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Minimal Surfaces in the Hyperbolic
Space and Radial-Symmetric Solu-
tions of the Cosh-Laplace Equation

by

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Minimal surfaces in the hyperbolic space and radial-symmetric solutions of the cosh-Laplace equation

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Abstract

An example of the constant mean curvature surface embedded into three-dimensional hyperbolic space, studied in the paper, is generated by a special (radial-symmetric) solution of corresponding Gauss-Codazzi equation. This solution appears to be the third Painlevé transcendent, which, in its turn, is explicitly described in a framework of isomonodromic deformation method. As a result the preimages of all umbilic lines of the surface are effectively constructed together with qualitative behavior of the surface at infinity.

0 Introduction

Classical differential geometry provides a number of applications for integrable non-linear equations of the Sine-Gordon type. One of them is linked with the elliptic cosh-Gordon (also called cosh-Laplace) equation

$$u_{z\bar{z}} = \cosh u \quad (0.1)$$

which appears as Gauss-Codazzi equation for an immersion of a constant mean curvature (CMC) surface into hyperbolic space H^3 . The real-valued solution $u(z, \bar{z})$ stands for the induced metrics $e^u dz d\bar{z}$ on the surface, whereas the immersion itself is given by the components of the Ψ -function involved in the Lax pair for equation (0.1)

$$\begin{cases} \Psi_z = U\Psi \\ \Psi_{\bar{z}} = V\Psi \end{cases} \quad (0.2)$$

Among the class of exact solutions $u(z, \bar{z})$ found by the inverse scattering technique, the finite-gap quasiperiodic ones have got a concrete geometric application: they describe the so called Willmore tori [?]. These are a doubly periodic CMC surfaces M in a hyperbolic space, realized as suitably glued two copies of $\mathbb{R}_+^3$ half-spaces with Poincare metrics and distinguished by the condition of minimizing the functional

$$W = \int_M H^2 dS, \quad (0.3)$$

where H is the mean curvature and dS is an area element.

In general the surfaces are not tori. Moreover, due to the maximum principle there are no compact minimal surfaces in H^3 , because any real-valued solution of equation (0.1) blows up beyond a certain domain in $(z, \bar{z})$ -plane. The lines where $u \rightarrow \infty$ also have clear geometric interpretation: they are preimages of the *umbilic lines* on the surface [?]. So the need to study singular solutions of equation (0.1) is well-founded and singularity lines together with a solution over there produces an interesting geometric stuff.

Leaving apart finite-gap solutions we turn to *radial-symmetric* solutions of equation (0.1). Choosing $H = 0$ as a parameter of immersion map (see below §1) this solution produces an example of Willmore CMC surface.

Another strong motivation of this choice is a reduction of the cosh- Laplace equation (0.1) to a special case of the third Painlevé equation (PIII)

$$u'' + \frac{1}{x} u' = \cosh u, \quad (0.4)$$

where $x = |z|$, $' = d/dx$.

A powerful tool is known to integrate the equation (0.4) – *isomonodromic deformation method (IDM)*, which is best suited for the current geometric application. The reason is a specific compatibility condition similar to (0.2) which produces a Ψ - function, providing in its turn an immersion map for the CMC surface. The first integrals of PIII equation (0.4) come as invariants of the surface, parametrizing both the 'central point' at $x = 0$ and the umbilic lines at $x = a_n$, where a_n is a singularity point of the induced metrics $e^u dz d\bar{z}$. It is possible even to get explicit formulas for the distribution of a_n at infinity in terms of the initial condition at the origin (see §3).

The structure of the paper will be as follows. In the first paragraph we remind the basic formulas of the theory of CMC surfaces. Then in §2 we trace quite schematically the application of the IDM to integration of the PIII equation and represent the asymptotic connection formulas for the singularity points. Finally, in §3 the immersion map will be analysed in the vicinity of the umbilic lines.

1 Constant mean curvature surfaces in H^3

Here we sketch briefly basic geometric formulas for CMC surfaces following the paper [?].

Consider a hyperbolic space Q

$$\{F, F\} = -1,$$

which is embedded into the Lorentz space $\mathbb{R}^{3,1}$. The metric of $\mathbb{R}^{3,1}$ induces a positively definite metric on Q

$$\{a, b\} = a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4.$$

If M is a smooth orientable surface in Q , then metric $\{, \}$ induces a complex structure of a Riemann surface $\mathcal{R}$ on M . Consider a conformal parametrization of M

$$F : \mathcal{R} \mapsto M \subset Q \subset \mathbb{R}^{3,1},$$

which poses normalization constraints

$$\{F_z, F_z\} = \{F_{\bar{z}}, F_{\bar{z}}\} = 0.$$

It is natural to complete them with additional relations involving normal vector N

$$\{F_z, F_{\bar{z}}\} = \{F_{\bar{z}}, N\} = \{F, N\} = 0, \quad \{N, N\} = 1.$$

Let us introduce also the notations

$$\{F_z, F_z\} = 2e^u, \quad \{F_{z\bar{z}}, N\} = 2H^he^u, \quad \{F_{zz}, N\} = A^h,$$

where H^h and A^h are called respectively the mean curvature and the Hopf differential.

The variation of the basis $F, F_z, F_{\bar{z}}, N$ with respect to a motion along the surface yields the equations

$$\sigma_z = U\sigma, \quad \sigma_{\bar{z}} = V\sigma, \quad \sigma = (F, F_z, F_{\bar{z}}, N)^T$$

where U, V are explicit 4×4 - matrices compiled of $A^h, H^h, u_z, u_{\bar{z}}$ and e^u functions (see [?]).

The compatibility condition

$$U_z - V_{\bar{z}} + [U, V] = 0$$

together with the CMC constraint

$$H^h = \text{const}$$

leads to the following Gauss-Peterson-Codazzi equations

$$u_{z\bar{z}} + 2((H^h)^2 - 1)e^u - A^h \bar{A}^h e^{-u}/2 = 0,$$

$$A_{\bar{z}}^h = 0.$$

Assuming $(H^h)^2 < 1$ under the scaling

$$z_H = \delta_H z, \quad e^{2i\phi} A = \delta_H^{-1} A^h \quad \delta_H = \sqrt{1 - (H^h)^2},$$

one comes to the system

$$u_{z_H \bar{z}_H} - 2e^u - \frac{1}{2} A \bar{A} e^{-u} = 0, \quad (1.2a)$$

$$A_{\bar{z}_H} = 0. \quad (1.2b)$$

However this system admits more suitable Lax representation.

Theorem 1 ([?]). *A CMC surface in Q , parametrized conformally by (1.1a), generates a holomorphic quadratic differential $A^h(dz)^2$. The induced metric $u(z, \bar{z})$ satisfies (1.2) which is a compatibility condition of the system*

$$\Phi_{z_H} = U\Phi, \quad \Phi_{\bar{z}_H} = V\Phi,$$

$$U = \frac{1}{2} \begin{pmatrix} 0 & 2\lambda e^{u/2} \\ A e^{-u/2} & u_{z_H} \end{pmatrix}, \quad V = \frac{1}{2} \begin{pmatrix} u_{\bar{z}_H} & -2\bar{A} e^{-u/2} \\ \frac{2}{\lambda} e^{u/2} & 0 \end{pmatrix}. \quad (1.3)$$

Proceeding to integration of (1.2) within the framework of inverse scattering technique, one should fix normalization and reduction conditions providing real-valued solution $u(z, \bar{z})$ in a certain functional class. Since $u \in \mathbb{R}$ the $U - V$ -pair satisfies the reduction

$$\overline{U(-\bar{\lambda}^{-1})} = \sigma_2 V(\lambda) \sigma_2, \quad (1.4)$$

which yields

$$\Phi(\lambda) = \sigma_2 \overline{\Phi(-\bar{\lambda}^{-1})} M(\lambda)$$

with a matrix M independent of $z, \bar{z}$. Here and below σ_α are standard Pauli matrices

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Keeping in mind geometric origing of $u(z, \bar{z})$, one can take an arbitrary normalization of the Φ - function in (1.3), since the immersion map for the CMC surface is given by the following

Theorem 2 ([?]). Take $A^h(z_H)(dz_H)^2$ as a holomorphic quadratic differential on R and $u(z_H, \bar{z}_H)$, $\Phi(z_H, \bar{z}_H, \lambda)$ - as the solutions of equations (1.2a), (1.3) respectively. Denote

$$\Phi_0 = \Phi(z_H, \bar{z}_H, \lambda = e^{-q} e^{2i\phi}),$$

then the components of the moving frame $\sigma = (F, F_z, F_{\bar{z}}, N)$ are given as

$$F = \frac{\Phi_0^* \Phi_0}{\sqrt{\det \Phi_0 \det \Phi_0^*}}, \quad N = \frac{\Phi_0^* \sigma_3 \Phi_0}{\sqrt{\det \Phi_0 \det \Phi_0^*}}, \quad (1.5)$$

where $\Phi_0^* = \overline{\Phi_0^T}$ and

$$H = \tanh q.$$

Note that the sign of $\sqrt{\det \Phi_0 \det \Phi_0^*}$ defines the choice of a sheet on Q , where part of the surface lies.

Let us restrict ourselves to the most interesting case of the minimal surfaces $H = 0$. Since $q = 0$ and λ runs over the unit circle we do not distinguish

$$z_H = z.$$

It is suitable also to use a Poincare model of a hyperbolic space

$$H_{\pm} = \left\{ (G_1, G_2, G_3) \in \mathbb{R}^3 \mid \pm G_3 > 0 \right\}.$$

If we denote by $Q_{\pm}$ the upper ($F_0 \geq 1$) and the lower ($F_0 \leq 1$) sheets of Q , then the map

$$S : Q_{\pm} \mapsto H_{\pm}$$

is given by already defined conformal map

$$S : (F_0, F_1, F_2, F_3) \mapsto \left(\frac{F_1}{F_0 - F_3}, \frac{F_2}{F_0 - F_3}, \frac{1}{F_0 - F_3} \right).$$

The coordinates on H provide further simplification of the immersion map, namely

$$G_1 + i G_2 = \frac{a \bar{b} + \bar{d} c}{b \bar{b} + d \bar{d}}, \quad G_3 = \frac{\Delta}{b \bar{b} + d \bar{d}}, \quad (1.6)$$

where

$$\Phi_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \Delta = \sqrt{\det \Phi_0 \det \Phi_0^*}.$$

2 Isomonodromic deformation method and singularities of the PIII function

The general solution of the third Painlevé equation

$$u'' + \frac{1}{x}u' = \cosh u, \quad (2.1)$$

forms a meromorphic function $\exp u(x)$ (see [?]), so that $u(x)$ has logarithmic singularities

$$u(x) = -2 \log(x - a) + 2 \log 2 - \frac{(x - a)}{a} + \frac{b(x - a)^2}{a^2} + O((x - a)^3), \quad x \rightarrow a, \quad (2.2)$$

where a is a coordinate of the pole and b is the second arbitrary constant, fixing the solution at the singularity.

The origin $x = 0$ stands apart from (2.2) singularity ansatz, since the origin is not a "movable" singularity of the equation (2.1). The asymptotics of the solution there reads ([?], Section 11)

$$u(x) = r \log x + s + O(x^{2-|r|}), \quad x \rightarrow 0, \quad (2.3)$$

where r, s are arbitrary real-valued constants, $|r| < 2$. The asymptotics (2.3) can be naturally considered as the initial condition for the solution of (2.1).

The basic analytic problem is the study of the pole distribution of the solution, i.e. the functions

$$a = a(r, s), \quad b = b(r, s), \quad (2.4)$$

especially for the far-away poles $a \rightarrow \infty$.

The isomonodromic deformation method (IDM), which seems to be best suited to solve the problem, is based on the compatibility condition of the two linear systems

$$\Psi_\lambda = A\Psi, \quad (2.5a)$$

$$\Psi_x = U\Psi, \quad (2.5b)$$

where

$$A(\lambda) = -\frac{x^2}{16} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{4\lambda} \begin{pmatrix} 0 & xu' \\ xu' & 0 \end{pmatrix} + \frac{1}{\lambda^2} \begin{pmatrix} \sinh u & -\cosh u \\ \cosh u & -\sinh u \end{pmatrix}, \quad (2.6a)$$

$$U(\lambda) = \frac{1}{8} \begin{pmatrix} -x\lambda & 4u' \\ 4u' & x\lambda \end{pmatrix}, \quad (2.6b)$$

where $\Psi = \Psi(\lambda, x, u)$ is 2×2 - complex-valued matrices, x, u are real-valued parameters. The equations (2.5), (2.6) are derived from the standard L - A - pair equations (see [?]) for $u'' + \frac{1}{x}u' + \sin u = 0$ under the change of variables

$$u \mapsto iu + \frac{\pi}{2}, \quad x \mapsto xe^{-\pi i/4}.$$

Near the irregular points $\lambda = 0, \lambda = \infty$ one can define two solutions Ψ and Φ of the system (2.5a) normalized by the asymptotics

$$\Psi \rightarrow \exp\left(-\frac{x^2\lambda}{16}\sigma_3\right), \quad \lambda \rightarrow \infty, \quad (2.7a)$$

$$\Phi \rightarrow \left(\cosh\left(\frac{u}{2} - \frac{\pi i}{4}\right) + \sigma_1 \sinh\left(\frac{u}{2} - \frac{\pi i}{4}\right)\right) \exp\left(-\frac{i}{\lambda}\sigma_3\right), \quad \lambda \rightarrow 0. \quad (2.7b)$$

The connection matrix Q links together the solutions Ψ and Φ

$$Q = \Phi^{-1}\Psi, \quad (2.8)$$

while the Stokes matrices arise for each of the solutions as one circles around the irregular points. Let Ψ_1 (Ψ_2) be the solutions of the equation (1.1a) in the right (left) half-planes λ with asymptotics (2.7a) as $\arg \lambda = \frac{\pi}{2}$ ($\arg \lambda = \frac{3\pi}{2}$). The pair of solutions Φ_1 (Φ_2) is defined in upper (lower) half-planes λ by the asymptotics (2.7b) as $\arg \lambda = 0$ ($\arg \lambda = \pi$). Then the constraints hold

$$\Psi_1(\lambda) = \Psi_2(\lambda) \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}, \quad (2.9a)$$

$$\Phi_1(\lambda) = \Phi_2(\lambda) \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}, \quad (2.9b)$$

where T is a Stokes multiplier which is obviously independent of λ . In the case of $u(x)$ being a solution of the PIII equation (0.1) it is also independent of x together with the connection matrix Q .

Theorem 3. (Flaschka - Newell [?]) *The function $u = u(x)$ satisfies PIII equation (2.1) iff the connection matrix (2.8) Q and the Stokes multiplier T , defined by (2.9), do not depend on x .*

The proof of the theorem follows immediately from the fact of compatibility of two equations (2.5a) and (2.5b) on the Ψ - function, which coincides with the PIII equation (2.1) on $u(x)$.

As the monodromy data Q and T are known one can solve the inverse monodromy problem, i.e. find the solution Ψ of the equation (2.5a) by solving an appropriate integral equation. Then the derivative $u'(x)$ is expressed as a "residue at infinity" of the Ψ - function

$$u'(x) = \frac{x}{2} \lim_{\lambda \rightarrow \infty} \lambda \Psi_{12}(\lambda, x) \exp\left(-\frac{x^2\lambda}{16}\right), \quad (2.10)$$

which relates the asymptotics of Ψ at infinity with the coefficient of order $O(\lambda^{-1})$ at the equation (2.5a).

The first step to construct the connection formulas (2.4) is to express the monodromy data through the Cauchy initial condition (2.3) as $x \rightarrow 0$. In fact, this problem was already solved in [?], [?]. Namely, for any $r, s \in \mathbb{C}$, $|r| < 2$ the connection matrix was calculated in explicit form

$$Q = \frac{1}{2\pi i} \begin{pmatrix} A - B & Ae^{-\frac{i\pi r}{4}} + Be^{\frac{i\pi r}{4}} \\ Ae^{\frac{i\pi r}{4}} + Be^{-\frac{i\pi r}{4}} & A - B \end{pmatrix}, \quad (2.11)$$

where

$$A = 2^{-3r/2} e^{-s/2} e^{i\pi r/8 - i\pi/4} \Gamma^2\left(\frac{1}{2} - \frac{r}{4}\right), \quad (2.12)$$

$$B = 2^{3r/2} e^{s/2} e^{-i\pi r/8 + i\pi/4} \Gamma^2\left(\frac{1}{2} + \frac{r}{4}\right). \quad (2.13)$$

The Stokes multiplier T was also found in [?]:

$$T = 2i \sin \frac{\pi r}{4}. \quad (2.14)$$

Fixing this monodromy data Q and T as first integrals of the PIII equation (2.1), consider the solution at a singular point $x = a$ and solve the direct monodromy problem with two unknown constants a and b , entering the asymptotics

$$u(x) = -2 \log \varepsilon + 2 \log 2 - \frac{\varepsilon}{a} + \frac{b}{a^2} \varepsilon^2 + O(\varepsilon^3), \quad (2.15)$$

where $\varepsilon = x - a$, $\varepsilon \rightarrow 0$.

Unfortunately, the solution doesn't look like the explicit expressions (2.11) - (2.14). In fact, it is given through Mathieu functions $W, \bar{W}$, which satisfy an equation

$$\frac{d^2 W}{d\lambda^2} = \left(\frac{a^4}{256} + \frac{3b - 2}{8\lambda^2} - \frac{1}{\lambda^4} \right) W, \quad (2.16)$$

normalized by the asymptotics at infinity

$$\begin{aligned} W(\lambda) &\rightarrow \exp\left(\frac{a^2}{16}\lambda\right), \\ \bar{W}(\lambda) &\rightarrow \exp\left(-\frac{a^2}{16}\lambda\right), \quad \lambda \rightarrow \infty, \quad \arg \lambda = \frac{\pi}{2}. \end{aligned} \quad (2.17)$$

The monodromy data is recovered through the following constraints (see [?])

$$\left(W(\lambda), W(-\lambda) \right) = -\frac{a}{4\sqrt{i}} \lambda \left(W\left(\frac{16i}{a^2\lambda}\right), \bar{W}\left(-\frac{16i}{a^2\lambda}\right) \right) Q, \quad (2.18a)$$

$$(W(\lambda e^{-i\pi}), \overline{W}(\overline{\lambda} e^{i\pi})) = (\overline{W}(\overline{\lambda}), W(\lambda)) \begin{pmatrix} 1 & 0 \\ -T & 1 \end{pmatrix}, \quad \lambda \in \Omega_1^\infty. \quad (2.18b)$$

Remark. The latter equation (2.18b) can be rewritten in terms of the Floquet exponent for the classical Mathieu function in a following way. Introducing the new variables

$$\lambda = \frac{4\sqrt{i}}{a} e^{i\zeta}, \quad W(\lambda) = \sqrt{\lambda} V(\zeta) \quad (2.19)$$

one gets from (2.18) the Mathieu equation

$$\frac{d^2 V}{d\zeta^2} + \left(\frac{3b}{8} - \frac{a^2 i}{8} \cos 2\zeta \right) V = 0. \quad (2.20)$$

which has a solution

$$V(\lambda) = e^{i\zeta/2} \left(e^{i\mu\zeta} Y_1(\zeta) + e^{-i\mu\zeta} Y_2(\zeta) \right),$$

where Y_1, Y_2 are π -periodic functions and μ is a Floquet exponent. This yields the constraint

$$T = 2i \cos \mu\pi. \quad (2.21)$$

The functional equations (2.18), (2.20) on the parameters a and b are highly transcend, so there are apparently no explicit solutions for any seems to be a functional equation expressing the Floquet exponent μ through a and b by an infinite chain fraction (see [?], p.143).

By chance, an effectivization of these constraints can be achieved in a case of large a . That means a construction of a distribution of poles for the PIII equation (2.1) as $x \rightarrow \infty$. The poles are parametrized by an integer n and for $n \rightarrow \infty$ the parameters a_n, b_n are expressed explicitly through the initial data r and s .

Let us sketch briefly the procedure refering the paper [?] for details. Assuming $b = O(a^2)$, $a \rightarrow \infty$, rewrite equation (2.16) in a variable $z = -a\lambda/4$

$$\frac{d^2 W}{dz^2} = \frac{a^2}{16} \left(1 + \frac{6b-4}{a^2 z^2} - \frac{1}{z^4} \right) W.$$

Define the approximate (with respect to small parameter a^{-1}) WKB-solution

$$W_{WKB}^+(z) = (q(z))^{-1/4} \exp\left(-\frac{a}{4} \int_{z_0}^z \sqrt{q(\zeta)} d\zeta\right) (1 + o(1)), \quad z > z_0, \quad (2.22)$$

$$W_{WKB}^-(z) = |q(z)|^{-1/4} \exp\left(i\frac{a}{4} \int_{z_0}^z \sqrt{|q(\zeta)|} d\zeta\right) (1 + o(1)), \quad z < z_0,$$

where z_0 is a simple turning point - the only positive root of $q(z)$

$$q(z) = 1 + \frac{6b-4}{a^2 z^2} - \frac{1}{z^4}.$$

The monodromy equation (2.18a) transforms into a quasiclassical scattering for the WKB - solution (2.20) ([?]). It is solved explicitly in terms of the phase integrals

$$\begin{aligned} Q_{12} &= \exp\left(\frac{a}{4}(I^-(z_0) - I^+(z_0))\right), \\ Q_{22} &= \exp\left(-\frac{a}{4}(I^-(z_0) + I^+(z_0)) - \frac{\pi i}{2}\right), \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} I^+(z_0) &= \int_{z_0}^{+\infty} [\sqrt{q(\zeta)} - 1] d\zeta - z_0, \\ I^-(z_0) &= -i \int_{-\infty}^{z_0} [\sqrt{-q(\zeta)} - \zeta^{-2}] d\zeta - i \frac{z_0^3}{3}. \end{aligned}$$

The right-hand side of the equation (4.10a) contains a large parameter a , while the left-hand side is constant, i.e. of order $O(1)$ with respect to a . Hence, the leading term of real-valued multiplier of a has to be zero, whereas the imaginary one has to be proportional to 2π . This gives the asymptotic conditions on $z_0 = z_0(a, b)$:

$$I^+(z_0) = O(a^{-1}), \quad \frac{a}{4} I^-(z_0) = 2\pi n + O(1), \quad n \in \mathbb{Z}. \tag{2.24}$$

$$z_0 = -\frac{1}{2} \ln \mu + \frac{\nu}{a} + O(a^{-2}), \quad a \rightarrow \infty.$$

The numeric solution of functional equations (2.24) gives an approximate values of z_0, I^+

$$\mu = 2.17966043, \quad I^+\left(\frac{1}{2} \ln \mu\right) = 1.55463.$$

Finally, the second equation (2.24) yields the approximate asymptotics for a_n, b_n in terms of the monodromy data

$$a_n = 11.4314n - 4.41773 \ln |q| + 1.81936 \arg q + O(n^{-1}), \quad n \rightarrow \infty, \tag{2.25}$$

$$\begin{aligned} b_n &= 18.7399n^2 + n(5.96507 \arg q - 14.4843 \ln |q| + \\ &\quad 2.73832) + O(1), \quad n \rightarrow \infty. \end{aligned} \tag{2.26}$$

Inserting here the exact expressions (2.11) - (2.13) for Q_{12}, Q_{22} , one gets necessary connection formulas for the distribution of poles for PIII equation in terms of its initial condition (2.3).

3 Asymptotics of the immersion map near the umbilic lines

Apply the asymptotic formulas (2.25) - (2.28) to describe a behavior of the CMC-surface near an umbilic line, i.e. the image of a singular point $x = a$.

First we need to link the Φ -function (1.3) - (1.5) for the cosh-Laplace equation with the Ψ -function (2.5) - (2.7) for the PIII equation. This is given by the following constraint

$$\Phi_0 = \begin{pmatrix} \zeta & \zeta \\ \frac{2}{\zeta}e^{v/2} & -\frac{2}{\zeta}e^{v/2} \end{pmatrix} \cdot \Psi\left(x, \lambda = \frac{4}{x}e^{i(\beta+\gamma)}\right), \quad (3.1)$$

where

$$\begin{aligned} \zeta &= z^{-\frac{m}{4}}e^{i(\gamma-\pi)/2}, \quad e^{v/2} = \frac{1}{2}e^{u/2}|z|^{m/2}, \\ z &= \frac{m+2}{2}\left(xe^{i\beta}\right)^{\frac{2}{m+2}}, \quad \bar{z} = \frac{m+2}{2}\left(xe^{-i\beta}\right)^{\frac{2}{m+2}}, \\ x &= \frac{4}{m+2}|z|^{\frac{m+2}{2}}, \quad m = 0, 1, 2, \dots \end{aligned} \quad (3.2)$$

The integer m marks the $m+2$ -fold radial symmetry of the surface since the above mentioned reduction (1.4) leads to the invariance under the rotation on the angle $2\pi/(m+2)$.

It is convenient to put $r = 0$ in (2.3) i.e. to study non-singular at the origin solution of PIII equation (2.1). The change of variables (3.2) produces then a special solution of (2.1), (0.1) with $r = \frac{2m}{m+2}$.

The principal term of the Ψ -function (1.3) for the limit case $x - a_n = \varepsilon \rightarrow 0$ is given by (see [?])

$$\Psi = -\frac{4}{\varepsilon a \lambda} \begin{pmatrix} \overline{W}(\lambda) & W(\overline{\lambda}) \\ \overline{W}(\lambda) & W(\overline{\lambda}) \end{pmatrix} + O(1). \quad (3.3)$$

Since the parameter λ in (3.1) runs over the circle exactly according to (2.21), the function $W(\lambda)$ turns into quasiperiodic Mathieu function satisfying equation (2.22). Putting together formulas (1.6), (3.1) and (3.3), one gets for the limit $x \rightarrow a_n$

$$G_1 + iG_2 = -e^{-i\beta} \frac{V(\beta)}{\overline{V}(\beta + \pi)}, \quad 0 \leq \beta \leq 2\pi, \quad (3.4a)$$

$$G_3 = 0, \quad (3.4b)$$

where V satisfies the Mathieu equation (2.20).

Theorem 4. *The umbilic line (3.4a) is a unit circle for any pole $x = a_n$.*

Proof. The Floquet exponent μ for the solution $V(\beta)$ is the first integral of the PIII equation since $T = 2i \cos(\pi\mu)$ is x -independent due to the Flaschka - Newell

theorem (see Theorem 3 in §1). On the other hand, according to (2.14) for the choice $r = 0$ one has $T = 0$, which yields

$$\mu = n + \frac{1}{2}, \quad n = 0, \pm 1, \pm 2, \dots$$

this provides the function V to be π -periodic, so that

$$|G_1 + iG_2| = 1,$$

which proves the theorem.

The result reveals a global structure of the surface: it has infinitely many self-intersections along umbilic lines, which coincide with the unit circle. It is natural to assume, that parts of the surface, parametrized by an annular domain $a_n < x < a_{n+1}$, should not much differ from each other at least for large n . The computer simulation confirms this statement. To proceed with the job it is convenient for numeric computations to change radial variable

$$x \mapsto u, \quad x = - \int_u^\infty \frac{dt}{\sqrt{2 \sinh t - 2A}},$$

then the matrix (3.1) has the form

$$\Phi_0(z, \bar{z}, \lambda = 1) = i \begin{pmatrix} \frac{R_+}{M} z^{-m/4} & -\frac{R_+}{N} z^{-m/4} \\ e^{u/2} \frac{R_-}{M} z^{m/4} & e^{u/2} \frac{R_-}{N} z^{m/4} \end{pmatrix} \cdot \exp \left\{ -\frac{i\sigma_3}{8} \int_u^\infty \frac{dt}{\sqrt{\sinh t - A}} \int_0^\beta \sqrt{A - i \sin t} dt \right\},$$

where

$$\begin{aligned} R_\pm &= e^{i\beta} \sqrt{2} \left[\sqrt{\sinh u - A} \pm \sqrt{i \sin t - A} \right] \pm e^{2i\beta} - e^u, \\ N &= 2e^{i\beta} \sqrt{2i \sin 2\beta - 2A}, \\ M &= \sinh u - e^{2i\beta} - e^{i\beta} \sqrt{2i \sin 2\beta - 2A}, \\ \ln \mu &< u < \infty, \quad A = \frac{\mu - \mu^{-1}}{2}. \end{aligned}$$

Applying then the immersion formulas (1.6) with the entries given above, one can get a picture of the surface. The numeric simulation is based on a plot of bundle of lines – the images in (G_1, G_2, G_3) – coordinates of the circles $x = \text{const}$, $0 \leq \beta < 2\pi$.

A corresponding profile of the surface looks like a cap with a circular edge $G_3 = 0$. Due to the construction of Poincaré map (see §1) the pass over an absolute changes the sign of G_3 coordinate. A plot of the two pieces of the surface is glued analytically along the umbilic line $G_3 = 0$. Note that initial data (3.5) for the PIII equation generates the first pole $a_0 \sim 50$, so the asymptotic ansatz (3.4) is valuable. The same approximation is true for all other poles as well and the picture does not differ much.

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