Solving the Constraints of General Relativity

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ABSTRACT

I show in this letter that it is possible to solve some of the constraints of the SO(3)-ADM formalism for general relativity by using an approach similar to the one introduced by Capovilla, Dell and Jacobson to solve the vector and scalar constraints in the Ashtekar variables framework. I discuss the advantages of this approach and compare it with similar proposals for different Hamiltonian formulations of general relativity.
The main purpose of this letter is to show that it is possible to solve some of the constraints of general relativity in the SO(3)-ADM formalism [1] by using techniques similar to those introduced by Capovilla, Dell and Jacobson [2] to solve the vector and scalar constraints in terms of Ashtekar variables. More specifically, I will find that it is possible to solve both the “Gauss law” (the generator of the internal SO(3) rotations) and the scalar constraint by introducing suitable $3 \times 3$ SO(3) matrices and imposing simple conditions on them. This result shows that the first step in Ashtekar’s procedure to derive the new variables formalism—the introduction of an internal SO(3) symmetry—has interesting ramifications even outside the Ashtekar variables framework. Also, it suggests some interesting relationships between three dimensional diffeomorphisms and the SO(3) transformations. Finally it may be useful for numerical relativists because it reduces the number of constraint equations that must be solved, it does not require the introduction of reality conditions for Lorentzian signature space times and the evolution equations can be written in terms of the new fields that will be introduced later.

The conventions and notation used throughout the letter are the following. Tangent space indices and SO(3) indices are represented by lowercase Latin letters from the beginning and the middle of the alphabet respectively. The 3-dimensional Levi-Civita tensor density and its inverse are denoted by $\tilde{\eta}^{abc}$ and $\eta_{abc}$ and the internal SO(3) Levi-Civita tensor by $\epsilon_{ijk}$. The variables in the SO(3)-ADM phase space are a densitized triad $\tilde{E}_i^a$ (with determinant denoted by $\tilde{E}$) and its canonically conjugate object $K^i_a$ (closely related to the extrinsic curvature). The (densitized) three dimensional metric built from the triad is denoted $\tilde{q}^{ab} \equiv \tilde{E}_i^a \tilde{E}^{bi}$ and its determinant $\tilde{q}$ so that $q^{ab} = \frac{\tilde{q}^{ab}}{\tilde{q}}$. I will use also the SO(3) connection $\Gamma^i_a$, compatible with the triad. The variables in the Ashtekar phase space are $\tilde{E}_i^a$, again,

1I represent the density weights by the usual convention of using tildes above and below the fields.
and the $SO(3)$ connection $A^i_a$. The curvatures of $A^i_a$ and $\Gamma^i_a$ are respectively given by $F^i_{ab} \equiv 2\partial_{[a} A^i_{b]} + \epsilon^i_{j k} A^j_a A^k_b$ and $R^i_{ab} \equiv 2\partial_{[a} \Gamma^i_{b]} + \epsilon^i_{j k} \Gamma^j_a \Gamma^k_b$. Finally, the action of the covariant derivatives defined by these connections on internal indices are

$$\nabla_a \lambda_i = \partial_a \lambda_i + \epsilon_{ijk} A^j_a \lambda^k \quad \text{and} \quad D_a \lambda_i = \partial_a \lambda_i + \epsilon_{ijk} \Gamma^j_a \lambda^k.$$ 

The compatibility of $\Gamma^i_a$ and $\tilde{E}^a_i$ means $D_a \tilde{E}^h_i \equiv \partial_a \tilde{E}^h_i + \epsilon_i^{jk} \Gamma^j_{ab} \tilde{E}^h_k + \Gamma^h_{ac} \tilde{E}^c_i - \Gamma^c_{ac} \tilde{E}^h_i = 0$.

The $SO(3)$-ADM constraints are [1]

\begin{align}
\epsilon_{ijk} K^j_a \tilde{E}^a_k &= 0 \\
D_a \left[ \tilde{E}^a_i K^k_b - \delta^a_i \tilde{E}^a_k K^k_b \right] &= 0 \\
-\zeta \tilde{q} R + \frac{2}{\tilde{q}} \tilde{E}^a_k \tilde{E}^b_d K^c_k K^b_d &= 0
\end{align}

where $R$ is the scalar curvature of the three-metric $q_{ab}$ (the inverse of $q^{ab}$). and the variables $K_{ai}(x)$ and $\tilde{E}^h_j(y)$ are canonical; i.e., they satisfy

\begin{align}
\left\{ K^i_a(x), K^j_b(y) \right\} &= 0 \\
\left\{ \tilde{E}^a_i(x), \tilde{E}^b_j(y) \right\} &= \delta^a_i \delta^b_j \delta^3(x, y) \\
\left\{ \tilde{E}^i_a(x), \tilde{E}^h_j(y) \right\} &= 0
\end{align}

By choosing $\zeta = +1$ or $\zeta = -1$ we can describe both Lorentzian and Euclidean signature space-times. The constraints (1-3) generate internal $SO(3)$ rotations, diffeomorphisms and time evolution respectively.

For non-degenerate metrics we can multiply (3) by $\sqrt{\tilde{q}}$ to get

$$-\zeta q R + 2 \tilde{E}^a_k \tilde{E}^b_d K^c_k K^b_d = 0$$

Using now

$$\tilde{q} R = -\epsilon_{ijk} \tilde{E}^a_i \tilde{E}^b_j R_{abk}$$

They may be extended to act on tangent indices by introducing a space-time torsion-free connection; for example the Christoffel symbols $\Gamma^i_{ab}$ built from $q^{ab}$. All the results presented in the paper will be independent of such extension.
equation (5) can be rewritten in the form

\[
e^{ijk} E_i^a E_j^b \left( \zeta R_{abk} + \epsilon_{klm} K_{ai} K_{bm} \right) = 0
\]  

(7)

We follow now a procedure very closely related to the one used by Capovilla, Dell and Jacobson to solve the scalar and vector constraints in the Ashtekar formalism. First we define the \(3 \times 3\) matrix \(\Psi_{ij}\) (for non-degenerate triads) as

\[
\tilde{E}^a_i \Psi_{ij} \equiv \eta^{abc} \left( \zeta R_{bcj} + \epsilon_{jkl} K^k_i K_l^j \right)
\]  

(8)

Introducing (8) in (7) we get immediately the condition \(tr\Psi = 0\). We consider now the Gauss law Eq. (1). In order to solve it we need to write \(K_{ai}\) as a function of \(\tilde{E}^a_i\) and \(\Psi_{ij}\). Defining

\[
\tilde{S}_i^a \equiv \tilde{E}^a_{ij} \Psi_{ji} - \zeta \tilde{\eta}^{abc} R_{bcj}
\]  

(9)

we have

\[
K_a^i = \frac{1}{2\sqrt{2\tilde{S}}} \eta_{abc} e^{ijk} \tilde{S}_j^b \tilde{S}_k^c
\]  

(10)

where \(\tilde{S}\) is the determinant of \(\tilde{S}_i^a\). This last expression is only valid when \(\tilde{S} \neq 0\) (or equivalently \(\det K \neq 0\)). Introducing now (10) in the Gauss law (1) we get

\[
\eta_{abc} \tilde{S}_j^b \tilde{S}_k^c \tilde{E}^a_{ij} = 0
\]  

(11)

Multiplying this last expression by \(\eta_{def} \epsilon^{kpq} \tilde{S}_p^d \tilde{S}_q^e\) and taking into account, again, that we require \(\tilde{S} \neq 0\) we find the equivalent condition

\[
\tilde{E}_i^a \tilde{S}_i^a [j = 0]
\]  

(12)

By using now the Bianchi identity \(\tilde{E}_i^a R_{ab}^i = 0\) we finally obtain \(\Psi_{[ij]} = 0\). The results derived above mean that by taking a symmetric and traceless \(\Psi_{ij}\) it is possible to solve both the Gauss law and the scalar constraint. The only equation left to solve
is the vector constraint. This is a nice result; we can say that, although introducing an internal $SO(3)$ symmetry seems to complicate the theory unnecessarily, using the previous reasoning not only we can solve the additional constraint that generates the new internal symmetry, but also the scalar constraint.

We can write the vector constraint in terms of $\tilde{E}_i^a$ and $\Psi_{ij}$ by using (10) and multiplying by $\tilde{S}^{3/2}$. Proceeding in this way we find that it is equivalent to

$$
\tilde{E}_i^a \tilde{S}_k^b \tilde{S}_k^{c1} \tilde{S}_k^{c2} \tilde{S}_k^{c3} \left( 4 \eta_{c1 c2 c3} \eta_{e f} \epsilon^{k_1 k_2 k_3} \epsilon^{i j k} - 3 \eta_{c1 c2 c3} \eta_{a} \epsilon^{k_1 k_2 j} \epsilon^{i k k_3} \right) D_{[i} \tilde{S}_{j]}^c = 0
$$

where we must now write $\tilde{S}_i^a$ in terms of $\tilde{E}_i^a$ and $\Psi_{ij}$. As we can see this is a complicated expression (although some simplification may be achieved by imposing that $\Psi_{ij}$ must be symmetric and traceless). It is a third order partial differential equation in the triad fields (due to the derivatives of the curvature) and first order in $\Psi_{ij}$.

In spite of not being a simple expression, Eq. (13) has some features that make it interesting. First of all it is written explicitly in terms of $\Psi_{ij}$ and $\tilde{E}_i^a$ only; in fact it is a polynomial equation in this variables. In some other cases (that I will discuss later) where some of the constraints can be solved by using a procedure very similar to the one described above, the equation that is left cannot be simply written in terms of $\Psi_{ij}$ and $\tilde{E}_i^a$ (although this can be achieved, in principle, by solving an additional differential equation). Second, Eq. (13) can be used for both Lorentzian and Euclidean signatures just by selecting $\zeta = -1$ or $\zeta = +1$. This is an advantage over some related results because there is no need to implement the reality conditions. Actually, it is probably fair to say that this is the way to explicitly incorporate the reality conditions in this type of solution to the constraints of general relativity.

In the following I will compare the result derived above with some closely related approaches to solve the constraints of general relativity in different Hamiltonian forms.

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3This fact is one of the reasons that seem to have prevented the use of the neat idea of Capovilla, Dell and Jacobson in numerical relativity.
ulations. I consider first the familiar Capovilla-Dell-Jacobson approach [2]. Starting from the constraints of general relativity in the Ashtekar formulation

\[ \nabla_a \tilde{E}^a_i = 0 \]  
\[ F^i_{ab} \tilde{E}^b_i = 0 \]  
\[ \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j F^c_{abk} = 0 \]

they make the ansatz

\[ \Psi_{ij} \tilde{E}^a_j = \tilde{\eta}^{abc} F^c_{bci} \]

Introducing this in (14) and (15) they immediately get, for non-degenerate triads, that \( \Psi_{ij} \) must be symmetric and traceless. The Gauss law gives then the equation

\[ (\nabla_a \Psi^{-1}_{ij}) \tilde{\eta}^{abc} F^c_{bci} = 0 \]

where use has been made of the Bianchi identity \( \nabla_a (\tilde{\eta}^{abc} F^c_{bci}) = 0 \). If we compare this result with the one presented in this letter we notice several interesting things. First of all, we can solve the scalar constraint in both cases, but in one of them we solve also the Gauss law whereas in the other we solve the vector constraint. This suggests some hidden relationship between three dimensional diffeomorphisms and internal \( SO(3) \) gauge transformations. Second, even though (18) is simple in the Euclidean case, the more relevant Lorentzian signature case is more difficult to deal with because of the reality conditions (whose implementation is not straightforward). In this respect our formulation is interesting because it is valid for both Euclidean and Lorentzian space-times and no reality conditions need to be taken into account. The equations that must be solved in both cases are very similar because they differ only in the value of the parameter \( \zeta \).

Let us consider now the real formulation in terms of Ashtekar variables for Lorentzian space-times discussed in [3]. The phase space of that formulation is the usual Ashtekar
phase space but the fields are now real. The Gauss law and the vector constraints are still given by (14) and (15), whereas the scalar constraint can be written as

$$\epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b (F_{abk} - 2R_{abk}) = 0$$  \hspace{1cm} (19)$$

Making the ansatz

$$\Psi_{ij} \tilde{E}_j^c = \tilde{\eta}^{abc} (F_{abi} - 2R_{abi})$$  \hspace{1cm} (20)$$

the vector and scalar constraints are solved if $$\Psi_{ij}$$ is chosen to be symmetric and traceless as in the usual Capovilla-Dell-Jacobson case. The Gauss law gives the equation

$$\nabla_c \left[ \Psi_{ij}^{-1} \tilde{\eta}^{abc} (F_{abj} - 2R_{abj}) \right] = 0$$  \hspace{1cm} (21)$$

Since we want to have an equation written only in terms of $$\Psi_{ij}$$ and $$A_{\alpha j}$$ we must remove from (21) the terms involving the triads $$\tilde{E}_i^a$$ -i.e. the term $$\tilde{\eta}^{abc} R_{abk}$$- by using (20). In practice this amounts to solving a system of coupled partial differential equations. Comparing this result with the one presented in this letter we see, again, that it is the vector constraint and not the Gauss law that is solved by choosing a symmetric $$\Psi_{ij}$$ and also that we must solve a system of partial differential equations instead of the single equation (13).

Finally I consider the two connection formulation of general relativity discussed in [4]. The phase space in that formulation is spanned by two $$SO(3)$$ connections $$\Lambda_a^i$$ and $$\Lambda_a^i$$ with curvatures given by $$\tilde{B}_i^a \equiv \eta^{abc} \tilde{F}_{bci}$$ and $$\tilde{B}_i^a \equiv \eta^{abc} \tilde{F}_{bci}$$. The Poisson brackets of the basic variables are

$$\{ \Lambda_a^i(x), \Lambda_b^j(y) \} = 0$$

$$\{ \tilde{\Lambda}_a^i(x), \tilde{\Lambda}_b^j(y) \} = \frac{1}{4\epsilon} \left( \epsilon_a^i \epsilon_b^j - 2\epsilon_a^i \epsilon_b^i \right) \delta^3(x, y)$$  \hspace{1cm} (22)$$

$$\{ \tilde{\Lambda}_a^i(x), \tilde{\Lambda}_b^j(y) \} = 0$$

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where \( e^i_a = \frac{1}{2} \Lambda^i_a - \frac{1}{2} \Lambda^i_a \) and \( \hat{e} \equiv \det e^i_a \). Finally the constraints are given by

\[
\epsilon_{ijk} e^j_a B^k_a = 0 \tag{23}
\]

\[
\epsilon_{ijk} e^j_a B^k_a = 0 \tag{24}
\]

\[
e^i_a B^a_i = 0 \tag{25}
\]

If we define now the \( 3 \times 3 \) matrices \( \Psi_{ij}^1 \) and \( \Psi_{ij}^2 \)

\[
\Psi_{ij}^1 \equiv e^i_a B^a_j \tag{26}
\]

\[
\Psi_{ij}^2 \equiv e^i_a B^a_j \tag{27}
\]

we find that all the constraints are solved if we take a symmetric \( \Psi_{ij}^2 \) and a symmetric and traceless \( \Psi_{ij}^1 \)!. This means, that by coordinatizing the phase space by using the 18 variables per space point given by \( \Psi_{ij}^1(x) \) and \( \Psi_{ij}^2(x) \), instead of the two connections \( \Lambda^i_a \) and \( \Lambda^i_a \) it is possible to solve the constraints is a trivial way. The problem is that in order to effectively use this formulation we should write the evolution equations in terms of the fields \( \Psi_{ij}^1(x) \) and \( \Psi_{ij}^2(x) \) (since we know the action of the diffeomorphism and vector constraints it would suffice to be able to write the time evolution of \( \Psi_{ij}^1(x) \) and \( \Psi_{ij}^2(x) \) in terms of themselves). This seems to require the inversion of the equations (26) and (27); i.e. writing the connections in terms of \( \Psi_{ij}^1(x) \) and \( \Psi_{ij}^2(x) \), a task not yet completed and probably difficult. This is not a problem in any of the formulations discussed above; in all of them it is straightforward to write the evolution equations is terms of the relevant fields \( \Psi_{ij} \) and \( \tilde{E}^a_i \) (or \( A_{ai} \)).

Another issue in the two-connection formulation as compared to the \( SO(3) \)-ADM approach discussed in this letter is the implementation of the reality conditions. The formulation given by (23-25) is valid only for Euclidean space-times or complex gravity. We must either impose reality conditions (and this requires also knowing \( \Lambda_{ai} \) and \( \Lambda_{ai} \) in terms of \( \Psi_{ij}^1 \) and \( \Psi_{ij}^2 \)) or use a modified Hamiltonian constraint and real fields.
Although a suitable Hamiltonian constraint for Lorentzian signature space-times is known in terms of two real connections [5], it is not straightforward to write it as a function of $\frac{1}{2} \Psi_{ij}(x)$ and $\frac{1}{2} \Psi_{kl}(x)$ and so the simplicity of the solution to the constraints discussed above for the Euclidean case is lost. This two connection formulation may be useful to get some information about the relationship between the vector constraint and the Gauss law because the constraints generating three dimensional diffeomorphisms and internal $SO(3)$ rotations (23) and (24) have the same structure (they are symmetric under the interchange of $\hat{A}_{ai}$ and $\hat{A}_{ai}$). This, in turn, may help to explain why we can either solve the vector constraint or the Gauss law by using the Capovilla-Dell-Jacobson method in the different Hamiltonian formulations.

In conclusion, the solution to the constraints of the $SO(3)$-ADM formalism presented in this letter seems to provide a convenient way to solve some of the constraints of the Hamiltonian formulation of general relativity. In our opinion it has several advantages with respect to similar approaches in the several Hamiltonian formulations presented above. There is no need to implement any reality conditions neither for Euclidean nor Lorentzian signatures, the constraint that is left to solve gives a polynomial equation involving $\Psi_{ij}$ and $E^a_i$, and the evolution equations can be written in a straightforward way in terms of this fields.

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References


