659 m</td>
</tr>
<tr>
<td>Dipole operating temperature</td>
<td>1.9 K (-271.3°C)</td>
</tr>
<tr>
<td>Number of magnets</td>
<td>9593</td>
</tr>
<tr>
<td>Number of main dipoles</td>
<td>1232</td>
</tr>
<tr>
<td>Number of main quadrupoles</td>
<td>392</td>
</tr>
<tr>
<td>Number of RF cavities</td>
<td>8 per beam</td>
</tr>
<tr>
<td>Nominal energy, protons</td>
<td>6.5 TeV</td>
</tr>
<tr>
<td>Nominal energy, ions</td>
<td>2.56 TeV/u (energy per nucleon)</td>
</tr>
<tr>
<td>Nominal energy, protons collisions</td>
<td>13 TeV</td>
</tr>
<tr>
<td>No. of bunches per proton beam</td>
<td>2808</td>
</tr>
<tr>
<td>No. of protons per bunch (at start)</td>
<td>$1.2 \times 10^{11}$</td>
</tr>
<tr>
<td>Number of turns per second</td>
<td>11245</td>
</tr>
<tr>
<td>Number of collisions per second</td>
<td>1 billion</td>
</tr>
</tbody>
</table>

Figure 1.4. Schottky signal obtained from a RF cavity [51].

and, as last, also in the LHC (CERN [6]). Furthermore, many studies have been done in order to define the nature and the source of this noise. In particular, it is interesting to report that what has been observed in RHIC are harmonics at 60 Hz instead of 50 Hz [8, 9]. Another interesting observation, still made at RHIC, is that the spectral power of the main harmonics in the horizontal plane is uncorrelated with coupling strength. Instead, power in the vertical plane increases if the coupling strength increases, and it is almost absent when the machine is well decoupled. Those observations strongly suggest how the main harmonics depend somehow on the power supply and, further, that the excitation seems to be just in the horizontal plane.

As far as CERN is concerned, in the Summary of the 220th LMC Meeting held on 3rd June 2015, it was decided to invest more energy to study the origin of the 50 Hz to better manage it for the next upgrade of the LHC (HL-LHC) [2]. Without

---

5The High Luminosity LHC (HL-LHC) would be able to achieve instantaneous luminosities a
going into too many details, we would like to stress some points that have been done on this front. In particular, to disprove the instrumental nature of the observation, has been observed that harmonics are visible only in the presence of the beam and, moreover, as Figure 1.5 shows, the phase advance of the noise lines matches the one of the betatron motion ($\approx 100^\circ$). This is another strong evidence that the problem does not come from the instrumentation but the multiple components of the 50 Hz are intrinsic in the beam (for more details see [24]).

1.3 Motivations of study

After this brief introduction, it is necessary to report the reasons that led to face this thesis work. As we mentioned before, one of the main present goal for CERN is to improve the performance of LHC and prepare it at the upgrade in HL-LHC. In particular, this 50 Hz excitation can lead to emittance blow-up and shortening of the beam lifetime [30]. In this context, an experiment was set up at CERN in order to observe the impact of dipolar noise on the beam developed with controlled ADT excitations [25]. In Figure 1.6 it is shown how the lifetime can decrease by injecting external noise (in this case controlled noise has been used, as mentioned before). Thus, it is essential to understand the tolerance that we could have for HL-LHC. Furthermore, experimentally it was observed a cluster of harmonics that arise at 8 kHz. In Figure 1.7 the measure made in the horizontal plane by the Beam Position Monitor (BPM) [16], for Fill 7343 in the first beam, is shown.

---

Figure 1.5. Phase advance of the 2.35 and 2.95 kHz lines between the damper pickups in LHC [44].

<table>
<thead>
<tr>
<th>B1H</th>
<th>2.35 kHz</th>
<th>2.95 kHz</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta\phi_{2.35\text{kHz}} = 103.3^\circ$</td>
<td>$\Delta\phi_{2.95\text{kHz}} = 102.5^\circ$</td>
</tr>
</tbody>
</table>

---

6Beam Position Monitors (BPM) are the non-destructive diagnostics used most frequently at almost all linacs, cyclotrons and synchrotrons. A BPM normally provides information about the beam phase and beam transverse position by using position sensitive detectors (PSDs).
Note how getting closer to the $8k\,Hz$, a cluster of harmonics at the mains frequency arises from the background noise. This phenomenon is not well understood yet but it seems that noise is interfering constructively or destructively with the beam depending on the frequency. In this thesis, we would try to give some tools to improve the understanding of the issue.

**Figure 1.6.** Lifetime of the Beam affected by controlled noises.

**Figure 1.7.** Cluster of harmonics at $8k\,Hz$ measured with ADT-ObsBox.
1.4 Workflow

Before entering in the heart of the work, it is appropriate to illustrate how the work has been articulated. Figure 1.8 shows the workflow followed for this thesis. In particular, after a brief introduction on transverse beam dynamics, two different problems will be faced in parallel. Firstly, we will answer to the question of how a particle orbit behaves under the effect of a turn-dependent dipolar kick. Afterwards, we will use the mathematical formalism of an analytical approach to observe what happens in the LHC if we consider the contribution of a single arc. In parallel, the dipoles chain equivalent model has been analysed to compute the phase shift between two consecutive dipoles. Finally, we will proceed to combine the two analyses carried out to understand, starting from the noise spectrum, what is the maximum displacement that the particles would have with a noisy LHC arc at 50 Hz noise frequency.
1. Introduction

Transverse beam dynamics introduction

- Dipolar kick: analytical approach
- Analysis of a perturbed arc
- Interference pattern
- Displacement due to 50 Hz noise in the beam spectrum in an arc of LHC

LHC sector circuit model

- Current in each aperture of the dipoles
- Phase shift between two consecutive dipoles

Conclusion

Figure 1.8. Structure of the thesis work.
Chapter 2

Transverse Beam Dynamics

This chapter aims to introduce the reader to the transverse beam dynamics in order to get sufficient knowledge to understand the transverse behavior of a particle under the effect of different magnetic fields. In particular, after a brief overview of the different beam orbits, introducing the useful Ferret-Serret framework, we will move on to describe the particle motion under the effect of a dipolar and quadrupolar magnetic field. Furthermore, we will see the strong analogy that occurs between beam dynamics and linear optics reaching a closed equation for the transfer motion in case of a periodical lattice.

2.1 Particle Accelerator Overview

An accelerator lattice is composed by a series of magnetic, electrostatic and electromagnetic elements separated by drift spaces. In most cases, the lattice is dominated by magnetic dipoles and quadrupoles that constitute the linear lattice and guide the beam in the same way as lenses do with light. The line in which the main dipoles are aligned is called central or reference orbit and it is the trajectory followed by the reference particle. Actually, due to the non-ideality of the machine, particles behave differently than the reference particle. In this case, the enforced periodicity induced from the synchrotron lattice defines a new equilibrium orbit and obliges the new trajectory to be closed. For this reason, it is often called the closed orbit. Moreover, particles having the same momentum as the reference particle, but with small spatial deviations will oscillate about the equilibrium orbit with what are known as betatron oscillations. Instead, particles with a different momentum will have a different closed orbit that will be referred to as an off-momentum or off-axis closed orbit. Off-momentum particles with small spatial displacements will perform betatron oscillations about their off-momentum closed orbit.

A formalism very common to describe the motion of the particles travelling in the accelerator is that of linear optics. It is important to underline that with this description we assume that the variation of the particle coordinates depends linearly on the coordinates themselves. The number of coordinates needed to describe the particle motion will depend on the degrees of freedom we are considering. Several reference systems can in principle be chosen, e.g., a laboratory reference, but we

1Field-free zone
can simplify the formalism by expressing the motion as relative to a given particle, the reference particle [48]. In other words, we choose a reference system that is co-moving with the reference trajectory using the so called Frenet-Serret coordinate system [47] (for more details see Appendix A). In Figure 2.1 two reference systems are shown: the laboratory reference system \( \{X, Y, Z\} \) in blue and the co-moving reference system \( \{x(s), y(s), z(s)\} \) in black. It is worth noting that the latter depends on \( s \), the longitudinal abscissa on the reference orbit (black line). Note that according to the definition, the reference orbit is described by the fixed point \((0, 0, 0)\) with respect to the \( \{x(s), y(s), z(s)\} \) frame.

2.2 Particle Motion in Dipole and Quadrupole Magnets

In the case of on-momentum particle with \( p = p_0 \), let’s consider the transverse components (perpendicular respect to the particle motion) of the magnetic field \( B_x \) and \( B_y \). They can be expanded by means of Taylor series up to the quadrupole component (truncated at the first order).

\[
B_y = \mp B_0 + \frac{\delta B_y}{\delta x} x = \mp B_0 + B_1 x; \quad B_x = \frac{\delta B_y}{\delta x} y = B_1 y, \tag{2.1}
\]

where \( B_0 \) is the dipole field defining a closed orbit and the quadrupole gradient function \( B_1 = \frac{\delta B_y}{\delta x} \) is evaluated at the closed orbit. It is worth to recall an “engineering”
formula which relates the momentum of the particles to its **magnetic rigidity** or reluctance to be deviated by the magnetic field, defined as:

$$B\rho = \frac{p}{|q|} \quad (2.2)$$

where \( p \) and \( q \) are respectively the momentum and the charge of the considered particle. The magnetic rigidity quantifies how much the beam tends to be bent. The transverse motion of a particle in a periodic lattice is described by the Hill equation \( [21] \):

$$\frac{d^2 \xi}{ds^2} + K_\xi(s)\xi = \frac{1}{\rho_0(s)} \frac{\Delta p}{p_0}. \quad (2.3)$$

where \( \xi \) is a general coordinate (it represents \( x \) in the horizontal plane and \( y \) in the vertical plane) and \( K_\xi(s) \) is the effective focusing function (with dimension \([m^{-2}]\)) that takes into account the focusing of different magnetic field orders. Note that (2.3) is similar to the equation of the harmonic oscillator, with the exception that \( K_\xi \) depends on \( s \) and it is periodic with period \( C \) (\( K_\xi(s) = K_\xi(s+C) \)). The aforesaid equation may be deduced classically by means of the Lorentz force (Appendix C) or by the relativistic Hamiltonian (Appendix B) in the well known Frenet-Serret coordinate system.

According to the magnetic forces involved in the lattice, the (2.3) can be made explicit in different ways depending on which plane we are considering. As we mentioned previously, in order to use the optical formalism, we need that the particle coordinates vary linearly in the lattice. Therefore, for our purposes, we consider only magnetic fields generated by dipoles and quadrupoles. Hence, the function \( K_\xi \) in the (2.3) assumes the values shown in [Table 2.1](#).

<table>
<thead>
<tr>
<th>Element</th>
<th>( K_x )</th>
<th>( K_y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic combined-function with horizontal bend</td>
<td>( \frac{1}{\rho^2} - k )</td>
<td>( k )</td>
</tr>
<tr>
<td>Magnetic combined-function with vertical bend</td>
<td>( -k )</td>
<td>( \frac{1}{\rho^2} + k )</td>
</tr>
<tr>
<td>Pure magnetic quadrupole</td>
<td>( -k )</td>
<td>( k )</td>
</tr>
<tr>
<td>Pure magnetic horizontal bend</td>
<td>( \frac{1}{\rho^2} )</td>
<td>0</td>
</tr>
<tr>
<td>Pure magnetic vertical bend</td>
<td>0</td>
<td>( \frac{1}{\rho^2} )</td>
</tr>
<tr>
<td>Drift space</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

In [Table 2.1](#) the quantity \( 1/\rho^2 \) represents the weak focusing effect induced by dipoles. Note that it depends quadratically by the curvature radius. Hence, in order to keep the beam focused in large machines, it is quite convenient to use dedicated quadrupolar components (strong focusing) \([13]\).

It is essential to stress that in a quadrupole, where \( 1/\rho = 0 \), we have \( K_x = -K_y \), i.e., a horizontally focusing quadrupole is also a vertically defocusing quadrupole.

---

\(^2\)George William Hill (March 3, 1838 – April 16, 1914) was an American astronomer and mathematician. Aside from its original application to lunar stability, the Hill equation appears in many settings including the modelling of a quadrupole, mass spectrometer, quantum optics of two-level systems, and in accelerator physics.
and vice versa. For this reason is necessary to alternate focusing quadrupole in the horizontal plane and focusing quadrupole in the vertical plane, in order to keep the beam squeezed along its orbit.

### 2.2.1 Dipoles

Consider a circular accelerator with for particles of momentum $p$ and charge $q$ with $N$ dipoles of length $L$ (or effective length $l$, i.e., measured on beam path). By referring to [Figure 2.2](#) we can define:

$$
\theta = \frac{2\pi}{N} \quad \rightarrow \text{Dipole bending angle}
$$

$$
\rho = \frac{l}{\theta} \quad \rightarrow \text{Dipole bending radius}
$$

$$
Bl = \frac{2\pi p}{N q} \quad \rightarrow \text{Integrated dipole strength}
$$

![Figure 2.2. Bending of the trajectory due to a dipole of magnetic field $B$ and length $l$.](image)

From the [(2.4)](#), it is clear that, for a given bending angle and beam magnetic rigidity, by choosing a dipole field, the dipole length is imposed and vice versa. Furthermore, the higher is the field, the shorter or smaller number of dipoles can be used. Therefore, the circumference is influenced by the field choice.

### 2.2.2 Quadrupoles and Linear Optics

A conventional warm quadrupole is composed of four shaped poles symmetrically disposed around the beam direction of propagation. As mentioned before, it is used to focus, or defocus, the beam and it does not act on the particles moving on the reference trajectory. In fact, quadrupoles are aligned on the reference trajectory in order to obtain field-free region in the middle. Observing [Figure 2.3](#) the components generated in each pole cancel each other out in the middle. The field configuration is such that we have focusing on the horizontal plane and defocusing in the vertical one.
and, in order to switch the focusing plane, the quadrupole needs to be rotated 90°. Hence, they exert a linearly-increasing Lorentz force, through a linearly-increasing magnetic field in the following way:

\[
B_x = G y \quad \Rightarrow \quad F_x = -qv_z B_y = -qv_z G x
\]
\[
B_y = G x \quad \Rightarrow \quad F_y = +qv_z B_x = +qv_z G y ,
\]

where \( G \) is the gradient of the quadrupole magnet defined as:

\[
G = \frac{2 \mu_0 n I}{r_{\text{aperture}}^2} \left[ \frac{T}{m} \right] = \frac{B_{\text{poles}}}{r_{\text{aperture}}^2} \left[ \frac{T}{m} \right],
\]

where \( \mu_0, n \) and \( I \) represent respectively the magnetic permeability, number of the magnetic coils and the current. In order to give a feeling on the orders of magnitude involved on this kind of machines, LHC main quadrupole magnets have \( G \approx 25 \ldots 235 \left[ \frac{T}{m} \right] \).

Generally, the guide field is constructed of magnetic segments, in each of which \( K(s) \), in (2.3), may be taken as a constant so that the integration can be made algebraically for each segment and the motion can be pieced together from such solutions:

\[
K > 0 : \quad x = a \cos(\sqrt{K} s + b) \\
K = 0 : \quad x = as + b \\
K < 0 : \quad x = a \cosh(\sqrt{-K} s + b)
\]

where \( a \) and \( b \) are constants in each segment and may be determined from the values of \( x \) and \( x' \) at the entrance to the segment. According with the so called ray transfer matrix analysis \[50\], we want express the solution in the following way:

\[
\begin{pmatrix} x \\ x' \end{pmatrix}_2 = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_1.
\]
For instance, in the case of a focusing element \((K > 0)\) the transfer matrix is:

\[
T = \begin{pmatrix}
\cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}l) \\
-\sqrt{|K|} \sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l)
\end{pmatrix}
\] (2.9)

where \(l\) is the quadrupole length. In the thin lens approximation this means that \(Kl\) is small compared to \(1/l\), and we get:

\[
T_{\text{Focus}} = \begin{pmatrix}
1 & 0 \\
-Kl & 1
\end{pmatrix};
\] (2.10)

Note that if we define the focal length of a quadrupole as \(f = (Kl)^{-1}\), substituting in (2.10), we obtain the same optical matrix of a focusing thin lens. Hence, the transverse beam dynamic is described by the same law that describes geometrical optics. That is the reason why, usually, it is called optical lattice. Following the same procedure for \(K < 0\) and \(K = 0\), we can compute the matrices respectively for the defocusing quadrupole and for the drift space.

\[
T_{\text{Focus}} = \begin{pmatrix}
1 & 0 \\
-Kl & 1
\end{pmatrix}; \quad T_{\text{Drift}} = \begin{pmatrix}
1 & l \\
0 & 1
\end{pmatrix}; \quad T_{\text{Defocus}} = \begin{pmatrix}
1 & 0 \\
\frac{1}{f} & 1
\end{pmatrix}.
\] (2.11)

In order to have an overall focusing, it is necessary to alternate the gradients forming the so-called \textbf{FODO} cell \[^{[20]}\]. In Figure 2.5 is shown a simple example of how particles behave in the trace-space following the optical path in Figure 2.4.

![Figure 2.4. Focusing and defocusing lenses](image)

Note that, through the optical path the beam rotates in the trace-space but, in general, the beam never presents a laminar behaviour and each particle has a different trajectory. Furthermore, we will see how each particle, at a given location \(s\) of the machine, forms in the phase-space an ellipse when plotted for several turns (it will be clearer in further sections).

\[^{[3]}\text{This matrix does not have a modulus of unity. Then the phase-space is not conserved. This issue is due to approximations. To solve this problem we have to put } t_{12} = 0 \text{ keeping the integral of the gradient in term } t_{21} \text{ to give the approximation for a thin quadrupole lens of zero-length.}\]
2.3 Linear Betatron Motion

Assuming $\xi = x$, the solution of transverse motion for on-momentum particles for the (2.3) is

$$x(s) = \sqrt{2J_x} \beta_x(s) \cos[\mu_x(s) + \delta],$$

(2.12)

where, $J_x$ (particle action) and $\delta$ (particle phase) depend on the initial conditions. The function $\beta_{x,y}(s)$, which is called the betatron function, is related with the effective focusing function $K(s)$ by means the following equation:

$$2\beta\beta'' - \beta^2 + 4\beta^2 K = 4.$$  

(2.13)

For simplicity, we dropped the subscript taking into account that the following equations hold for both planes $x$ and $y$.

The betatron function describes completely the lateral focusing properties of the guide field. For definition, it must be periodic ($\beta(s_0) = \beta(s_0 + C)$), where $C$ is the ring circumference, and it must be always positive-definite.

---

Note that in some text $x$ is defined as:

$$x(s) = \sqrt{2J_x} \beta_x(s) \cos[\mu_x(s) + \delta]$$

. It depends on $J_x$ definition.
As an particle makes one complete revolution in a circular machine starting at, for instance $s = 0$, its oscillation phase $(\mu(s_0) + \delta)$ advances by:

$$\psi(C) = \int_0^C \frac{ds}{\beta(s)}.$$  \hspace{1cm} (2.14)

In any complete revolution the phase increases by this same amount. This phase advance is an important parameter of a storage ring, as LHC, and is usually written as $2\pi Q$.

We define $Q_i$, with $i = x, y$, as the tune number and it represents the number of oscillations per revolution in the respective plane. In other words, by definition, we have:

$$Q_i = \frac{1}{2\pi} \int_0^C \frac{ds}{\beta_i(s)} = \frac{1}{2\pi} \oint \frac{ds}{\beta_i(s)}.$$  \hspace{1cm} (2.15)

For our purposes, it is essential to underline that if $Q_i$ is an integer, the betatron oscillation would ideally become periodic, repeating itself each revolution. However, there will be surely an imperfection in the guide field (even small). This imperfection will act as a perturbation, which is synchronous with the oscillation frequency, leading to a resonance excitation of the oscillations and an exponential growth of the amplitude. This resonances may occur when:

$$mQ_x + nQ_y = r,$$  \hspace{1cm} (2.16)

where $m, n, r$ are integers. We will look further into this aspect in the next chapter.

### 2.3.1 Twiss Functions

Considering the x-plane (the procedure will be similar in y-plane), from $\beta_x(s)$ we can define other two useful functions:

$$\alpha_x(s) = -\frac{1}{2} \frac{d\beta_x(s)}{ds} = -\frac{\beta'_x}{2}, \quad \gamma_x(s) = 1 + \frac{\alpha_x^2(s)}{\beta_x(s)}.$$ \hspace{1cm} (2.17)

where $\alpha_x(s), \beta_x(s)$ and $\gamma_x(s)$ are the Twiss functions$^5$ and they depend only on the machine optics.

Computing the derivative of $(2.12)$ respect to $s$, leads to:

$$\begin{cases} x(s) = \sqrt{\beta_x(s)} J_x \cos (\mu_x(s) + \mu_{x,0}) \\ x'(s) = -\sqrt{\beta_x(s)} \left\{ \alpha_x(s) \cos (\mu_x(s) + \mu_{x,0}) + \sin (\mu_x(s) + \mu_{x,0}) \right\} \end{cases}$$ \hspace{1cm} (2.18)

From the first equation of $(2.18)$:

$$\cos (\mu(s) + \mu_0) = \frac{x(s)}{\sqrt{J_x \beta_x(s)}}.$$  \hspace{1cm} (2.19)

Remembering the Twiss functions$^5$ and solving for $J_x$, we obtain:

$$J_x = \gamma_x(s) x(s)^2 + 2 \alpha_x(s) x(s) x'(s) + \beta_x(s) x'(s)^2.$$  \hspace{1cm} (2.20)

$^5$Note that we have different Twiss functions for each reference plane.
This equation is a parametric representation of an ellipse in the $xx'$ space and $J_x$ is a constant of the motion known as the Courant-Snyder invariant or action. In other words, for each trajectory $J_x$ is known and it is independent on $s$. This a consequence of a theorem for classical mechanics, the Liouville’s theorem. This theorem is valid also for the non-linear system, but for linear machine it implies that, if there is no beam energy change, the area of the ellipse in the phase space is constant. This means that the area of Figure 2.6 given by $\pi J_x = \varepsilon$, is an intrinsic beam parameter and cannot be changed by the focal properties of the lattice. It is called Emittance ($\varepsilon$); it is a measure for the average spread of particle coordinates in position-and-momentum phase space and has the dimension of length (or length times angle depending on the definition).

![Figure 2.6. Phase-space](image)

Note from the (2.20) that the shape and the orientation of the ellipse are given by the Twiss functions. Then, a large $\beta$-function corresponds to a large beam size and a small beam divergence and wherever $\beta$ reaches a maximum or a minimum, $\alpha = 0$.

### 2.3.2 Transfer Matrix in a Periodic Lattice

Without entering into details, the direct consequence of Liouville’s theorem is to consider in the optics formalism only matrices that satisfy the Symplectic condition. An arbitrary $2n \times 2n$ transfer matrix ($M$) is said to be symplectic if the following condition is verified:

\[
M^T J M = J,
\]

where $M$ is the symplectic matrix instead $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ is the anti-symmetric matrix.

The symplectic linear transformation preserves the phase-space areas.
that all the matrices in (2.11) satisfy this condition) but is essential to underline
that this is a necessary but not sufficient condition.

From the (2.18), using some trigonometric formulas, we can find the following
transfer matrix (see Appendix D):

$$M_{0\rightarrow s} = \begin{pmatrix}
\sqrt{\beta_0} (\cos \mu_s + \alpha_0 \sin \mu_s) \\
\sqrt{\beta_s \beta_0} \sin \mu_s \\
\sqrt{\beta_s} \beta_0 \sin \mu_s \\
\sqrt{\beta_0} (\cos \mu_s - \alpha_s \sin \mu_s)
\end{pmatrix}, \quad (2.21)
$$

where $\alpha_0$ and $\beta_0$ are the Twiss parameters computed at the starting point (note that
the determinant of the aforesaid matrix is equal to one). Then, we can compute the
single-particle trajectories between two locations using simply the equation:

$$\begin{pmatrix}
x \\
x'
\end{pmatrix}_s = M_{0\rightarrow s} \begin{pmatrix}
x \\
x'
\end{pmatrix}_0. \quad (2.22)
$$

Considering an entire turn in a periodic lattice leads to have in (2.21) $\beta_0 \equiv \beta_s$ and
$\alpha_0 \equiv \alpha_s$ since the starting point and the end point coincide. Therefore, we can
define the **One turn map** as:

$$M_{OTM} = \begin{pmatrix}
\cos \mu_L + \alpha_s \sin \mu_L \\
-\gamma_s \sin \mu_L \\
\beta_s \sin \mu_L \\
\cos \mu_L - \alpha_s \sin \mu_L
\end{pmatrix}, \quad (2.23)
$$

where $\mu_L$ is the phasae-advance for one turn in the lattice and, from the (2.15), it
can be replaced by $\mu_L = 2\pi Q_x$. 
Chapter 3

Analytical Study of Harmonic Dipolar Field Perturbations

In this chapter, we will consider the excitation of the particle trajectory if we introduce different kinds of perturbations. In particular, as first step of the analysis, we will describe how the particle closed orbit changes under the effect of time-constant dipolar perturbations. Therefore, the particle orbit is no longer coinciding with the reference orbit. Afterwards, the study will proceed considering an harmonic dipolar perturbation. The aim is to achieve a closed-form expression allowing to recover the particle position spectrum from the kick noise spectrum.

3.1 Normalised Coordinates

For our perturbative analysis, it is quite convenient to consider normalised coordinates. This leads us to simplify the problem converting the Hill’s equation into that of an harmonic oscillator and the phase-space in a pure circular rotation. Therefore, we want to express the transfer matrix (2.21) in the following way:

\[
M_{s_1 \rightarrow s_2} = \bar{P}(s_2) \begin{pmatrix} \cos \mu & \sin \mu \\ - \sin \mu & \cos \mu \end{pmatrix} \bar{P}^{-1}(s_1),
\]

(3.1)

where \( \bar{P}(s_2) \) and \( \bar{P}(s_1)^{-1} \) are respectively:

\[
\bar{P}(s_2) = \begin{pmatrix} \sqrt{\beta_2} & 0 \\ - \frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix}; \quad \bar{P}^{-1}(s_1) = \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ \alpha_1 \frac{1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}.
\]

(3.2)

In this way, we can move from the physical phase-space to the normalised phase-space where the periodic motion is just a clockwise rotation of the angle \( \mu \). Hence, the purpose is to consider the computation of the particle motion in the normalised space and, afterwards, to go back in the physical-space with the new coordinates, as shown in Figure 3.1.
3. Analytical Study of Harmonic Dipolar Field Perturbations

\[
\begin{align*}
\bar{X}(s_1) & \xrightarrow{M(\Delta\mu)} \bar{X}(s_2) \\
\bar{P}^{-1}(s_1) & \xrightarrow{} \bar{P}(s_2) \\
X(s_1) & \rightarrow \bar{X}(s_2)
\end{align*}
\]

Figure 3.1. From the physical to the normalised phase-space.

Henceforth, if not differently stated, we are referring to the transfer matrix in the
normalised-space simply with \( M \).

\[
M = \begin{pmatrix}
\cos \mu & \sin \mu \\
-\sin \mu & \cos \mu
\end{pmatrix}. \tag{3.3}
\]

3.2 Closed Orbit from Single Dipole Kick

Let us consider a single dipole kick \( \theta = \delta x' \) that provides an instantaneous bend-
ing to the particle. Moreover, let us consider an ideal periodic lattice with the
aforementioned kick located in \( s_0 \) as shown in Figure 3.2.

\[
\text{C}
\]

\[
\cdots \hspace{1cm} \text{Kick } \theta \hspace{1cm} \cdots
\]

\[
s_0 \hspace{2cm} s_0 + C
\]

Figure 3.2. Ideal lattice with single kick.

In order to find the new orbit induced by the kick, we can solve the problem
considering a linear time-invariant system with a perturbation. Referring to Figure 3.3
we look for a closed orbit solution, that is \( X_N = X_{N-1} \) (represented as a feedback
loop), where the block represents the transfer matrix (3.3).

From that, it yields:

\[
\bar{X}_N = M \bar{X}_N + D \Rightarrow \bar{X}_N = \left[1 - M \right]^{-1} D, \tag{3.4}
\]

where \( X_N = (x_N, x'_N)^T \) and \( D = (0, \theta)^T \). Solving the (3.4) for \( \bar{X}_1 \) leads to:

\[
\begin{cases}
\bar{x}(s_0) = \frac{\theta}{2} \cot (\pi Q_x) \\
\bar{x}'(s_{0+}) = \frac{\theta}{2}
\end{cases} \tag{3.5}
\]

\footnote{Note that the subscript \( n \) was dropped since the equation does not depend on the turn.}
Therefore, the (3.5) represents the initial conditions of the closed orbit at the location of the kick. For the generic location $s$ around the ring the orbit distortion is written as:

$$\bar{x}(s) = \theta \frac{\cos(\pi Q - |\mu(s) - \mu_0|)}{2 \sin(\pi Q)}.$$  \hspace{1cm} (3.6)

Note the $\sin(\pi Q)$ in the denominator, which is related the value of the maximum distortion amplitude. Indeed, if $Q$ is an integer, the betatron oscillation diverges to infinity leading to an instability. As far as the phase-space is concerned, Figure 3.4 shows how the dipole kicks add-up in consecutive turns for $Q = n$, where $n$ is an integer. Therefore, as we said previously, integer tune excites orbit oscillations leading to the resonance and then the orbit becomes unstable. However, if $Q \neq n$, in particular, if $Q = n/2$, dipole kicks get cancelled in consecutive turns. In other words, closed orbit distortion is most critical for tunes close to integer.

As the final step, we can return easily in the physical-space by means of the (3.2),
obtaining the well known equation:

\[ x(s) = \theta \frac{\sqrt{\beta(s) \beta_0}}{2 \sin(\pi Q)} \cos (\pi Q - |\mu(s) - \mu_0|). \tag{3.7} \]

Furthermore, we know that in reality dipole errors are distributed around the accelerator. Since we are in a linear system, the closed orbit in presence of multiple dipole kicks along \( s \), i.e.:

\[ x(s) = \frac{\sqrt{\beta(s)}}{2 \sin(\pi Q)} \sum_i \theta(i) \sqrt{\beta(i)} \cos(\pi Q - |\mu(s) - \mu(i)|). \tag{3.8} \]

### 3.3 Harmonic Dipolar Perturbation

Let us consider now an harmonic dipolar Perturbation \( \Theta(n) \) located in \( s = s_0 \). Differently from the previous case, the perturbation is dependent on the turn and therefore our system is not time-invariant anymore. However, we want to describe the particle motion in this new configuration.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.5}
\caption{Lattice}
\end{figure}

With this purpose, we refer to Figure 3.5 in the normalised space. The particle trajectory is described by:

\[ \bar{X}_N = M \bar{X}_{N-1} + D_N, \tag{3.9} \]

where \( D_N = \begin{pmatrix} 0 \\ \Theta(N) \end{pmatrix} \) is the perturbation, \( \bar{X}_N = \begin{pmatrix} \bar{x}(N) \\ \bar{x}'(N) \end{pmatrix} \) is the vector of the trace-space coordinates and \( N \) is the turn number. Thanks to the periodicity of \( M \) (\( M \) in fact is still time-invariant), we can expand the equation (3.9) in the following way:

\[ \bar{X}_N = M^N \bar{x}_0 + M^{N-1}D_1 + M^{N-2}D_2 + \cdots + M^{N-N}D_N. \tag{3.10} \]

In order to compute the power of the matrix \( M \), it is possible to express it as a diagonal-factorization in the following way:

\[ M = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}, \tag{3.11} \]
where \( \lambda_i \) are the eigenvalues of \( M \) and \( P \) is the matrix composed by its eigenvectors. Furthermore, we can express the power of the matrix, seen in (3.10), as:

\[
M^N = \left( \prod_{i=1}^{2n} \right) PGP^{-1} \times \left( \prod_{i=2}^{N} \right) PGP^{-1} = PG^N P^{-1}.
\] (3.12)

It is interesting to see that stability depends only on the eigenvalues of \( M \). Let’s recall some useful properties:

- For a real matrix the eigenvalues, if complex, appear in complex conjugate pairs.
- For a symplectic matrix \( M \):

\[
\prod_{i}^{2n} \lambda_i = 1
\] (3.13)

This allows us to consider for the symplectic matrix \( M \) the eigenvalues \( \lambda_1 = e^{-i\mu} \) and \( \lambda_2 = e^{i\mu} \). Hence, the power of the matrix \( G \) can be easily expressed as a simple scalar multiplication, in the following way:

\[
G^N = \left( \begin{array}{cc}
    e^{-iN\mu} & 0 \\
    0 & e^{iN\mu}
\end{array} \right).
\] (3.14)

Going further, the matrix \( P \) and \( P^{-1} \) can be computed through the eigenvectors of the matrix \( M \):

\[
P = \begin{pmatrix}
i & -i \\
1 & 1
\end{pmatrix}; \quad P^{-1} = \begin{pmatrix}
-\frac{i}{2} & \frac{1}{2} \\
\frac{i}{2} & \frac{1}{2}
\end{pmatrix}.
\] (3.15)

Therefore, the (3.12) becomes:

\[
M^N = \begin{pmatrix}
\cos(N\mu) & \sin(N\mu) \\
-\sin(N\mu) & \cos(N\mu)
\end{pmatrix}
\] (3.16)

Note that the kick modifies locally just the angle. Therefore, in (3.10), the elements \( M_{11} \) and \( M_{21} \) are not involved in the product. Hence, according with the previous considerations, the (3.9) becomes:

\[
\begin{cases}
\ddot{x}(N) = \sum_{n=0}^{N} \Theta(n) \sin ((N-n)\mu) \\
\dot{x}'(N) = \sum_{n=0}^{N} \Theta(n) \cos ((N-n)\mu)
\end{cases}.
\] (3.17)

Note that the previous equation is general and holds for any kind of perturbation starting at \( n = 0 \). More in general, we are interested to extend \( n \to -\infty \) in order to consider periodic \( \Theta(n) \). In this case, the (3.17) becomes:

\[
\begin{cases}
\ddot{x}(N) = \sum_{n=-\infty}^{N} \Theta(n) \sin ((N-n)\mu) \\
\dot{x}'(N) = \sum_{n=-\infty}^{N} \Theta(n) \cos ((N-n)\mu)
\end{cases}.
\] (3.18)

Finally, following the steps shown in Figure 3.1, we can return in the physical space by means of the transformation matrix \( \bar{P} \).
3.3.1 Example: sinusoidal perturbation

Assume a ring affected by a dipolar kick of the following type:

\[ \Theta(n) = \theta \cos(2\pi Q_p n), \] (3.19)

where \( n, \theta \) and \( Q_p \) are respectively the number of turn considered, the kick amplitude and the number of noise oscillations per turn. The (3.18) becomes:

\[
\begin{align*}
\bar{x}(N) &= \sum_{n=\infty}^{N} \theta \cos(2\pi Q_p n) \sin ((N-n)\mu) \\
\bar{x}'(N) &= \sum_{n=\infty}^{N} \theta \cos(2\pi Q_p n) \cos ((N-n)\mu).
\end{align*}
\] (3.20)

The aforesaid series converges to:

\[ \bar{x}(N) = \frac{\theta \cos(2\pi NQ_p) \sin(2\pi Q)}{2(\cos(2\pi Q_p) - \cos(2\pi Q))}. \] (3.21)

Note that for \( Q_p = Q \pm \frac{m}{2} \), where \( m \) is an integer, we have a resonance condition in which the betatron oscillation diverges.

<table>
<thead>
<tr>
<th>Table 3.1. Parameters.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>( \theta )</td>
</tr>
<tr>
<td>( Q )</td>
</tr>
<tr>
<td>( N )</td>
</tr>
<tr>
<td>( Q_p )</td>
</tr>
</tbody>
</table>

Figure 3.6. Comparison between the (3.17) and a Python tracking code.
Let us consider, for instance, the parameters in Table 3.1. In Figure 3.6 the comparison between the (3.17) and a tracking code written in Python is shown and the agreement is perfect [38]. As we mentioned previously, we expected a resonance in $Q_p = 62.27 \pm \frac{m}{2}$ with $m = 1, 2,\ldots$ as arbitrary integer. Indeed, in Figure 3.7 the betatron oscillation, for 50 turns is shown. For $Q_p = 6.73$ and $Q_p = 7.27$, the linear increase of the oscillations is visible.

![Figure 3.7. Plotting of (3.17) for different $Q_p$.](image)

As final check, we can have a look at the (3.21) observing the closed orbit evolution. In Figure 3.8 the closed orbit is plotted for different kick frequencies. Note that, in this case, the closed orbit varies with the turn but, for integer values of $Q_p$, the perturbation becomes a fixed kick inducing the closed orbit to be independent of time. Indeed, if we consider zero frequency (in this case even for $Q_p$ integer), the (3.21) returns to be the closed orbit equation seen previously in the case of single fixed kick (3.5).

\[
\bar{x}_{CO} = \theta \frac{\sin (2\pi Q)}{2 \left(1 - \cos (2\pi Q)\right)}. \tag{3.22}
\]

Recalling some trigonometric properties:

\[
\sin (2\pi Q) = 2 \sin (\pi Q) \cos (\pi Q);
\]

\[
1 - \cos (2\pi Q) = 2 \sin^2 (\pi Q),
\]

and substituting in (3.22) we find back the (3.5). It is worth reminding that, one may convert the metric of the problem in the physical space units using the $\bar{P}$ from (3.2). In this case, observation point and kick location coincide, therefore

\footnote{Python is an interpreted, high-level, general-purpose programming language.}
$x(N) = \beta_0 \bar{x}(N)$. Note that in general we need also the $\bar{P}$ to get the kick in the normalised coordinated. Therefore, the aforesaid $\theta$, used in the previous equations, refer to the normalised space.

![Figure 3.8. Closed orbit for different $Q_p$.](image)

3.3.2 Multiple kicks

Firstly, let’s consider two perturbations: $\Theta_1(n)$ positioned in $s = s_1$ and $\Theta_2(n)$ positioned in $s_2 = s_0 + C = s_0$, as shown in Figure 3.9.

![Figure 3.9. Ring with two kicks.](image)

where $\phi_{s_1} + \phi_{s_2} = 2\pi Q$. In order to simplify the notation, from here on, we will indicate the respective phase advance with $\phi_1$ and $\phi_2$. Since we are in a linear system, we can just compute the superposition of the two effects. Therefore, considering the first kick, we have:

$$\bar{X}_N = M_{s_1 \rightarrow s_0} M_{s_0 \rightarrow s_1} \bar{X}_{N-1} + M_{s_1 \rightarrow s_0} D_N.$$  

(3.23)
Thanks to the periodicity, we can expand the aforesaid equation in the following way:

\[ \bar{X}_N = M^N \bar{X}_0 + M_{s_1 \rightarrow s_0}[M^{N-1}D_1 + M^{N-2}D_2 + \cdots + M^{N-N}D_N], \tag{3.24} \]

where, as usual, we consider the initial condition equal to \( X_0 = (0, 0)^T \). Note that (3.24) and (3.10) are identical, except for a semi-rotation of \( \phi_2 \). Therefore, taking into account what just mentioned, the betatron oscillation, under the effect of \( \Theta_1(n) \), is written as:

\[ \bar{x}(N) = \sum_{n=-\infty}^{N} \Theta_1(n) \sin (2\pi Q(N-n) + \phi_2); \tag{3.25} \]
\[ \bar{x}'(N) = \sum_{n=-\infty}^{N} \Theta_1(n) \cos (2\pi Q(N-n) + \phi_2). \tag{3.26} \]

Applying the superposition principle leads to:

\[ \bar{x}(N) = \sum_{n=-\infty}^{N} \left[ \Theta_1(n) \sin (2\pi Q(N-n) + \phi_2) + \Theta_2(n) \sin (2\pi Q(N-n)) \right]; \tag{3.27} \]
\[ \bar{x}'(N) = \sum_{n=-\infty}^{N} \left[ \Theta_1(n) \cos (2\pi Q(N-n) + \phi_2) + \Theta_2(n) \cos (2\pi Q(N-n)) \right]. \tag{3.28} \]

As a consequence of the superposition, there are, in general, two resonance conditions: we will have as many resonances as many distinct perturbations in the ring. Therefore, following the last example, if we consider \( \Theta_1(n) = \theta \cos (2\pi Q_{p_1} n) \) and \( \Theta_2(n) = \theta \cos (2\pi Q_{p_2} n) \), the conditions become:

\[ Q_{p_1} = Q \pm \frac{m}{2}; \quad Q_{p_2} = Q \pm \frac{n}{2}, \]

where \( m \) and \( n \) are integers.

In the case of multiple kicks, is possible to follow the same approach. Let us consider this time \( M \) kicks arranged along the machine. After one turn the particle position is the result of the superposition of all kicks during the run. This leads to:

\[ x(N) = \sum_{n=-\infty}^{N} \sqrt{\beta_0} \sum_{m=1}^{M} \sqrt{\beta_m} \Theta_m(n) \sin (2\pi Q(N-n) + \sum_{j=m+1}^{M} \phi_j \delta_{-1}(M-m)). \tag{3.29} \]

This time, for completeness, the betatron oscillation has been written in the physical trace-space.
Chapter 4

LHC Main Dipolar Chain Perturbation and Interference Pattern

We are going to analyse an LHC sector under the hypothesis that each dipole induces a perturbation $\Theta_m(n)$, where $m = 1, 2 \ldots, 154$ represents the number of the chosen dipole, to the betatron oscillation, as seen in the previous chapter. The goal is to see how their effects can combine together increasing or decreasing the horizontal displacement depending on the noise phase-shift.

4.1 Analysis Setup

As we mentioned in the first chapter, LHC is composed of 8 sectors (arcs) each of them having 154 main dipoles. Of course, in the sectors also quadrupoles and higher order multipoles are present but, for our purpose, we are neglecting all higher-order terms considering only the main dipoles chain. Furthermore, we make the hypothesis that, except for the arc under consideration, the ring is ideal.

Let’s consider, for instance, the first sector of LHC. The next step is to choose wisely an appropriate amplitude of the noise that fit coherently with the quantities involved in the machine (educated guess). Each dipole has a bending angle equal to:

$$\vartheta = \frac{2\pi}{154 \times 8} \approx 5 \text{ mrad} \quad (4.1)$$

Hence, if we suppose that the stability of the dipoles is $10^{-4}$, it is reasonable to expect a magnitude of the perturbation on the order of nrad. Thus, we take into account an error of $\theta = 1 \text{ nrad}$. We consider now that each dipole of the arc is affected by a perturbation of the following form:

$$\Theta_m = \theta \cos (2\pi Q_p + m\psi). \quad (4.2)$$

Furthermore, we have chosen a noise phase-shift equal to $m\psi$ in order to have equal phase shift between two consecutive dipoles. Taking advantage to the MAD-X
simulation code (*Methodical Accelerator Design*) \(^{32}\) used via the python module \texttt{cpymad}\(^{1}\), we were able to recover the following parameters of the LHC lattice during the injection beam mode\(^{2}\): 

Table 4.1. LHC main parameters computed using MAD-X during the injection beam mode.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>26.66 km</td>
</tr>
<tr>
<td>Horizontal Tune $Q_x$</td>
<td>62.27</td>
</tr>
<tr>
<td>Vertical Tune $Q_y$</td>
<td>60.295</td>
</tr>
<tr>
<td>$\beta_x^{\text{max}}$</td>
<td>590.17 m</td>
</tr>
<tr>
<td>$\beta_y^{\text{max}}$</td>
<td>641.69 m</td>
</tr>
</tbody>
</table>

In Figure 4.1 we reported the phase advance and the betatron oscillation along the first arc in LHC simulated using MAD-X. As expected, $\beta_x$ and $\beta_y$ alternate their maxima as seen in the theory of chapter 2. Furthermore, note from Figure 4.1a that the phase-advances are almost linear along $s$ with slightly different values. We want to remember that the phase-advances in one turn define resonance conditions as seen in (2.16).

In Figure 4.1a we reported the phase advance and the betatron oscillation along the first arc in LHC simulated using MAD-X. As expected, $\beta_x$ and $\beta_y$ alternate their maxima as seen in the theory of chapter 2. Furthermore, note from Figure 4.1a that the phase-advances are almost linear along $s$ with slightly different values. We want to remember that the phase-advances in one turn define resonance conditions as seen in (2.16).

![Figure 4.1. Phase-advance and betatron oscillations](image.png)

Going further, in Figure 4.2 the setup considered for the analysis is reported. As we mentioned at the beginning of this section, we considered only one arc (in blue) with 154 perturbation kicks (one for each dipole) in the form described in (1).

\(^{1}\text{cpymad is a Cython (an optimising static compiler for Python programming) binding to MAD-X for giving full control and access to a MAD-X interpreter within the python environment.}\)

\(^{2}\text{The injection beam mode represents the stage in which the machine proved to be able to have a circulating beam with an appropriate lifetime and it is ready to accept higher intensities needed for physics.}\)
4.2 Interference Pattern

The rest of the ring is supposed to be ideal and our observation point will be located in the same position of $\beta_{\text{max}}^x$ (black line) in order to have the first estimate on maximum displacement that occurs in the machine.

4.2 Interference Pattern

Summarising, we are considering for simplicity perturbations with same noise amplitude and same noise frequency but $\psi_m = m\psi$ (phase-shift) with $m = 1, 2 \ldots, 154$. This means perturbations are equally spaced from each other. In this context, we have for each noise kick that:

$$x(s, N) = \sqrt{\frac{\beta_p(s)}{\beta_{\text{max}}^x}} \sum_{n=-\infty}^{N} \beta_p(s) \Theta_m(n) \sin(2\pi Q(N - n) + \Phi_m),$$

(4.3)

where $\Phi_m$ is the phase-advance concerning the $m$-th dipole. In order to simplify the reading, we consider $\mu_p = 2\pi Q_p$ and $\mu_L = 2\pi Q$. The (4.3), considering the (4.2), leads to:

$$x(s, N) = \sqrt{\frac{\beta_p(s)}{\beta_{\text{max}}^x}} \frac{(\sin(\Phi_m + \mu_p N + m\psi - \mu_L) + \sin(\Phi_m - \mu_p N + \mu_L - m\psi)) + 4(\cos(\mu_L) - \cos(\mu_p))}{4(\cos(\mu_L) - \cos(\mu_p))} - \sqrt{\frac{\beta_p(s)}{\beta_{\text{max}}^x}} \frac{(\sin(\Phi_m - \mu_p(N + 1) - m\psi) + \sin(\Phi_m + \mu_p(N + 1) + m\psi))}{4(\cos(\mu_L) - \cos(\mu_p))}.$$

(4.4)

In Figure 4.3a the superposition of (4.4) for all 154 dipoles at the turn $N = 100$ with $Q_p = 0.22$ is reported as a function of the noise phase-shift. It is interesting to note that, depending on the phase shift, the betatron oscillation amplitude increases or decreases forming an interference pattern. However, the pattern changes periodically with the turn (Figure 4.3b).

According to our goals, we are interested in quantifying the maximum displacement that could occur in the machine, therefore, we need to find the arguments that maximise the numerator of (4.4). In Figure 4.3b the superposition made in the previous
figure for many turns is shown. Note how varying the turn, the values assumed by $|x|$ are always inside the shape, thus, in order to find the maximum, we computed the envelope of the aforesaid figure as shown in Figure 4.3c. In the latter figure, is interesting to observe how small perturbations, with order of amplitude of nrad, can lead to a betatronic oscillation of the order of micro-meters (in particular four orders of magnitude larger). Note that we expected a phase-shift of few degrees, and this is the reason why we have chosen a range of phase-shift between of $0^o < \psi < 12^o$.
in any case, the consistency of this choice, will be verified in the following chapter.

Finally, for completeness, in Figure 4.4 the behaviour of the betatronic oscillation depending on the phase-shift and on the turn number is shown. Note that, as we mentioned before, it has periodic behaviour along $N$. Moreover, if we consider the perpendicular plane with respect to the turn, we obtain the Figure 4.3c. As far as the Figure 4.5 is concerned, it shows the betatron oscillation close to the resonance condition. As expected, even if we have 154 perturbations, we observe only one point of singularity since we considered the same perturbation frequency for each dipole.

This chapter raises the question: which is the phase shift between the dipoles? We will face at this issue in the following chapters.

Figure 4.4. Betatronic oscillation as a function of the phase-shift and the turn number
Figure 4.5. Resonance for $Q = 62.27$
Chapter 5

Main Dipoles Power Supply

As far as the power supply of the magnets is concerned, we want to analyse directly the current and the voltage induced by the power convert (PC) during several FILLs in order to quantify the electrical parameters involved in the machine. Indeed, from the electrical point of view, the main dipoles are basically crossed by current through superconductive coils which have, albeit very small (ideally zero), a certain resistivity. In particular, after a brief introduction about PC, the chapter initially will report the relevant parameters in a whole FILL in order to become more familiar with different beam modes\(^1\). Afterwards, we will move on to analyse the data provided by the machine in order to quantify the resistance and inductance of the system which will be necessary to build a dipoles chain equivalent circuit model (see the next chapter).

5.1 LHC Power Converters

5.1.1 Architecture

The task of a power converter is to process and control the flow of electric energy by supplying voltages and currents in a form that is optimally suited for loads. In the case of a particle accelerator, the main loads are the magnets and the radio-frequency system. It is worth to recall that beam energy is proportional to the magnetic field since the latter is generated by the current circulating in the magnet coils.

In principle there are two broad categories of power supplies: linear regulated power supply which use silicon controlled rectifier (SCR) technology, and switched mode power supply (SMPS)\(^2\). In the case of LHC main dipoles it was decided to adopt the first one since, at the time of the LHC commissioning, SMPS was a fairly recent technology and was not completely clear if magnets, with their large currents (~11.5kA), could be powered. However, SMPS had the advantage of not introducing any appreciable ripple below the so called chopping frequency (higher than 10 kHz)\(^3\). Moreover, since the request of power to be supplied to the

\(^1\)In LHC there are two general modes: the accelerator mode and the beam mode. The accelerator mode provides a general overview of the machine activity (e.g. proton physics, access, shutdown, etc.), while the beam mode provides the state of the machine with regard to the machine cycle (e.g. injection, ramp, etc.).
magnets was high and without precedent, it was decided to split the circuit (for the first time) in 8 sectors, each powered by a different PC.

In principle, referring to Figure 5.1, the PC of the main dipoles is divided in three main sub-systems:

- A complex digital control electronics in charge of the high precision current loop and of the communication with the Cern Control Room (CCC).
- Power part acting as a voltage source.
- High precision current sensor DC Current transducer (DCCT), which measures the output current.

What is of interest for us is the power part in which the voltage source is contained. The task of the voltage source is to receive the mains power and convert it to a suitable DC output voltage, controlled by a reference input. In particular, the eight main dipole voltage sources consist of thyristor, line-commutated, power converters to which a parallel injection active filter is added to improve mains rejection and ripple performance. A parallel topology, as shown in Figure 5.2, is used and it consists of two sub-systems, each containing a six-pulse thyristor rectifier, converting AC in DC, and a passive filter to reduce the ripple amplitude generated.

---

21.2GJ stored energy per sector.
3Direct current.
4It is a solid-state semiconductor device and it acts exclusively as a bi-stable switch, conducting when the gate receives a current trigger and continuing to conduct as long as the voltage across the device is reverse biased or till the voltage is removed.
5Alternating current.
by the rectifiers. Furthermore, the active filter, connected in the passive filter capacitor branch, has a working range of $\sim 4\%$ of the total output voltage. This leads rejection of main transients and gives a wide dynamic range for the control loops [6, 39].

The rectifiers are phase-shifted by $30^\circ$ and connected in parallel in order to have, as a whole, a twelve-pulse system that performs much better in terms of rectification efficiency$^6$ and ripple content compared with the common three-pulse or six-pulse structures.

The rectifiers are phase-shifted by $30^\circ$ and connected in parallel in order to have, as a whole, a twelve-pulse system that performs much better in terms of rectification efficiency$^6$ and ripple content compared with the common three-pulse or six-pulse structures.

\[ f_r = pf_{\text{mains}}, \quad (5.1) \]

where $p$ is the number of pulses and $f_{\text{mains}}$ is the frequency of the AC source. Therefore, since mains frequency is $f_{\text{mains}} = 50Hz$, we expect a fundamental ripple frequency equal to $f_r = 600Hz$. For completeness, in Table 5.1 the main output performance parameters for the main dipole voltage sources reported in the LHC design report [6] is shown.

In this context, it is interesting to mention also that, during the 2018 proton run at CERN, it has been shown that the status of the active filters has a direct impact on the beam spectrum. In particular, disabling both at flat bottom and top energy each filter, it was observed an increase or decrease of the amplitude of the harmonics in the Beam spectrum [25]. This suggests that, actually, $PC$ plays a key role in the $50Hz$ excitation but, in order to observe how the perturbation propagates in the machine till exciting the beam, we need to build an equivalent model circuit of the

$^6$It is the ratio between DC output power and the input power from the AC supply.
Table 5.1. Main output parameters for the main dipole voltage sources.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Voltage ripple without active filter</td>
<td>$1 \times 10^{-3} @ 50 \text{Hz} ; 5 \times 10^{-3} @ 600 \text{Hz}$</td>
</tr>
<tr>
<td>Voltage ripple with active filter</td>
<td>$25 \times 10^{-6} @ 50 \text{Hz} ; 0.5 \times 10^{-3} @ 600 \text{Hz}$</td>
</tr>
<tr>
<td>Mains rejection without active filter (5 – 20 Hz)</td>
<td>$10 – 20 \text{dB}$</td>
</tr>
<tr>
<td>Mains rejection with active filter (5 – 20 Hz)</td>
<td>$50 – 60 \text{dB}$</td>
</tr>
<tr>
<td>Small signal bandwidth without active filter</td>
<td>$70 \text{Hz}$</td>
</tr>
<tr>
<td>Small signal bandwidth with active</td>
<td>$5 \text{kH}$</td>
</tr>
</tbody>
</table>

dipoles chain. This equivalent model circuit will be representative of the load in Figure 5.2 and it will be faced in the next chapter.

5.1.2 PC Behaviour and Parameters Involved

Figure 5.3 and Figure 5.4 show respectively the particle energy with the current provided to the first sector and the two beams intensity with the energy through different beam modes (time-table of the beam modes are reported in Table 5.2).

Figure 5.3. Energy and first sector current in FILL 7320.

In agreement with the above mentioned, note how the energy of the particle, in Figure 5.3, is proportional to the current during the entire FILL. There is just a discrepancy at the beginning since the beam had not yet been injected. Indeed, note that in Figure 5.4, before increasing the energy, beams are injected in both directions bunch-by-bunch according to a pre-chosen rule (for more details see [28, 5]). After the injection (precisely at 14:07:33), the PC starts the RAMP process providing energy to the particles and reaching $6.5 \text{TeV}$ ($13 \text{TeV}$ in the centre-of-mass).

7The center-of-mass energy of a system of particles is the energy measured in the center-of-mass reference frame which is the weighted average position of all the mass in the system and is the point that moves with the total momentum of the system, as if the total mass of the system were...
5.1 LHC Power Converters

Figure 5.4. Energy and Intensity of the beams in FILL 7320.

Table 5.2. Beam Modes in Fill 7320 made in 2018-10-19.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Start Time</th>
<th>End Time</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>FILL</td>
<td>12:45:03.492000103+00:00</td>
<td>06:01:32.407000065+00:00</td>
<td>17:16:28</td>
</tr>
<tr>
<td>SETUP</td>
<td>13:17:13.173000097+00:00</td>
<td>13:25:33.641999960+00:00</td>
<td>00:08:20</td>
</tr>
<tr>
<td>INJPROT</td>
<td>13:25:33.642999887+00:00</td>
<td>13:33:41.921999931+00:00</td>
<td>00:08:08</td>
</tr>
<tr>
<td>INJPHYS</td>
<td>13:33:41.923000097+00:00</td>
<td>14:04:53.861999989+00:00</td>
<td>00:31:11</td>
</tr>
<tr>
<td>PRERAMP</td>
<td>14:04:53.862999916+00:00</td>
<td>14:07:33.638999939+00:00</td>
<td>00:02:39</td>
</tr>
<tr>
<td>RAMP</td>
<td>14:07:33.6400000105+00:00</td>
<td>14:27:55.875000000+00:00</td>
<td>00:20:22</td>
</tr>
<tr>
<td>FLATTOP</td>
<td>14:27:55.875999928+00:00</td>
<td>14:30:27.203000069+00:00</td>
<td>00:02:31</td>
</tr>
<tr>
<td>SQUEEZE</td>
<td>14:30:27.203999996+00:00</td>
<td>14:41:22.252000092+00:00</td>
<td>00:10:55</td>
</tr>
<tr>
<td>ADJUST</td>
<td>14:41:22.253000021+00:00</td>
<td>14:49:07.568000078+00:00</td>
<td>00:07:45</td>
</tr>
<tr>
<td>STABLE</td>
<td>14:49:07.569000006+00:00</td>
<td>05:57:39.144999981+00:00</td>
<td>15:08:31</td>
</tr>
<tr>
<td>BEAMDUMP</td>
<td>05:57:39.145999908+00:00</td>
<td>05:58:11.927999973+00:00</td>
<td>00:00:32</td>
</tr>
<tr>
<td>RAMPDOWN</td>
<td>05:58:11.928999901+00:00</td>
<td>06:01:32.407000065+00:00</td>
<td>00:03:20</td>
</tr>
</tbody>
</table>

The main magnet parameters seen by the power supplies are:

- Inductance.
- Resistance.
- Current limits.
- Voltage limits.

In Figure 5.5 we reported the same FILL shown previously from the point of view of current and voltage, till the STABLE beam mode. In order to quantify concentrated in that point.
the aforesaid parameters, we can consider, in first approximation, that current and voltage are related in the following way:

\[ V = RI + L \frac{dI}{dt}, \]  

(5.2)

where \( R \) is the resistance induced by the resistivity of the wires. Since the wires are made in superconducting materials, we expect that \( R \) is very low.

**Superconductive Wire Resistance**

Since the inductance is related to the variation in time of the current, we can think to compute the resistance computing the ratio between voltage and current in the two plateau (blue line) showed in Figure 5.5. Therefore, Figure 5.6 and Figure 5.7 show the two resistances values and its probability density in the different LHC sectors. For simplicity, we named \( R_{\text{Down}} \) and \( R_{\text{Up}} \) respectively the resistances computed at \textit{PRERAMP} and at \textit{FLATTOP}.

As expected, in both cases, the resistance is of the order of \( m\Omega \) in every sector. Note that, a clear spread is measured in the lowest plateau due to the fluctuations at low currents. Instead, in the case of \( R_{\text{Up}} \) (Figure 5.7), the variance is much lower. Figure 5.8 reports the comparison of the two distributions in order to verify more clearly what we mentioned before. We appreciate, then, that the respective values of the resistance are \( R_{\text{Up}} = (0.940 \pm 0.019)m\Omega \) and \( R_{\text{Down}} = (0.90 \pm 0.12)m\Omega \). Therefore, taking into account the previous values, we decided to consider, as our resistance, the average of \( R_{\text{Up}} \), since it is subject to less uncertainty.

\[ ^8 \text{As temperature decreases, a superconducting material’s resistance gradually decreases until it reaches a critical temperature. At this point, the resistance drops off near to zero} \]
5.1 LHC Power Converters

Figure 5.6. Resistance measured in \textit{PRERAMP}; Fills: 7250-7335

Figure 5.7. Resistance measured in \textit{FLATTOP}; Fills: 7250-7335
5. Main Dipoles Power Supply

![Comparison between the resistances measured in the two plateau.](image)

Figure 5.8. Comparison between the resistances measured in the two plateau.

**Inductance of the Single Arc**

Once computed the resistance, we are able to compute the inductance by means of (5.2). Figure 5.9 reports the behaviour of the inductance, in each sector and for different FILLs, as a function of the current. Note that, even at zero frequencies, the magnets have different values due to their non-ideality. Furthermore, it is interesting to observe that inductances exhibit a decreasing behaviour for high current since the iron starts to saturate. If the current had increased further, we would have observed a plateau in which the inductances are completely saturated. This is the reason why we have to consider a current limit in the operation of the magnets. In conclusion, we evaluate an inductance equal to \( L = (15.176 \pm 0.064) \text{H} \), which represents the value for the whole sector. Thus, for each dipole we have \( L_{\text{dipole}} = L/154 \approx 98 m\text{H} \).
Figure 5.9. Inductance measured in different FILLs as a function of the current.
Chapter 6

Circuit Models of Superconducting DipoleMagnets

For a simple interpretation of the behaviour of LHC main dipoles, in terms of signals, it is quite convenient to model an equivalent circuit in order to approach the problem with the common and simplest rules of circuit theory. Therefore, we start to study how to build a structure of the magnet circuit model and, afterwards, we move to analyse an equivalent circuit proposed by Technology Department (TE) at CERN for the description of fast power abort on the quench detection [42]. In particular, by means of this analysis, we will be able to quantify the phase-shift between the currents of each magnet aperture in order to estimate the betatronic excitation due to the power supply. In other words, we are going to determine in which part of Figure 4.3c we should fall.

6.1 Structure of the LHC Magnet Circuit Model

Figure 6.1 shows a typical superconducting dipole including: a beam pipe\(^1\) a superconducting coil for conducting high density current to generate the magnetic fields; a laminated collar for holding the magnet together; a laminated iron yoke forming the magnetic field path and a cryogenic shell required for low temperature operation. In principle, for pure DC, dipole magnets have to appear as a pure inductance since, as we have seen in the previous chapter, the resistivity of the coils are negligible. However, LHC has PCs which work as a twelve-phase system with 50Hz signals leading currents containing AC components. In this context, eddy currents\(^2\) may be induced in the coil and in other metallic elements leading a resistive component in the load seen from the PC.

We can equivalently describe the eddy current phenomenon as a coupled inductive circuit \(^7\) as shown in Figure 6.2. The primary inductor \((kL)\) represents the inductance of the coils, while the second one represents the one turn eddy current

---

1. The beam pipes provide a path for the particle beams
2. Eddy currents are loops of electrical current induced within conductors by a changing magnetic field in the conductor according to Faraday’s law of induction
path in the beam pipe with $R_s$, its resistance contribution, when the inductor becomes frequency dependent. Note that at very low frequency, i.e. $\omega k L \ll R_s$, we return in the ideal case in which the magnet is perfectly represented by an inductor $L$. On the other hand, if $\omega k L \approx R_s$, the inductor impedance starts to be reduced. The third case occurs when $\omega k L \gg R_s$ where the inductor results constant but reduced by a factor $(1 - k)$ since $kL$ is negligible compared to the energy lost in the resistive beam pipe ($R_s$).

We can do a further step forward considering in the analysis also the coil-to-ground and turn-to-turn parasitic capacitance that affect the quality of the magnet field in the beam pipe [46, 54]. Therefore, considering these effects, we obtain the circuit reported in Figure 6.3.

A high frequency resonance occurs between the magnet inductance, the coil-to-
ground capacitances and the turn-to-turn capacitance. In order to understand the eddy currents and the parasitic capacitances effects with frequency it would be worth to characterise all the electrical parameters by means of a measuring instrument as a network analyser [33, 23] which is capable to measure them over the entire operating frequency range. Unfortunately, during this thesis project this was not possible. Therefore, we concentrated the effort on the analysis of an equivalent circuit modelled by the TE department at CERN developed for a description of fast power abort after a quench detection.

6.2 50 Hz Equivalent Circuit

In Fig. 6.4 a sketch of the equivalent circuit [40, 42], mentioned on the previous section and made by TE department, is shown. The circuit reports all the 154 superconducting twin-aperture dipole magnets in a sector. In general, each dipole has an inductance at nominal field of 98.7mH, so the total inductance of each circuit is 15.2H and its total stored energy is 1.1GJ at a nominal current of 11850A. The circuit in Fig. 6.4 consists of: a power supply; a crowbar which is an electrical circuit used for preventing an overvoltage condition of the PC; a filter composed by \( L_{\text{filter}}; C_{\text{filter}}, R_{\text{filter}} \); on the right mesh, a series of components which represent all the 154 before mentioned dipoles (M001 until M154). Therefore, each dipole implicitly describes a circuit as in Fig. 6.5 in which the two inductances represent the two apertures of the magnet. It is important to emphasise that this circuit holds just at low frequencies, around 50Hz, instead for high frequencies its correctness is no longer guaranteed. In fact, Figure 6.5 represents the simplification at low frequency of the two apertures in which, in parallel to each dipole, a resistor \( R_P \) was taken into account in order to smooth transient voltage oscillations. Further forward we will continue with the analysis of the phase shift in high frequencies in order to verify if, in these conditions, it’s correct to assume that phase shift is dominated by the length travelled by current (as intuitively one might think) instead of lumped elements. In that case, we should consider the formalism...
of transmission lines in which the delay must also be taken into account.

Figure 6.4. Schematic of the LHC main dipoles circuit [41].

Figure 6.5. Circuit representation of a dipole.

Finally, the reader can find in Table 6.1 a summary of the parameters involved in Figure 6.4.

6.2.1 Phase shift due to the lumped elements

The purpose is to solve the circuit Figure 6.4 taking into account that each dipole is represented by the Figure 6.5 in order to find the phase-shift between two consecutive
Table 6.1. Equivalent circuit parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of superconductive dipoles</td>
<td>154</td>
</tr>
<tr>
<td>Inductance $L_{mag}$ of each magnet</td>
<td>98mH</td>
</tr>
<tr>
<td>Capacitance to ground $C_g$ of each magnet</td>
<td>98mH</td>
</tr>
<tr>
<td>Parallel resistance $R_{//}$ of each magnet</td>
<td>100Ω</td>
</tr>
<tr>
<td>Capacitance $C$ of the PC filter</td>
<td>110mF</td>
</tr>
<tr>
<td>Inductance $L$ of the PC filter</td>
<td>284µH</td>
</tr>
<tr>
<td>Resistors $R$ in the filter branches (8x in parallel)</td>
<td>27mΩ</td>
</tr>
<tr>
<td>Resistance $R_{EE}$ of the extraction resistor</td>
<td>147mΩ</td>
</tr>
</tbody>
</table>

magnet apertures. The way in which we will face the issue is to reduce the circuit in a succession of series and parallel impedance as shown in Figure 6.6. In this way we can easily implement a Python code in order to compute the voltage and the current in each node. Afterwards, by means of these values, compute the current flowing in the apertures (represented by the inductance) of each magnet in order to find the phase current.

Before starting the circuit analysis, it is worth to introduce the $\Delta - Y$ transform which will be very useful for our purposes [49]. This transformation allows us to change the topology of the circuit, from a $\Delta$ shape to a $Y$ shape (and vice versa), keeping the same value of the voltages at the end of the impedances. In other words, we are carrying out a linear transformation, in which the voltages are the invariant quantities (as we have done in the beam transfer dynamic chapter in which the invariant was the Courant-Snyder parameter $J$). Therefore, this means that in Fig. 6.7 $V_A$, $V_B$, $V_C$ are the same in each configuration. What is changing are the impedances involved; in particular, referring to the figure, if we wanted to pass from a $\Delta$ shape to a $Y$ shape, we have to do the following transformations:
Instead, in order to convert from \( Y \) shape to \( \Delta \), we have to consider the following one:

\[
R_{AB} = \frac{R_{AR_B} + R_{AR_C} + R_{RB_C}}{R_C}, \\
R_{BC} = \frac{R_{AR_B} + R_{AR_C} + R_{RB_C}}{R_A}, \\
R_{AC} = \frac{R_{AR_B} + R_{AR_C} + R_{RB_C}}{R_B}.
\]  

\[ (6.2) \]

**Figure 6.7.** \( \Delta - Y \) transform.

In our case, the piece of circuit that we consider is shown in Figure 6.8. It is clear that this is a \( Y \) network (or \( T \) network). Therefore, we are interested in transforming it in a \( \Delta \) network. Using the (6.2) we obtain:
In Figure 6.9 the new topology after the linear transformation is shown. The impedance $Z_{AB}$, $Z_{BC}$ and $Z_{AC}$ are already known from (6.3) while the voltages $V_A$ and $V_C$ are unknowns (note that $V_C = 0$ because it is connected to the ground). In this new configuration, it is possible to further simplify the circuit by computing the parallel of $R_{//}$ and $Z_{AC}$ and, afterwards, if we concatenate the previous circuit for 154 times (it means for each dipole) we reach the desired configuration shown in Figure 6.6. At this point, a Python code was implemented in order to find

$$Z_{AB} = \frac{2L}{(j\omega)C} + (j\omega)^2L$$

$$Z_{BC} = \frac{2L}{(j\omega)C} + (j\omega)^2L$$

$$(6.3)$$

$$Z_{AC} = j\omega C(2\frac{L}{C} + (j\omega)^2L)$$

the values of the voltages in each node $V = [V_1; V_2; \ldots; V_n]$. It can easily be done
solving the circuit in Figure 6.9 as the n-th node in which on the right-hand we have the equivalent impedance seen from the right side; instead, on the left-hand, we considered the equivalent voltage generator computed with the Thevenin’s theorem \[22\]. After calculating \( V \), and therefore having all the voltages on each mesh (this means to know \( V_A \) and \( V_C \)), we can return to the original topology (Fig 6.5) in order to compute the current in each aperture \( I_{L_n} \).

Figure 6.10. Reduction of the circuit seen in Fig 6.6 considering a generic node \( n \). \( V_{left} \) and \( Z_{left} \) are calculated with the Thevenin’s theorem. \( Z_{right} \) is the equivalent impedance seen from the right side.

Referring to the Fig 6.11, we have that:

\[
I_{in} = I_{R//} + I_{A1};
I_{out} = I_R + I_{A2} = I_{in} - (j\omega C)V_x \Rightarrow
\]

\[
I_{A2} - I_{A1} = -j\omega CV_x.
\]
It is also true that:

\[ I_{A1} = \frac{V_A - V_x}{j\omega L}, \]
\[ I_{A2} = \frac{V_x - V_C}{j\omega L}. \]  

(6.6)

Replacing the (6.6) in the (6.5), we get the voltage at the node \( V_x \):

\[ V_x = \frac{V_A + V_C}{j\omega L} \left[ \frac{2}{j\omega L + j\omega C} \right]^{-1}; \]  

(6.7)

instead, the current at the second aperture is given by the (6.6). We are able now to compute every \( I_{A2_n} \) in the main circuit (Figure 6.4) and, in order to find the phase shift between two consecutive dipoles, we have to calculate the imaginary part of the difference between two nearby apertures. In other words:

\[ \Delta \phi_n = \phi_{I_{A2_n}} - \phi_{I_{A2(n-1)}}. \]  

(6.8)

\[ \Delta \Psi \text{ [deg]} \]

\[ f=25[H \text{z}] \]
\[ f=50[H \text{z}] \]
\[ f=100[H \text{z}] \]
\[ f=200[H \text{z}] \]

\[ 0 \]
\[ 20 \]
\[ 40 \]
\[ 60 \]
\[ 80 \]
\[ 100 \]
\[ 120 \]
\[ 140 \]
\[ 160 \]

\[ 0 \]
\[ -2 \]
\[ -4 \]
\[ -6 \]
\[ -8 \]
\[ -10 \]

\[ \text{Dipole} \]

\[ \text{f=25[H z]} \]
\[ \text{f=50[H z]} \]
\[ \text{f=100[H z]} \]
\[ \text{f=200[H z]} \]

\[ \text{Figure 6.12. Phase shift between consecutive dipoles.} \]

In Fig 6.12 the behaviour of the phase shift as a function of the considered dipole with respect to the rest of the chain in the arc is reported. We can easily note that by increasing the frequency, we also increase the phase shift significantly but we must not forget that the model above 50Hz is not reliable any more. As said before, the frequency of our interest is the 50Hz (yellow line). Therefore, in this case, the phase shift in each dipole it’s about 2°, which is not negligible if we think that it is introduced by a single dipole and not by its totality. For the sake of completeness, in Fig 6.13 we shown how the phase shift varies by changing the resistance; in particular, it should be noted how the depressions in the figure are emphasised by the value of the resistance itself.
6. Circuit Models of Superconducting Dipole Magnets

6.2.2 Phase shift due to the delay on the path

If we shift to the high frequencies, the delay in the phase of the electrons crossing the wires starts to be considerable compared with a particle crossing the dipole. As a first approximation, let us assume that we are travelling with the speed of light. As it is known from the theory, the phase delay due to the path is given by:

$$\Delta X = \frac{\lambda \Delta \phi}{2\pi} \Rightarrow \Delta \phi = \frac{2\pi \Delta X}{\lambda}; \tag{6.9}$$

where $\Delta X$ is the difference between two paths, $\lambda$ is the wavelength in which the information on the frequency is enclosed ($\lambda = \frac{v}{f}$) and $\Delta \phi$ is the sought phase shift. Knowing that each dipole is about 15 m, let’s compute the path that the current has to cross through the wires. With reference to Fig. 6.14 which represents the cross section of a dipole, we have: $15 + 9 + 16 = 40$ windings for each quadrant $\Rightarrow 40 \times 4 = 160$ in the totality. Going further, multiplying by the length of the dipole $160 \times 15 m = 2400 m$ and for the whole arc $\Rightarrow 2400 m \times 154 \times 2 = 739200 m$.

![Cross section of the LHC dipole](image)

**Figure 6.14.** Cross section of the LHC dipole

Fig. 6.15 reports the comparison between the phase shift due to the path compared to that of the circuitry. It should be noted that, as it was supposed, at the low frequency it is completely dominated by the lumped elements instead, at high frequency, a greater contribution of the delay of the path is present even though there is still a significant contribution by the circuit.
Phase-shift in a dipole

\[\text{Phase Shift [deg]}\]

\[\text{Hz}\]

Figure 6.15. Comparison between phase shift due to delay and circuitry
Chapter 7

Conclusion and Future Steps

7.1 Overview

This thesis has investigated the spurious harmonics that contaminate the beam spectrum providing an analytical approach.

Assuming that these harmonics are direct consequence of the power converter frequency (there is strong evidence that suggests this assumption [25, 24], an arc of LHC was modeled, employing MAD-X, in order to study the behaviour of the betatronic oscillation under the hypothesis that each dipole of the chain is perturbed by the frequency of the network (50Hz).

Furthermore, in order to answer at the question of what phase-shift do we expect in the dipoles chain, a preliminary equivalent circuit model of the LHC dipole string was studied.

7.2 Difficulties and Strategy

As we have seen in [chapter 1], this issue has been well known for years. With the advent of more and more performing diagnostic technologies and with the request for higher luminosities accelerator, this phenomenon can no longer be overlooked.

To date, an equivalent model, which describes the dipoles chain coherently up to the frequency range of our interest (around 8kHz), does not exist. I believe that it is of primary importance to synthesise a valid equivalent circuit not just for better understanding the cluster of harmonics at 8kHz (mentioned in [chapter 1] and shown in [Figure 1.7]) but also for operational purposes. In fact, the circuit allows us to study the resonant frequencies and the impedance of the magnet string and this gives us the means of investigating the low frequency tune modulation caused by the power converter ripple, which is responsible for reducing the dynamic aperture in presence of non-linear fields [45], and the tracking of the ideal current throughout the magnet string for beam stability purposes.

Despite the request, unfortunately, it wasn’t possible to do measurements in firsthand on the dipoles to characterise the component. Therefore, it seemed appropriate to continue the study with an already available circuit which works around 50Hz.
7.3 Results

The project addressed during this time at CERN proved to be quite ambitious for just a master’s thesis work. However, interesting results have been achieved. Firstly we found a closed solution for the betatron oscillation under the hypothesis of perturbed dipoles, and we could study the case with the parameters involved in LHC. In this context, we achieved orders of a magnitude consistent with experimental results. Afterwards, a simulation has been set up, employing MAD-X, in order to estimate the response in the x-plane under the effect of an entire perturbed arc in LHC. In particular, in chapter 4 an interference pattern due to the combination of all 154 perturbed dipoles is shown. Therefore, depending on the phase-shift between the dipoles, different heights of displacement peaks on the x-plane are appreciated. Finally, with a view to future and more accurate transmission line model, an electrical equivalent circuit of a LHC magnets string (working at not higher than 50 Hz) was studied in order to estimate the phase-shift of the current presents in each aperture of the dipoles.

7.4 Future Steps

One possible future step of this work is to synthesise the aforesaid transmission line. One way to do this is to measure the response, in terms of amplitude and phase, of the magnet under test by means of a vector analyser and, afterwards, to plot the Bode diagram. The next step is to find a transfer function that is capable to fit at the trend of the Bode diagram depending on the poles and the zeros assigned at the transfer function. A similar way of proceeding is well described by F. Bourgeois and K. Dahlerup-Petersen in [4]. In particular, they show how to model the magnet as a lumped transmission line composed by a series of LC cells and how it behaves very much like a pure transmission line. Moreover, an advantage is that simplified RLC magnet models can easily be checked against sophisticated ones.

In conclusion, the hypothesis of the aforesaid 8kHz cluster of tones is an interplay between noise from the main bends circuit, impedance due the resistive wall and damper. I believe that for the LHC application, an accurate transmission line approach would provides the answer to this issue and to several questions from many different fields as well.
Appendix A

Frenet Serret coordinate system

Let us consider a reference system \((X,Y,Z)\) and let’s write it in terms of a new reference system \((x,s,y)\) fixed with the reference particle as is shown in the Figure A.1:

\[
\begin{align*}
X &= (\rho + x) \cos \theta = (\rho + x) \cos \left( \frac{\theta}{\rho} \right) \\
Y &= (\rho + x) \sin \left( \frac{\theta}{\rho} \right) \\
Z &= y
\end{align*}
\]

(A.1)

Figure A.1. Frenet Serret coordinate system.

We defined the position vector \(\vec{r} = \rho \hat{e}_x + x \hat{e}_x + y \hat{e}_y\) in terms of the new coordinates. We assume that \(\rho\) is the accelerator curvature radius which in general can vary along \(s\) but we assume that inside the magnets it is constant. In other words we have

\[
\frac{\partial \rho}{\partial s} = 0.
\]
If we consider now the infinitesimal displacement of the position vector \( d\vec{r} \), referring to Figure A.2, we have:

\[
d\vec{r} = \rho d\hat{e}_x + dx\hat{e}_x + xd\hat{e}_x + dy\hat{e}_y,
\]

moreover we have:

\[
d\hat{e}_x = d\theta\hat{e}_s = \frac{ds}{\rho}\hat{e}_s.
\]

Substituting the previous one in \( d\vec{r} \), we obtain:

\[
d\vec{r} = dx\hat{e}_x + (1 + \frac{x}{\rho})ds\hat{e}_s + dy\hat{e}_y
\]

We can now compute the speed:

\[
\vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\hat{e}_x + (1 + \frac{x}{\rho})\dot{s}\hat{e}_s + \dot{y}\hat{e}_y.
\]

In conclusion, the use of the metric coefficients \( h_1 = 1, h_2 = 1 + \frac{x}{\rho}, h_3 = 1 \) allows us to use all the operators in this new coordinates as \( \vec{\nabla}f, \vec{\nabla} \cdot \vec{A} \) and \( \vec{\nabla} \times \vec{A} \).

\[
\vec{\nabla}f = \frac{1}{h_1} \frac{\partial f}{\partial x} \hat{e}_x + \frac{1}{h_2} \frac{\partial f}{\partial s} \hat{e}_s + \frac{1}{h_3} \frac{\partial f}{\partial y} \hat{e}_y
\]

\[
\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial (h_2 h_3 A_x)}{\partial x} + \frac{\partial (h_1 h_3 A_s)}{\partial s} + \frac{\partial (h_1 h_2 A_y)}{\partial y} \right]
\]

\[
\vec{\nabla} \times \vec{A} = \frac{1}{h_2 h_3} \left[ \frac{\partial (h_3 A_y)}{\partial s} - \frac{\partial (h_2 A_s)}{\partial y} \right] \hat{e}_x + \frac{1}{h_1 h_3} \left[ \frac{\partial (h_1 A_s)}{\partial y} - \frac{\partial (h_3 A_x)}{\partial x} \right] \hat{e}_s + \frac{1}{h_1 h_2} \left[ \frac{\partial (h_2 A_x)}{\partial x} - \frac{\partial (h_1 A_s)}{\partial s} \right] \hat{e}_y
\]
Appendix B

Hamiltonian in the Frenet Serret coordinate system

We are interested to describe the transverse equation of motion. Therefore we consider only a transverse magnetic field neglecting $\vec{E}$ that is used for the longitudinal acceleration (the scalar potential $\Phi = 0$). This means $\vec{B}_s = (\vec{\nabla} \times \vec{A})_s = 0$ and $\vec{B} = B_x(x,y)\hat{e}_x + B_y(x,y)\hat{e}_y$. From the Appendix A we can write the curl of the potential vector in the new coordinates:

$$
\begin{align*}
B_x &= -\frac{\delta A_x}{\delta y} \\
B_y &= \frac{1}{1 + \frac{x}{\rho}} \frac{\delta (1 + \frac{x}{\rho}) A_s}{\delta x} = \frac{A_s}{\rho (1 + \frac{x}{\rho})} + \frac{\delta A_s}{\delta x}
\end{align*}
$$

(B.1)

Starting from the Lagrangian:

$$
L = -\sqrt{m_0^2 c^4 - m_0^2 c^2 v^2} - q\phi + q\vec{v} \cdot \vec{A}
$$

(B.2)

$$
L = -\sqrt{m_0^2 c^4 - m_0^2 c^2 v^2} \left( \dot{x}^2 + \left(1 + \frac{x}{\rho}\right)^2 \dot{s}^2 + \dot{y}^2 \right) + q \left( \dot{x} \hat{e}_x + \left(1 + \frac{x}{\rho}\right) \dot{s} \hat{e}_s + \dot{y} \hat{e}_y \right) \cdot (A_x \hat{e}_x + A_s \hat{e}_s + A_y \hat{e}_y)
$$

(B.3)

$$
L = -\sqrt{m_0^2 c^4 - m_0^2 c^2 v^2} \left( \dot{x}^2 + \left(1 + \frac{x}{\rho}\right)^2 \dot{s}^2 + \dot{y}^2 \right) + q \left( \dot{x} A_x + \left(1 + \frac{x}{\rho}\right) \dot{s} A_s + \dot{y} A_y \right)
$$

(B.4)

$$
\Pi_x = \frac{\partial L}{\partial \dot{x}} \quad \Pi_y = \frac{\partial L}{\partial \dot{y}} \quad \Pi_s = \frac{\partial L}{\partial \dot{s}}
$$

(B.5)

$$
\Pi_x = \frac{m_0 c^2 \dot{x}}{\sqrt{m_0^2 c^4 - m_0^2 c^2 v^2}} + q A_x = \frac{m_0 c^2 \dot{x}}{\sqrt{1 - \frac{v^2}{c^2}}} + q A_x = m_0 \gamma \dot{x} + q A_x
$$

$$
\Pi_y = m_0 \gamma \dot{y} + q A_y \quad \Pi_s = \left(1 + \frac{\rho}{\dot{x}} \right) \left[ \left(1 + \frac{\rho}{\dot{s}} \right) m_0 \gamma \dot{s} + q A_s \right]
$$

(B.6)
\[ H = \sqrt{m_0^2 c^4 + c^2 m_0^2 \gamma^2 v^2} + q\phi = \sqrt{m_0^2 c^4 + c^2 \left[ m_0^2 \gamma^2 x^2 + m_0^2 \gamma^2 \left( 1 + \frac{x}{\rho} \right)^2 s^2 + m_0^2 \gamma^2 y^2 \right]} = \]
\[ = c \sqrt{m_0^2 c^2 + (\Pi_x - qA_x)^2 + \left( \frac{\Pi_s}{1 + \frac{x}{\rho}} - qA_s \right)^2 + (\Pi_y - qA_y)^2} \] (B.7)

\[ H = c \sqrt{m_0^2 c^2 + (\Pi_x - qA_x)^2 + \left( \frac{\Pi_s}{1 + \frac{x}{\rho}} - qA_s \right)^2 + (\Pi_y - qA_y)^2} \] (B.8)

The equations of motion are then:
\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial \Pi_x} \quad \dot{\Pi}_x = -\frac{\partial H}{\partial x} \\
\dot{s} &= \frac{\partial H}{\partial \Pi_s} \quad \dot{\Pi}_s = -\frac{\partial H}{\partial s} \\
\dot{y} &= \frac{\partial H}{\partial \Pi_y} \quad \dot{\Pi}_y = -\frac{\partial H}{\partial y}
\end{align*}
\] (B.9)
Appendix C

Equation of motion in the transverse plane

In a circular machine we need a transverse deflecting force to keep the particles in orbit. This force is given by the Lorentz force that we recall below:

\[ \vec{F} = q \cdot (\vec{E} + \vec{v} \times \vec{B}). \tag{C.1} \]

We suppose all the particles having the same momentum. In cylindrical coordinates we have \( \rho = \rho_0 + x \rightarrow \dot{\rho} = \dot{x} \), \( \dot{\theta} = \omega \) and \( \rho \dot{\theta} = v = v_0 \). Moreover, we consider a magnetic field \( B = B_y \hat{e}_y = B_0 \hat{e}_y \).

![Coordinate system in a circular accelerator.](image)

The Lorentz force can be split in its three components and, according to the magnetic field, we obtain:

\[
\begin{align*}
F_\rho &= \frac{d}{dt}(m\dot{\rho}) - m\rho \ddot{\rho}^2 = q \left( \rho \dot{\theta} B_y - \dot{y} B_\theta \right) \\
F_\theta &= \frac{1}{\rho} \frac{d}{dt} \left( m\rho^2 \dot{\theta} \right) = -q \left( \dot{\rho} B_y - \dot{y} B_\rho \right) = 0. \\
F_y &= \frac{d}{dt}(m\dot{y}) = q \left( \dot{\rho} B_\theta - \rho \dot{\theta} B_y \right) = 0.
\end{align*}
\tag{C.2}
\]
From the first equation of (C.2) we have that the circular motion is perpendicular to the uniform field \( \Rightarrow -m\rho \dot{\theta}^2 = q\rho B_0 \). Then:

\[
\dot{\theta} = -\frac{q}{m} B_0 = \Omega_c, \tag{C.3}
\]

where \( \Omega_c \) is called cyclotron frequency.

Using the Newton’s second law \( \vec{F} = M\vec{a} \), we can rewrite the radial force:

\[
a_x = \ddot{x} - \omega_0^2 (\rho_0 + x) = \frac{qv_0 B_y}{m}. \tag{C.4}
\]

It is convenient to change coordinate and use a system that follows an ideal path along the accelerator (Frenet reference system). This means

\[
\frac{d}{dt} \equiv v_0 \frac{d}{ds}.\]

Therefore, applying the change, we obtain:

\[
\ddot{x} = v_0^2 x'' = \omega_0^2 (\rho_0 + x)^2 x'' \Rightarrow x'' - \frac{1}{\rho_0 + x} = \frac{q}{m v_0} B_y. \tag{C.5}
\]

Remembering that:

\[
\frac{q}{m} = -\frac{v_0}{\rho_0 B_0},
\]

and since \( x << \rho_0 \), we obtain:

\[
x'' - \frac{1}{\rho_0} \left( 1 - \frac{x}{\rho_0} \right) = -\frac{1}{\rho_0 B_0} B_y. \tag{C.6}
\]

Now expanding the field in a Taylor series up to the quadrupole component

\[
B_y \approx B_0 + \left( \frac{\partial B_y}{\partial x} \right)_{x=0} x + \ldots = B_0 - |B\rho| k x + \ldots \tag{C.7}
\]

where \( k = -\frac{1}{|B\rho|} \left( \frac{\partial B_y}{\partial x} \right)_{x=0} \). Substituting the B-field we get:

\[
x'' + \left( \frac{1}{\rho_0^2} - k \right) x = 0. \tag{C.8}
\]

The truncations of the above expansion are based on the central force and constant velocity approximations and they are critically balanced so that the phase space is conserved.

It’s possible to do the same analysis with particles having different momentum. In this context, during the computation, we must consider a momentum deviation \( v' = (v + \Delta v) \). The final equation is:

\[
\frac{d^2 x}{d s^2} + \left( \frac{1}{\rho_0^2} - k \right) x = \frac{1}{\rho_0} \frac{\Delta p}{p_0}. \tag{C.9}
\]

As far as the motion in the vertical plan is concerned, with a similar procedure we obtain the following equation:

\[
\frac{d^2 y}{d s^2} + ky = 0. \tag{C.10}
\]
Appendix D

Transfer Matrix

Starting from the (2.18), and remembering some useful trigonometric formulas:
\[
\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b \\
\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b,
\] (D.1)
we can obtain the following equations:
\[
x(s) = \sqrt{\beta_x(s) J_x} (\cos \mu_x(s) \cos \mu_{x,0} - \sin \mu_x(s) \sin \mu_{x,0}); \\
x'(s) = -\sqrt{J_x/\beta_x(s)} [\alpha_x(s) (\cos \mu_x(s) \cos \mu_{x,0} - \sin \mu_x(s) \sin \mu_{x,0}) + \\
+ \sin \mu_x(s) \cos \mu_{x,0} + \cos \mu_x(s) \sin \mu_{x,0}].
\] (D.2)

Considering the starting point, \(s(0) = s_0\) and putting \(\mu(0) = 0\) leads to:
\[
\cos \mu_0 = \frac{x_0}{\sqrt{\beta_0 J}}; \\
\sin \mu_0 = -\frac{1}{\sqrt{J}} \left(x'_0 \sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right). 
\] (D.3)

If we replace the last equations in (D.2), we obtain:
\[
x(s) = \sqrt{\frac{\beta_s}{\beta_0}} \{\cos \mu_s + \alpha_0 \sin \mu_s\} x_0 + \left\{\sqrt{\beta_s \beta_0} \sin \mu_s\right\} x'_0; \\
x'(s) = \frac{1}{\sqrt{\beta_s \beta_0}} \left\{\left(\alpha_0 - \alpha_s\right) \cos \mu_s - \left(1 + \alpha_0 \alpha_s\right) \sin \mu_s\right\} x_0 + \sqrt{\frac{\beta_0}{\beta_s}} \left\{\cos \mu_s - \alpha_s \sin \mu_s\right\} x'_0.
\] (D.4)

Hence, from the (D.4) we can define the following matrix:
\[
M = \left(\begin{array}{c}
\sqrt{\frac{\beta_0}{\beta_s}} \cos \mu_s + \alpha_0 \sin \mu_s \\
\left(\alpha_0 - \alpha_s\right) \cos \mu_s - \left(1 + \alpha_0 \alpha_s\right) \sin \mu_s
\end{array}\right) \left(\begin{array}{c}
\sqrt{\beta_s \beta_0} \sin \mu_s \\
\sqrt{\frac{\beta_0}{\beta_s}} \sin \mu_s
\end{array}\right).
\] (D.5)

Then, we can compute the single particle trajectories between two locations using the following equation:
\[
\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0.
\] (D.6)
Appendix E

Alternative Demonstration

\[ x(N) = \sum_{n=0}^{N} \Theta(n) \sin((N - n)\mu_L); \quad (E.1) \]

\[ x'(N) = \sum_{n=0}^{N} \Theta(N) \cos((N - n)\mu_L). \quad (E.2) \]

Let’s try to demonstrate the equation \( (E.1) \) through mathematical induction. For the base case it’s easy to see that the equation \( (E.1) \) will give the same solution previously seen. Assuming that the \( (E.1) \) holds, let’s move to the inductive step, therefore the equation must hold also for \( N+1 \):

\[
\begin{pmatrix}
x(N + 1) \\
x'(N + 1)
\end{pmatrix} =
\begin{pmatrix}
\cos \mu_L & \sin \mu_L \\
-\sin \mu_L & \cos \mu_L
\end{pmatrix}
\begin{pmatrix}
\sum_{n=0}^{N} \Theta(N) \sin((N - n)\mu_L) \\
\sum_{n=0}^{N} \Theta(N) \cos((N - n)\mu_L)
\end{pmatrix}
+ \begin{pmatrix}
0 \\
\Theta(N + 1)
\end{pmatrix},
\]

\[ (E.3) \]

\[ x(N + 1) = \cos \mu_L \sum_{n=0}^{N} \Theta(N) \sin((N - n)\mu_L) + \sin \mu_L \sum_{n=0}^{N} \Theta(N) \cos((N - n)\mu_L) \Rightarrow \]

\[ \Rightarrow x(N + 1) = \sum_{n=0}^{N} \theta_n \sin((N + 1 - n)\mu_L). \quad (E.4) \]

Considering the expression:

\[ \Theta(N + 1) \sin((N + 1) - (N + 1))\mu_L = 0, \quad (E.6) \]

and adding it to the \( (E.5) \), we obtain:

\[ x(N + 1) = \sum_{n=0}^{N} \Theta(n) \sin((N + 1 - n)\mu_L) + \Theta(N + 1) \sin((N + 1) - (N + 1))\mu_L \Rightarrow \]

\[ \Rightarrow x(N + 1) = \sum_{n=0}^{N+1} \Theta(n) \sin((N + 1 - n)\mu_L), \quad (E.7) \]

\textit{quod erat demonstrandum.}
Appendix F

Quadrupolar Perturbation

Let’s consider the case of a quadrupolar perturbation positioned in \( s = s_0 + C \). In this case we can’t just consider the position because the quadrupolar strength have effect on the second order moment. Then, there is no reason to consider the position \( x \) but we need to consider other observables like the envelope of the beam or the beta function. Going by steps, let’s see first what happens in the case of fixed error and then we will move to the case of an harmonic perturbation.

F.1 Time-Invariant Quadrupolar Noise

In terms of optics, we have that:

\[
X_N = M_{opt}X_{N-1} + \begin{pmatrix} 0 & \Delta kds & 0 \\ \Delta kds & 0 & 0 \end{pmatrix} M_{opt}X_{N-1} \Rightarrow \\
X_N = [I + \begin{pmatrix} 0 & \Delta kds & 0 \\ \Delta kds & 0 & 0 \end{pmatrix} M_{opt}X_{N-1} = \begin{pmatrix} 1 & 0 \\ \Delta kds & 1 \end{pmatrix} M_{opt}X_{N-1} = M_{trasf}X_{N-1},
\]

where \( M_{trasf} = \begin{pmatrix} 1 & 0 \\ \Delta kds & 1 \end{pmatrix} M_{opt} \) and \( M_{opt} \) is the well known rotation matrix. If we work in the physical space, by replacing, we get:

\[
M_{trasf} = \begin{pmatrix} \cos \mu_L + \alpha \sin \mu_L & \beta \sin \mu_L \\ \Delta kds \cos \mu_L + \alpha \sin \mu_L - \gamma \sin \mu_L & \Delta kds \beta \sin \mu_L + \cos \mu_L - \alpha \sin \mu_L \end{pmatrix}.
\]

(F.1)

If we compute the trace of the matrix and remembering that \( \mu_L = 2\pi Q_L \), we will obtain:

\[
Trace(M_{trasf}) = 2 \cos \mu_L + \Delta kds \beta(s) \sin \mu_L = 2 \cos (\mu_L + \Delta \mu),
\]

in other words, we impose that it is equal to a pure rotation with a certain phase advance perturbation from which we can compute that:

\[
\Delta \mu(s) = \frac{\Delta kds \beta(s)}{2}.
\]

(F.2)
Integrating the aforesaid equation along the length of the quadrupole, we get the tune shift due the quadrupolar error:

$$\Delta Q(s) = \int_{quad} \frac{\Delta k \beta(s) ds}{4\pi}.$$  \hfill (F.3)

Therefore, the quadrupolar error produces a tune shift which is proportional to the $\beta$-function at the location of the quadrupole. Once again, the field quality, power supply tolerances etc. are much tighter at places where $\beta$ is large. Hence, we can say that $\beta$ is a measurement of the sensitivity of the beam.

Passing to another observable, let’s consider the betatronic oscillation. In this case, we consider a generic position $s'$ for the quadrupole perturbation, which leads to have the following equation for one turn:

$$(X)_{s_0 \rightarrow s_0+C} = M_{s' \rightarrow s_0+C} \left( \begin{array}{cc} 1 & 0 \\ \Delta k ds & 1 \end{array} \right) M_{s_0 \rightarrow s'}(X)_{s_0},$$ \hfill (F.4)

where we want that:

$$M_{s' \rightarrow s_0+C} \left( \begin{array}{cc} 1 & 0 \\ \Delta k ds & 1 \end{array} \right) M_{s_0 \rightarrow s'} = M^*,$$

with $M^* = \begin{pmatrix} m_{11}^* & m_{12}^* \\ m_{21}^* & m_{22}^* \end{pmatrix}$. We are still in the linear case, then it’s reasonable to say that $m_{12}^*$ must be of the following form:

$$m_{12}^* = (\beta_0 + \Delta \beta) \sin (2\pi (Q + \delta Q))$$

$$= \beta_0 \sin (2\pi Q) + 2\pi \beta_0 \Delta Q \cos (2\pi Q) + \Delta \beta \sin (2\pi Q) + 2\pi \Delta \beta \Delta Q \cos (2\pi Q);$$ \hfill (F.6)

where, for the $[\text{F.6}]$, we used that $\cos (2\pi \Delta Q) \approx 1$ and $\sin (2\pi \Delta Q) \approx 2\pi \Delta Q$. Moreover, note that $\beta_0 \sin (2\pi Q) = m_{12}$ (where $m_{12}$ represents second element of the ideal rotation matrix) and $2\pi \Delta \beta \Delta Q \cos (2\pi Q)$ is negligible.

On the other hand, developing the transfer matrix of $[\text{F.4}]$, we obtain:

$$m_{12}^* = m_{12} - M_{12}(s_0 \rightarrow s') M_{12}(s' \rightarrow s_0 + C) \Delta k ds;$$ \hfill (F.7)

where $m_{12}$ represents second element of the ideal rotation matrix; instead, $M_{12}(s_0 \rightarrow s') M_{12}(s' \rightarrow s_0 + C)$ are equal respectively to:

$$M_{12}(s_0 \rightarrow s') = \sqrt{\beta_0} \beta(s') \sin \psi$$

$$M_{12}(s' \rightarrow s_0 + C) = \sqrt{\beta_0} \beta(s') \sin (2\pi Q - \psi)$$

Equating $[\text{F.6}]$ with $[\text{F.7}]$, we obtain:

$$\frac{1}{2} \Delta k ds \beta_0 \beta(s') \cos (2\pi Q) + \Delta \beta \sin (2\pi Q) = -\beta_0 \beta(s') \Delta k ds \sin \psi \sin (2\pi Q - \psi);$$ \hfill (F.8)
where in the left hand side of the equation, we used the (F.3). After some math and some rearrangements, we reach the final form:

$$\frac{\Delta \beta}{\beta_0} = -\frac{1}{2 \sin(2\pi Q)} \int_{quad} \beta(s') \Delta k(s') \cos(2\psi - 2\pi Q) ds'. \quad (F.9)$$

Note that we have a singularity in $Q = \frac{1 \pm m}{2}$ where $m$ is an integer. Therefore, integer tunes and half integer tunes need to be avoided for machine operation in order to avoid that beam envelope becomes unstable due the quadrupole error.

### F.2 Harmonic Perturbation

Let’s consider a perturbation of the following type:

$$\Delta G(N) = A_{\text{noise}} \cos(2\pi Q_{\text{noise}} N) \quad (F.10)$$

Using the transfer matrix (F.1) we are able to compute the $\beta$, $\alpha$ and $\gamma$ for each turn with the following equation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_N = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22} + m_{21}m_{12} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{N-1} \quad (F.11)$$

where the elements $m_{11}$, $m_{12}$, $m_{21}$ and $m_{22}$ are respectively the elements of the matrix (F.1). Let’s compute the previous equation for a pair of turns considering, for simplicity, the initial condition $\beta_0 = 1$, $\alpha_0 = 0$ and $\gamma_0 = 1$:

- @n = 1 ⇒ $\beta_1 = \cos(\mu L)^2 + \sin(\mu L)^2 = 1$
- @n = 2 ⇒ $\beta_2 = 1 + 0.5\Delta G(1)^2 - 0.5\Delta G(1)^2 \cos(2\mu L) + \Delta G(1) \sin(2\mu L)$
- ...

If we go further, the equations explode in complexity. Since we are considering the case of a perturbation, it’s reasonable to consider negligible degrees higher than the first one of $\Delta G(N)$ . Taking into account what said above, the equation will be modified as follows:

- @n = 1 ⇒ $\beta_1 = \cos(\mu L)^2 + \sin(\mu L)^2 = 1$
- @n = 2 ⇒ $\beta_2 \approx 1 + \Delta G(1) \sin(2\mu L)$
- @n = 3 ⇒ $\beta_3 \approx 1 + \Delta G(2) \sin(2\mu L) + \Delta G(1) \sin(4\mu L)$
- ...

For N turns, we can write it in a more compact way:

$$\beta(N) = 1 + \sum_{n=-\infty}^{N-1} \Delta G(N - n) \sin(2n\mu L), \quad (F.12)$$
and, in analogous way, we could find α:

\[
\alpha(N) = - \sum_{n=-\infty}^{N-1} \Delta G(N - n) \cos(2n\mu L). \tag{F.13}
\]

In Figure F.1 it’s shown how the formulas (F.12) and (F.13) very well approximate the expected trend of the tracking. We remind that this approximation holds only if \( A_{\text{noise}} \) is small and, furthermore, the smaller it is, the closer is the approximation to the true value.

Going further, it’s possible to observe that the series (F.12) converges to:

\[
1 + \frac{iA_{\text{noise}}e^{-4i\pi(N+1)Q_L}K}{4\left(1 + e^{2i\pi(2Q_L+Q_n)}\right)\left(1 + e^{2i\pi(2Q_L+Q_n)}\right)}, \tag{F.14}
\]

where \( K \) is the quantity:

\[
K = (-e^{4i\pi(2NQ_L+Q_L+Q_n)} + 2e^{2i\pi(4(N+1)Q_L+Q_n)} +
- e^{4i\pi(2N+1)Q_L} + e^{4i\pi(Q_L+Q_n)} + e^{4i\pi Q_L} - 2e^{2i\pi Q_n}). \tag{F.15}
\]

We have a resonance condition in which, for \( Q_n = \pm 2Q_L \), the Twiss parameters diverge, as shown in Fig F.2. Note that the tracking trend diverges more fastly than the formula’s one because, as say previously, the recovered formula is just an
approximation where we neglected all the quantities with degrees higher than two and, under the resonance condition, this quantity diverges as well. Thus, it starts to have a no negligible impact in the trade, as it is shown in the following figure.

**Figure F.2.** Comparison between the tracking code and the (F.12) under the resonance condition ($Q_n = 8.62; Q_L = 4.31; A_{noise} = 0.1[\text{Rad})$).
Bibliography


[38] Python. URL: https://www.python.org/.


