On problem of systems under influence of imperfect instruments

J. Formánek

*Nuclear Centre of the Charles University
V Holešovičkách 2, 180 00 Prague 8, Czech Republic*

fu 8446

Abstract

An algorithm for handling imperfect instruments is developed in the framework of quantum theory. As an illustration the problem of light passing through a set of imperfect polarizers is discussed. It is shown that the results obtained in this way are in agreement with experimental data.

1. Introduction

The whole framework of quantum theory is intimately related to the notion of "filters" as special devices which are able to "prepare" system in a very specific way. In most cases such filters exist only as an idealized limit of real devices. That is, of course, nothing new in physics - let us recall e.g. the notion of "point particle" in classical mechanics. On the other hand, one can easily misinterpret experimental data if proper attention is not paid to the fact that the devices actually used only approximate the ideal ones.

In this paper we indicate how the problem of imperfect instruments influence upon studied system could be handled in the framework of quantum theory. As an example transmittance of a setup composed of imperfect polarizers is discussed. It is shown that results are in agreement with the experimental data⁽¹⁾.

This paper is organized as follows: In Sect. 2 some basic points of quantum theory are recalled. Formalism for description of "instruments" is developed in Sect. 3. Its application to the case of real polarizers is presented in Sect. 4. Some of the results are compared with experimental data in Sect. 5. The last section is devoted to brief conclusions.

2. Quantum theory

Let us recall some of the basic assumptions of the quantum theory first:

i) The maximum information which can be provided about a given system (in a given instant t) is an information contained in a statement like: "If a measurement of the (specified) complete set of observables A would be performed (at the considered moment t), then the eigenvalue a would be (certainly) obtained".¹ This information represents a *pure state*.

Any less complete information represents a *mixed state*.

ii) There is a Hilbert space $\mathcal{H}$ attributed to the considered system.

iii) Each state can be completely described by a *density matrix*,² i.e. by

¹We use to say: "the system is in the eigenstate of A with eigenvalue a ". A could be a shorthand for more (compatible) observables. For simplicity sake we shall handle A as an observable with purely discrete spectrum.

²In mathematics the term "density matrix" is usually reserved to an operator for which besides (1), (2) also the "normalization condition" $SpW = 1$ holds. We shall call such operator a "normalized density matrix".

a positive operator W :

$$W = W^+ \quad (1)$$

$$Sp W^2 \leq (Sp W)^2 . \quad (2)$$

Evidently, if W is a density matrix and k any positive number then

$$W' \equiv kW$$

is again a density matrix. The density matrices W and W' describe the *same* state.

iv) A density matrix W describes a pure state when and only when the relation

$$Sp W^2 = (Sp W)^2 \quad (3)$$

holds.

In such a case

$$P \equiv \frac{1}{Sp W} W \quad (4)$$

is a projector to one-dimensional subspace of $\mathcal{H}$.

Therefore, the eigenstate of A with eigenvalue a_k is uniquely described by the projector

$$P_{a_k}(A) \equiv |a_k \rangle \langle a_k| \quad (5)$$

where $|a_k \rangle$ is any normalized vector from the before-mentioned subspace.

v) Any device which is able to "prepare" the system in the state $P_{a_k}(A)$ is called *filter* (corresponding to A) with all channels closed except the k -th one.

Let the system "enters" such filter in a state W , then - if it succeeds to pass through - it reappears in the state

$$W' \equiv P_{a_k}(A)W P_{a_k}(A) \quad (6)$$

and the probability of this transition is given by

$$w = \frac{Sp W'}{Sp W} . \quad (7)$$

vi) The time evolution of states (of any isolated system) is described by an unitary transformation

$$W(t) = U(t - t_0)W(t_0)U^+(t - t_0) \quad (8)$$

where

$$U(t) = \exp\{-iHt\} \quad (9)$$

and H is a self-adjoint operator (Hamiltonian of the system).

3. Instrument

Let there is another system ($\equiv$ "device") besides the considered one ($\equiv$ "system"). Let (at the moment t_0) the system is at a state $W^{(1)}(t_0)$ and the device at a state $W^{(2)}(t_0)$; $W^{(j)}(t_0)$ being normalized density matrices. The corresponding state of the "large system" ($\equiv$ system + device) is described by the normalized density matrix

$$\mathcal{W}(t_0) \equiv W^{(1)}(t_0) \otimes W^{(2)}(t_0). \quad (10)$$

At $t > t_0$ the system will be in the state

$$W^{(1)}(t) = S_p^{(2)} \mathcal{W}(t) \quad (11)$$

where

$$\mathcal{W}(t) \equiv \mathcal{U}(t - t_0) \mathcal{W}(t_0) \mathcal{U}^+(t - t_0), \quad (12)$$

$$\mathcal{U}(t) \equiv \exp\{-i\mathcal{H}t\} \quad (13)$$

and

$$\mathcal{H} = H + H^{(2)} + H_I \quad (14)$$

where H_I describes mutual system-device interaction.

If (in the case of initial state (10)) the system and device do not influence each other during the time interval (t_0, t_1) , then the density matrix (11) is identical with

$$W(t) \equiv U(t - t_0) W^{(1)}(t_0) U^+(t - t_0) \quad (15)$$

for $t \in (t_0, t_1)$.

Let us assume that ³

$$W^{(1)}(t) \neq W(t) \quad \text{for } t \geq t_1 \quad (16)$$

³i.e. at the moment t_1 the mutual system-device interaction becomes essential

but for $t \in (t_2, t_3)$:

$$\begin{aligned} t_2 &\equiv t_1 + \delta t \\ t_3 &\equiv t_2 + \Delta t \end{aligned} \quad (17)$$

the density matrix (12) can be expressed as

$$\mathcal{W}(t) = \mathcal{W}_1(t) + \mathcal{R}(t) \quad (18)$$

where the *density matrix* $\mathcal{W}_1(t)$ corresponds to a situation where the system "*passed through*" the device and the system with device *do not* influence each other.⁴

In such a case the probability to find the system behind the device can be expressed (for $t \in (t_2, t_3)$) as

$$w(t) = Sp \mathcal{W}_1^{(1)}(t) \quad (19)$$

⁴ $\mathcal{R}(t)$ need not represent a density matrix, e.g. in 1-D case for "system" $\equiv$ point-like particle, "device" $\equiv$ a macroscopic object, H_I = short range system-device interaction and

$$\mathcal{W}(t) = |\psi\rangle\langle\psi| \otimes W^{(2)}$$

where the density matrix $W^{(2)}$ corresponds to the device "in rest" localized at origin (i.e. the probability to find it at the distance $r > a$ away from the origin is negligible),

$$|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle$$

where $\langle x|\psi_1\rangle$ is a wave packet localized at $x = b > a$, moving forward and $\langle x|\psi_2\rangle$ is a wave packet localized at $x = -c < -a$, moving backward.

In this case

$$\mathcal{W}_1(t) \equiv |\psi_1\rangle\langle\psi_1| \otimes W^{(2)}$$

is a density matrix, whereas

$$\mathcal{R}(t) = (|\psi_2\rangle\langle\psi_2| + |\psi_2\rangle\langle\psi_1| + |\psi_1\rangle\langle\psi_2|) \otimes W^{(2)}$$

is not.

Any test of the system performed (at the moment t) behind the device may be analyzed using

$$W_1^{(1)}(t) \equiv Sp^{(2)} \mathcal{W}_1(t) = |\psi_1\rangle\langle\psi_1|$$

as a density matrix describing "state of the system which penetrated the device".

The interval Δt (17) is practically unlimited. (The problem related to spreading of wave packets could be interesting in principle but it has no bearing to the realistic situations which we are after here.)

where the density matrix

$$W_1^{(1)}(t) \equiv S p^{(2)} \mathcal{W}_1(t) \quad (20)$$

describes "the state of the system behind device".

On the other hand, this density matrix can be expressed also as (the system-device interaction is uneffective here)

$$W_1^{(1)}(t) = U(t - t_2) W_1^{(1)}(t_2) U^+(t - t_2). \quad (21)$$

Therefore, the change of the system's state caused by its "penetration through the device" can be expressed as a mapping (cf. (21) with (15))

$$W(t_1) \rightarrow W_1^{(1)}(t_1). \quad (22)$$

If the interval δt is short enough (i.e. if evolution of the *free* system's state $W(t_1)$ would be negligible during δt) one can replace (22) by

$$W(t_2) \rightarrow W'(t_2) \equiv W_1^{(1)}(t_2) \quad (23)$$

and the probability (19) can be expressed as

$$w(t) = S p W'(t_2). \quad (24)$$

Let us assume that the before-mentioned properties are met for some other initial states of the system too. Let

$$W^{(1)}(t_0; j) \quad j = 1, \dots, N \quad (25)$$

be corresponding normalized density matrices.

In all these cases the penetration through the device is described by the mapping

$$W(j) \rightarrow W'(j) \quad (26)$$

where

$$W(j) \equiv W(t_2; j); \quad W'(j) \equiv W_1^{(1)}(t_2; j). \quad (27)$$

Similarly one concludes that for the system which is at $t = t_0$ in the state

$$\begin{aligned} W^{(1)}(t_0) &\equiv \sum_{j=1}^N \pi_j W^{(1)}(t_0; j), \\ \pi_j &\geq 0, \quad \sum_{j=1}^N \pi_j = 1 \end{aligned} \quad (28)$$

the penetration probability is given by

$$w = \sum_{j=1}^N \pi_j w_j = SpW' \quad (29)$$

where

$$W' \equiv \sum_{j=1}^N \pi_j W'(j). \quad (30)$$

The penetration through the device is again described by the mapping

$$W \rightarrow W' \quad (31)$$

where

$$W \equiv U(t_2 - t_0)W^{(1)}(t_0)U^+(t_2 - t_0) \quad (32)$$

$$W' \equiv Sp^{(2)}\mathcal{U}(t_2 - t_0)W^{(1)}(t_0) \otimes W^{(2)}(t_0)\mathcal{U}^+(t_2 - t_0). \quad (33)$$

Therefore, the mapping (31) has to be linear (cf. (30)).⁵

Let $Q_l (l = 1, \dots, M)$ are bounded operators, then

i) the mapping

$$W \rightarrow W' \equiv \sum_{l=1}^M Q_l W Q_l^\dagger \quad (34)$$

is linear,

ii) image of any density matrix is again a density matrix.

Moreover, the inequality

$$SpW \leq SpW' \quad (35)$$

holds, provided the condition

$$\sum_{l=1}^M Q_l Q_l^\dagger \leq 1 \quad (36)$$

is satisfied.⁶

⁵It is, of course, a direct consequence of the superposition principle (linearity of the relations (32), (33)).

⁶Actually, it can be proved that practically any linear mapping which associates to each density matrix again a density matrix in such a way that the relation (35) holds is expressible in the form (34). (For more details see e.g.⁽²⁾.)

A device will be called an *instrument* when the change of the studied system's states caused by penetration through the device can be expressed as a mapping (34), (36).⁷ Therefore, each instrument is uniquely described by a given set of operators: $\{Q_l\}$.

The relations (34), (36) are invariant under any replacement

$$Q_l \rightarrow Q'_l \equiv e^{i\alpha_l} Q_l \quad (37)$$

where α_l are arbitrary real numbers. Therefore, the sets $\{Q_l\}$ and $\{Q'_l\}$ describe the same instrument.

On the other hand, the *same* technical setup represents *different* instruments from the point of view of different coordinate systems.

Let us recall that what is called the state W in one coordinate system is described as the state

$$\tilde{W} = U^\dagger W U \quad (38)$$

in another coordinate system, where the unitary operator U is determined by the relation between these two coordinate systems, e.g. if the second system is obtained by a rotation $(-\vartheta\vec{n})$ of the first one, then

$$U = \exp \{i\vartheta\vec{n}\vec{J}\} \quad (39)$$

where $\vec{J}$ is the total angular momentum of the studied physical system.

Therefore, if the change caused by penetration through a given technical setup is described by a mapping (34) in an original system, then the corresponding mapping in the rotated frame is given by

$$\tilde{W} \rightarrow \tilde{W}' = \sum_{l=1}^M \tilde{Q}_l \tilde{W} \tilde{Q}_l^\dagger, \quad (40)$$

where

$$\tilde{Q}_l \equiv U^\dagger Q_l U, \quad (41)$$

i.e. the same technical setup which we call an instrument $\{Q_l\}$ represents the instrument $\{\tilde{Q}_l\}$ in the rotated frame.

One can rephrase this conclusion as: The rotation $(\vartheta\vec{n})$ of a technical setup which represented an instrument $\{Q_l\}$ provides the instrument $\{\tilde{Q}_l\}$.

⁷Needless to say that not all devices (neither the macroscopic ones) represent instruments.

Let us illustrate our considerations by several well known *examples of instruments*:

i) Any filter with one open channel is an instrument - as can be immediately seen from (6), (7).

ii) Any filter (corresponding to a complete set of observables A) is an instrument $\{P_{\sum, a_j}(A)\}$, where the projector

$$P_{\sum, a_j}(A) \equiv \sum_j P_{a_j}(A) \quad (42)$$

with the sum running over the open channels. Therefore, *any* filter is an instrument represented by *one projector*.

It is easy to see that *any* instrument which is represented by a *single operator* has the following important property: The system which passed through such instrument is in a *pure* state, provided it entered in a *pure* state.⁸ We shall call such instruments "instruments of the first kind".

Therefore, instruments which are able to transfer the system from pure states to mixed ones (we shall call them "instruments of the second kind") has to be represented by sets $\{Q_l\}$ containing more than one operator. Any filter (with several open channels) supplemented by registration devices represents an instrument $\{P_l\}$ of this kind: l counts the registration devices and the projector

$$P_l \equiv \sum_j P_{a_j}(A) \quad (43)$$

where sum runs over those open channels which are "watched" by the l -th registration device.

Another example of the second kind instruments is provided by any filter in the case when the decision about sets of its open channels is taken by a random number generator. This is an instrument $\{\sqrt{\pi_l}P_l\}$, where π_l is the probability that it is the set of channels corresponding to the projector P_l which are the open ones.

⁸These two states need not be identical.

4. Imperfect polarizer

Idealized linear polarizer was mentioned as a suggestive example of a filter already by Dirac in his famous book ⁽³⁾. Of course no real polarizer represents such perfect device.

Let us recall that each linear polarizer is usually characterized by two real parameters k_j ($1 > k_1 > k_2 > 0$): $k_1, (k_2) \equiv$ transmittance of light polarized along (perpendicular to) the transmission axis.

We shall neglect any change of photon's momentum caused by the polarizer; i.e. our system consists from polarization degrees of freedom of a photon with specified momentum $\vec{k}$.

Any *pure* state of this system may be expressed as a superposition of two *helicity* states. Let us call the corresponding orthonormal vectors

$$|\lambda = \pm 1 \rangle \equiv |\pm \rangle. \quad (44)$$

We shall choose our coordinate axis $\vec{e}_3, \vec{e}_1$ parallel to $\vec{k}$ and the transmission axis respectively. The orthonormal vectors

$$|1 \rangle \equiv \frac{1}{\sqrt{2}}(|- \rangle + |+ \rangle) \quad (45)$$

and

$$|2 \rangle \equiv \frac{1}{\sqrt{2}}(|- \rangle - |+ \rangle) \quad (46)$$

then describe the linear polarization along $\vec{e}_1$ and $\vec{e}_2$ respectively.

We shall utilize the vectors (45), (46) as the basis of our system's Hilbert space whenever operators will be expressed as 2×2 matrices.

Any state can be described by a normalized density matrix

$$W(\vec{\xi}) \equiv \frac{1}{2}(1 + \vec{\xi}\vec{\sigma}) \quad (47)$$

where $\vec{\sigma}$ are Pauli matrices and

$$\vec{\xi} = \vec{\xi}^*; \quad |\vec{\xi}| \leq 1 \quad (48)$$

($|\vec{\xi}| = 1 \Leftrightarrow$ a pure state). In other words, any state is uniquely determined by 3 real parameters ξ_j ($j = 1, 2, 3$) : $\sum_j \xi_j^2 \leq 1$.

Especially, the linear polarization along $\vec{e}_1$ can be described by the vector (45) as well as by the projector

$$P_1 \equiv |1\rangle\langle 1| \quad (49)$$

which coincides with the density matrix (47) for $\xi_1 = \xi_2 = 0, \xi_3 = 1$:

$$P_1 = W_1 \equiv \frac{1}{2}(1 + \sigma_3). \quad (50)$$

Similarly, the $\lambda = +1$ helicity eigenstate is described by the projector

$$P_+ \equiv |+\rangle\langle +| \quad (51)$$

which coincides with the density matrix (47) for $\xi_1 = \xi_3 = 0, \xi_2 = 1$:

$$P_+ = W_+ \equiv \frac{1}{2}(1 + \sigma_2). \quad (52)$$

Unitary operator (39) achieves the form

$$U(\vartheta) = \cos \vartheta - i\sigma_2 \sin \vartheta. \quad (53)$$

Therefore, the linear polarization along the vector $\vec{n}$ which arised from $\vec{e}_1$ by $(\vartheta\vec{e}_3)$ rotation is described by the density matrix

$$P(\vartheta) = U^\dagger(\vartheta)P_1U(\vartheta) = \frac{1}{2} [1 + \sigma_3 \cos 2\vartheta - \sigma_1 \sin 2\vartheta]. \quad (54)$$

Especially, the polarization along

$$\vec{\varepsilon} \equiv \frac{1}{\sqrt{2}}(\vec{e}_1 + \vec{e}_2) \quad (55)$$

is described by the projector

$$P_\varepsilon \equiv P(\pi/4) = \frac{1}{2}(1 - \sigma_1) \quad (56)$$

identical with the density matrix (47) for $\xi_1 = -1, \xi_2 = \xi_3 = 0$.

Needless to say that the polarization along $\vec{e}_2$ is described by the projector

$$P_2 \equiv P(\pi/4) = \frac{1}{2}(1 - \sigma_3) \quad (57)$$

which coincides with the density matrix (47) for $\xi_1 = \xi_2 = 0$, $\xi_3 = -1$.

Let us assume that the polarizer is an instrument of the first kind. In such a case it can be described by a matrix

$$Q = a + \vec{\beta}\vec{\sigma} \quad (58)$$

where (cf. (37))

$$a = a^* \geq 0 \quad (59)$$

but the vector $\vec{\beta}$ could be complex in general.

A photon which passed through such polarizer is in the state

$$W'(\vec{\xi}) \equiv QW(\vec{\xi})Q^+ \quad (60)$$

where the normalized density matrix $W(\vec{\xi})$ describes the "impinging" photon state.

Especially, for initial state (50) (i.e. for incoming photon polarized along $\vec{e}_1$) one easily finds

$$\begin{aligned} W'_1 &= QW_1Q^+ = \frac{1}{2}\{a^2 + |\vec{\beta}|^2 + 2a\text{Re}\beta_3 + 2\text{Im}\beta_1\beta_2^* + \\ &+ \sigma_2[a^2 - |\vec{\beta}|^2 + 2|\beta_3|^2 + 2a\text{Re}\beta_3 + 2\text{Im}\beta_1^*\beta_2] + \\ &+ 2\sigma_1[\text{Re}\{(a + \beta_3)\beta_1^*\} + \text{Im}\{(a + \beta_3)\beta_2^*\}] + \\ &+ 2\sigma_2[\text{Re}\{(a + \beta_3)\beta_2^*\} - \text{Im}\{(a + \beta_3)\beta_1^*\}]\}. \end{aligned} \quad (61)$$

But photons polarized along the transmission axis should not lose their polarization due to passage through a "good polarizer". Therefore, for such a good polarizer the relation

$$W'_1 = k_1 W_1 \quad (62)$$

has to hold and

$$k_1 = SpW'_1 \quad (63)$$

is nothing else than the before-mentioned transmittance. Therefore, the matrix (58) for a good polarizer has to acquire the form (cf. (61), (62))

$$Q = a + b\sigma_3 \quad (64)$$

where

$$a = a^* \geq 0, \quad b = b_1 + ib_2, \quad b_j = b_j^*. \quad (65)$$

The mapping (60) then can be expressed as

$$W'(\vec{\xi}) = w(\vec{\xi})W(\vec{\xi}') \quad (66)$$

where

$$\begin{aligned} w(\vec{\xi})\xi'_1 &= \xi_1(a^2 - |b|^2) + 2\xi_2ab_2 \\ w(\vec{\xi})\xi'_2 &= \xi_2(a^2 - |b|^2) - 2\xi_1ab_2 \\ w(\vec{\xi})\xi'_3 &= \xi_3(a^2 + |b|^2) + 2ab_1 \end{aligned} \quad (67)$$

and the transition probability

$$w(\vec{\xi}) = a^2 + |b|^2 + 2\xi_3ab_1. \quad (68)$$

The condition

$$w(\vec{\xi}) \leq 1 \quad (69)$$

which should hold for any $\vec{\xi}$ satisfying (48) restricts possible values of the parameters a, b by the inequality⁹

$$(a + b_1)^2 + b_2^2 \leq 1. \quad (70)$$

For a photon initially polarized along $\vec{e}_1(\vec{e}_2)$ the probability (68) defines the transmittance $k_1(k_2)$, i.e.

$$\begin{aligned} k_1 &= |a + b|^2 \\ k_2 &= |a - b|^2. \end{aligned} \quad (71)$$

For an initially unpolarized photon (i.e. for $\vec{\xi} = 0$)¹⁰

$$w(0) = a^2 + |b|^2 = \frac{k_1 + k_2}{2} \equiv \tau. \quad (72)$$

Knowing the state of photons behind the polarizer one can predict results of any measurement performed *there*; e.g. the probability that the eigenvalue $\lambda = +1$ will be obtained by a helicity measurement is given by

$$w_+(\vec{\xi}) = \frac{SpW'(\vec{\xi})P_+}{SpW'(\vec{\xi})} = \frac{1}{2}(1 + \xi'_2). \quad (73)$$

⁹The formula (70) expresses, of course, nothing else than the condition (36) for the special case considered here.

¹⁰The r.h.s. of (68) is independent of ξ_1, ξ_2 . Therefore, the transition probability (72) should be observed for any initial state with vanishing 3rd component of the vector $\vec{\xi}$.

Similarly, the probability to find the linear polarization along $\vec{\epsilon}$ (cf. (55)) is given by

$$w_{\epsilon}(\vec{\xi}) = \frac{SpW'(\vec{\xi})P_{\epsilon}}{SpW'(\vec{\xi})} = \frac{1}{2}(1 - \xi'_1). \quad (74)$$

Especially for an initial photon with helicity $\lambda = +1$ (i.e. for $\xi_1 = \xi_3 = 0$, $\xi_2 = 1$) the probabilities (73), (74) acquire values (cf. (67))

$$w_+ = \frac{1}{2} \left(1 + \frac{a^2 - |b|^2}{\tau} \right) \quad (75)$$

and

$$w_{\epsilon} = \frac{1}{2} \left(1 - \frac{2ab_2}{\tau} \right) \quad (76)$$

respectively.

Therefore, the parameters a , b (which determine the considered instrument uniquely) can be evaluated from experimentally accessible quantities e.g. as ¹¹

$$a = (\tau w_+)^{\frac{1}{2}} \quad (77)$$

$$b_1 = \frac{\Delta}{4(\tau w_+)^{\frac{1}{2}}} \quad (78)$$

$$b_2 = \frac{1}{2} \left(\frac{\tau}{w_+} \right)^{\frac{1}{2}} (1 - 2w_{\epsilon}) \quad (79)$$

where

$$\Delta \equiv k_1 - k_2 > 0 \quad (80)$$

and the density matrix (66) achieves the form

¹¹One can see from (77) - (80), (72) that if an actual polarizer is indeed an instrument of the type considered here, then the experimentally determined values of δ , Δ , w_+ , w_{ϵ} have to respect the constrain

$$w_+ + \frac{1}{4w_+} \left[\left(\frac{\Delta}{2\tau} \right)^2 + (1 - 2w_{\epsilon})^2 \right] = 1.$$

The condition (70) is then satisfied automatically.

$$\begin{aligned}
W'(\vec{\xi}) = & \frac{1}{2} \left\{ \tau + \xi_3 \frac{\Delta}{2} + \sigma_3 \left[\xi_3 \tau + \frac{\Delta}{2} \right] + \right. \\
& + \sigma_1 \tau \left[\xi_1 (2w_+ - 1) - \xi_2 (2w_\epsilon - 1) \right] + \\
& \left. + \sigma_2 \tau \left[\xi_2 (2w_+ - 1) + \xi_1 (2w_\epsilon - 1) \right] \right\}. \quad (81)
\end{aligned}$$

Let us place another polarizer (an instrument of the same type) behind the first one. Let the direction of its transmission axis is obtained by rotation $(\vartheta_1 \vec{e}_3)$ of the first one; i.e. the second polarizer is an instrument

$$\begin{aligned}
Q'(\vartheta_1) & \equiv U^+(\vartheta_1)(a' + b'\sigma_3)U(\vartheta_1) = \\
& = a' + b'\sigma_3 \cos 2\vartheta_1 - b'\sigma_1 \sin 2\vartheta_1. \quad (82)
\end{aligned}$$

If a photon enters this polarizer in the state (81), then (provided it succeeds to get through) it would be in the state

$$W''(\vec{\xi}, \vartheta_1) \equiv w(\vec{\xi})Q'(\vartheta_1)W(\vec{\xi}')Q'^+(\vartheta_1) \quad (83)$$

behind it.

The density matrix (83) can be expressed as

$$W''(\vec{\xi}, \vartheta_1) = w(\vec{\xi}, \vartheta_1)W(\vec{\xi}''(\vec{\xi}, \vartheta_1)) \quad (84)$$

where

$$\begin{aligned}
w(\vec{\xi}, \vartheta_1) = & (\tau + \xi_3 \frac{\Delta}{2})\tau' + (\Delta + 2\xi_3\tau)\frac{\Delta'}{4} \cos 2\vartheta_1 - \\
& - \tau \left[\xi_1 (2w_+ - 1) - \xi_2 (2w_\epsilon - 1) \right] \frac{\Delta'}{2} \sin 2\vartheta_1 \quad (85)
\end{aligned}$$

determines the probability for a photon to pass through the considered sequence of two polarizers in the case when it enters the first one in the state (47).

It is easy to find also the vector $\vec{\xi}''(\vec{\xi}, \vartheta_1)$ as a function of $\vec{\xi}, \vartheta_1, \tau, \Delta, w_\epsilon, w_+, \tau', \Delta', w'_\epsilon, w'_+$.

Especially, for a photon which enters the considered setup of polarizers unpolarized (i.e. for $\vec{\xi} = 0$) the density matrix (83) achieves the form

$$\begin{aligned}
W''(0, \vartheta_1) = & \frac{1}{2} \left\{ \tau\tau' + \frac{\Delta\Delta'}{4} \cos 2\vartheta_1 + \right. \\
& + \sigma_3 \frac{1}{2} \left[\Delta\tau'(w'_+ + (1 - w'_+) \cos 4\vartheta_1) + \tau\Delta' \cos 2\vartheta_1 \right] - \\
& - \sigma_1 \frac{1}{2} \left[\tau\Delta' \sin 2\vartheta_1 + \Delta\tau'(1 - w'_+) \sin 4\vartheta_1 \right] + \\
& \left. + \sigma_2 \frac{1}{2} \Delta\tau'(2w'_e - 1) \sin 2\vartheta_1 \right\}. \tag{86}
\end{aligned}$$

If, moreover the second polarizers's transmission axis is perpendicular to the first one (i.e. for $\vartheta = \pi/2$) then the expression (86) simplifies to ¹²

$$\begin{aligned}
W''(0, \pi/2) = & \frac{1}{2} \left\{ \tau\tau' - \frac{\Delta\Delta'}{4} + \sigma_3 \frac{\Delta\tau' - \tau\Delta'}{2} \right\} \\
= & \frac{1}{4} \left\{ k_1 k'_2 + k_2 k'_1 + \sigma_3 [k_1 k'_2 - k_2 k'_1] \right\}. \tag{87}
\end{aligned}$$

Therefore, if the transmittances of both polarizers would be proportional, (i.e. $k'_j = \kappa k_j$), then any photon which passed through such a sequence of perpendicularly oriented polarizers should be unpolarized - provided it entered unpolarized the first one.

Let us add behind the second polarizer a third one which represents again an instrument of the same type, i.e. an instrument (cf. (82))

$$Q''(\vartheta) = a'' + b''\sigma_3 \cos 2\vartheta - b''\sigma_1 \sin 2\vartheta. \tag{88}$$

Photon which has succeeded to pass through a setup of such 3 polarizers sketched at Fig. 1, is in the state

$$\begin{aligned}
W''(\vec{\xi}, \vartheta_1, \vartheta) = & Q''(\vartheta) W''(\vec{\xi}, \vartheta_1) Q''^+(\vartheta) \\
= & w(\vec{\xi}, \vartheta_1, \vartheta) W(\vec{\xi}''(\vec{\xi}, \vartheta_1, \vartheta)) \tag{89}
\end{aligned}$$

¹²Generally the density matrix (83) (for fixed $\vec{\xi}$ and ϑ_1) depends on values of all 6 parameters ($a, b_1, b_2, a', b'_1, b'_2$) determining the two polarizers, but in the special case ($\vec{\xi} = 0, \vartheta_1 = \pi/2$) only 4 transmittances matter - as can be seen from (87).

where $\vec{\xi}$ determines the incoming photon state (47).

It is straightforward to express $w(\vec{\xi}, \vartheta_1, \vartheta)$, $\vec{\xi}'''(\vec{\xi}, \vartheta, \vartheta_1)$ as function of $\vec{\xi}, \vartheta_1, \vartheta$ and 9 parameters determining the three polarizers but the resulting formulae are pretty long. Let us, therefore, content ourself with the most easily measurable quantity, namely the transition probability $w(\vec{\xi}, \vartheta_1, \vartheta)$ in the case of initially unpolarized photons:

$$\begin{aligned} w(0, \vartheta_1, \vartheta) = & \left[\tau\tau' + \frac{\Delta\Delta'}{4} \cos 2\vartheta_1 \right] \tau'' + \\ & + \left[\tau\Delta' \sin 2\vartheta_1 + \Delta\tau'(1 - w'_+) \sin 4\vartheta_1 \right] \frac{\Delta''}{4} \sin 2\vartheta + \\ & + \left\{ \Delta\tau' \left[w'_+ + (1 - w'_+) \cos 4\vartheta_1 \right] + \tau\Delta' \cos 2\vartheta_1 \right\} \frac{\Delta''}{4} \cos 2\vartheta. \end{aligned} \quad (90)$$

Especially if the two polarizers are perpendicular to each other this probability is given by the expression

$$\begin{aligned} w(0, \pi/2, \vartheta) = & \left(\tau\tau' - \frac{\Delta\Delta'}{4} \right) \tau'' + (\Delta\tau' - \tau\Delta') \frac{\Delta''}{4} \cos 2\vartheta \\ = & \frac{1}{4} \left[(k_1 k'_2 + k_2 k'_1)(k''_1 + k''_2) + (k_1 k'_2 - k_2 k'_1)(k''_1 - k''_2) \cos 2\vartheta \right] \end{aligned} \quad (91)$$

which is completely determined (for fixed ϑ) by the transmittances.

Till now we have assumed that all considered polarizers are instruments of the *first kind*.¹³

Let us now briefly discuss the same problem in the case when the considered polarizers represent the simplest instruments of the second kind, namely, we shall assume that the first polarizer is described by a pair of operators

$$Q_l \equiv \sqrt{\pi_l} P_{(l)}, \quad l = 1, 2 \quad (92)$$

where $P_{(l)}$ are mutually orthogonal projectors and the nonnegative real numbers π_l satisfy the condition

$$\pi_1 + \pi_2 \leq 1. \quad (93)$$

¹³Let us recall that complete specification of such polarizer requires to supplement the transmittance k_1, k_2 by another real parameter (e.g. w_+ or w_-).

The remaining two polarizers are described similarly.

Without any loss of generality one can express the projectors as

$$P_{(l)} = \frac{1}{2} \left[1 - (-1)^l \vec{n} \cdot \vec{\sigma} \right] \quad (94)$$

where

$$\vec{n} = \vec{n}^*, \quad |\vec{n}| = 1. \quad (95)$$

It is easy to show that for a good polarizer (i.e. in order to satisfy the condition (62)) the constraint

$$n_1 = n_2 = 0 \quad (96)$$

must hold.

Therefore, one can replace the formulae (94) by (cf. (50), (57))

$$P_{(l)} = P_l. \quad (97)$$

The state of a photon behind such polarizer is described by the density matrix

$$\begin{aligned} W'(\vec{\xi}) &= \sum_l \pi_l P_l W(\vec{\xi}) P_l = \\ &= \frac{1}{4} \left\{ \pi_1 + \pi_2 + \xi_3(\pi_1 - \pi_2) + \sigma_3 [\pi_1 - \pi_2 + \xi_3(\pi_1 + \pi_2)] \right\}. \end{aligned} \quad (98)$$

The transition probability

$$w(\vec{\xi}) = Sp W'(\vec{\xi}) = \frac{1}{2} [\pi_1 + \pi_2 + \xi_3(\pi_1 - \pi_2)] \quad (99)$$

for $\xi_3 = +1$ (-1) has to coincide with the transmittance k_1 (k_2). Therefore, the parameters π_l determining our instrument have a simple physical meaning

$$\pi_l = k_l, \quad l = 1, 2. \quad (100)$$

Moreover, it is easy to show that in this case the probabilities w_+ , w_- are equal to each other, namely that

$$w_+ = w_- = 1/2. \quad (101)$$

The relations (100), (101) allow to express the density matrix (98) again in the form (81). Therefore, *all* previously derived formulae for density matrices in terms of $k_1, k_2, w_+, w_-, k'_1, \dots, w''_e$ remain valid also for the corresponding setups consisting of the second kind polarizers.¹⁴

5. Comparison with experiment

The experimental data⁽¹⁾ on ϑ -dependence of transmittance for a setup sketched at Fig. 1, with the first two polarizers mutually orthogonal ($\vartheta_1 = \pi/2$) in the case of initially unpolarized light are reproduced at Fig. 2.

The values of transmittances of individual polarizers - as specified at⁽¹⁾ are summarized in Tab. I.

One easily finds that for (mean values of) these transmittances the factor multiplying $\cos \vartheta$ at (91) is negative and its magnitude represents $\sim 4\%$ of the first one. It should result in an unpronounced minimum (maximum) for $\vartheta = 0$ ($\pi/2$) - in a qualitative agreement with the data.

The formula (91) predicts that the ϑ -dependence should have the form

$$I(\vartheta) = A + B \cos 2\vartheta. \quad (102)$$

The corresponding average value

$$\langle I \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} I(\vartheta) d\vartheta = A = (5.05 \pm 0.32) \cdot 10^{-3} \quad (103)$$

and the difference between extremes

$$\Delta I \equiv I(0) - I(\pi/2) = 2B = (-4.14 \pm 5.67) \cdot 10^{-4} \quad (104)$$

are both in good agreement with the data.

¹⁴In spite of the fact that polarizers of the second kind do *not* represent a special case of the first kind polarizers: For polarizers of the first kind the relations (101) are equivalent to the requirement $k_2 = 0$, whereas for the second kind polarizers they are valid for all possible values of transmittances k_1, k_2 .

6. Conclusions

Devices utilized for experimental studies usually only approximate the ones from which the definitions of the quantum theory's basic notions stem. We have developed - in the framework of quantum theory - an algorithm a p t for handling a large class of such devices ($\equiv$ instruments). Its ability to provide quantitative results was demonstrated on the problem of transmittances of a setup bild from imperfect polarizers. It was shown that these results nicely correspond to the experimental data⁽¹⁾ and therefore that the worry expressed at⁽⁴⁾: "That (i.e. the data⁽¹⁾) can be hardly derived theoretically if the light polarization is described with the help of quantum-mechanical approach" is unfounded.

References

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Figure captions

Fig. 1 Sketch of three linear polarizers setup. Empty arrows indicate the transmittance axes. Photons move from the left along $\vec{e}_3$ axis.

Fig. 2 The experimental data on ϑ -dependence of transmittance for the considered setup with $\vartheta_1 = \pi/2$ presented by Krása et al⁽¹⁾.

Tab. I. *Transmittances of polarizers*

j	1	2
k_j	0.893 ± 0.013	0.0146 ± 0.0012
k'_j	0.903 ± 0.013	0.0135 ± 0.0012
k''_j	0.767 ± 0.011	0.0331 ± 0.0018

Fig. 1

Fig. 2