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THE CASIMIR FORCE BETWEEN PLATES
WITH SMALL DEVIATIONS
FROM PLANE PARALLEL GEOMETRY

by

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Abstract

We present a general calculation of the corrections to the Casimir force between plates with small deviations from plane parallel geometry of arbitrary type. These corrections are considered up to the fourth order with respect to the relative amplitudes of the deviations. The developed formalism is applied to the cases of non-parallel plates and of large-scale as well as of short-scale deformations of the surfaces of different types. A number of particular examples is explicitly calculated. The method used is the developed earlier method of additive summation of the Casimir force potentials with a subsequent renormalization. The results obtained by this method are tested by comparison with the results from the Green functions method. A special investigation is performed on the role of mixed terms in the expansion with respect to the deviation amplitudes resulting from different plates. It is shown that such contributions are present already in the first non-vanishing correction to the Casimir force, so that this correction is non-additive.

1 Introduction

The Casimir effect is a consequence of the alteration of vacuum fluctuations in restricted quantization volumes or in spaces with non-Euclidean topology (see, e.g., the reviews ^{1,2} and the monograph ³). Casimir energy densities and forces must be taken into account in elementary particle physics and cosmology (MIT bag model ⁴, Kaluza-Klein theories ⁵, constraints on hypothetical long-range interactions ⁶, topologically non-trivial spaces ⁷). The Casimir effect plays also a significant role in statistical physics ⁸. Recently, it was discussed in connection with thin films ⁹ and wetting processes on alkali metals ¹⁰.

The calculation of the Casimir force is a rather complicated problem. The most straightforward way is the solution of the wave equation with appropriate boundary conditions and the construction of the Green function for the geometry under consideration ⁸. This can be done for several simplest configurations only (such as two plane parallel plates, a sphere, a small ball over a plane, two small balls, a dihedral angle, a torus, etc. – see, e.g., Refs. 1-3 and references therein). For configurations with small deviations from the mentioned above geometries approximative methods may be used (see, e.g., Refs. 11-15). However, the perturbation expansions with respect to a small parameter, characterizing the deviation from the basic geometry, based on the Green functions method ¹¹⁻¹³ turned out to be quite complicated and not very effective. Only a few particular results have been obtained by this method up to now.

The configuration, for which the Casimir force has been investigated first time, consists of two plane parallel metal plates. At once this is the simplest geometry. In the present paper we investigate the Casimir force for configurations with small deviations from that geometry. Our aim is to give a complete description for all nonstochastic deviations. Hereby we consider plates made out of arbitrary materials.

The exact result for the Casimir force per unit area acting between two plane parallel plates may be expressed in the form ⁸

$$F_0 = -\Psi(\epsilon_1, \epsilon_2) \frac{\pi}{10d^4} , \quad (1)$$

where d is the distance between the plates, $\epsilon_{1,2}$ are the dielectric permeabilities of the plates and we note

$$\begin{aligned} \Psi(\epsilon_1, \epsilon_2) = \frac{5}{16\pi^3} \int_0^\infty \int_1^\infty \frac{x^3}{p^2} \left\{ \left[\frac{(s_1 + p)(s_2 + p)}{(s_1 - p)(s_2 - p)} e^x - 1 \right]^{-1} \right. \\ \left. + \left[\frac{(s_1 + p\epsilon_1)(s_2 + p\epsilon_2)}{(s_1 - p\epsilon_1)(s_2 - p\epsilon_2)} e^x - 1 \right]^{-1} \right\} dp dx . \end{aligned} \quad (2)$$

Here, $s_{1,2} \equiv (\epsilon_{1,2} - 1 + p^2)^{1/2}$. Throughout the paper we use units with $\hbar = c = 1$.

For example, for perfect conductors we have $\epsilon_1 = \epsilon_2 \rightarrow \infty$ and $\Psi(\epsilon_1, \epsilon_2) \rightarrow \pi/24$. More information, a graphical representation for instance, concerning this function can be found in Refs. 8, 15.

Under small deviations from the plane parallel geometry we understand all kinds of deviations whose amplitude is small in comparison with the distance between the

plates. We divide the deviations into three classes. First, the plates may be non-parallel. Secondly, the plates may have a point dependent small curvature, representing a large-scale unevenness. Finally, the plates may be covered with small distortions which characteristic longitudinal scale is small in comparison with the size of the plates. Such distortions form a short-scale structure of the surface. All these deviations can be described by some function depending on the point on the plate, i.e., on two variables. The deviations from the plane parallel geometry give rise to corrections to the Casimir force (1). Such corrections must be taken into account in experiments on measuring the Casimir force and in some of the applications of the Casimir effect mentioned above.

The configuration of two non-parallel plates has been investigated in Ref. 13 using exact results obtained from the Green functions method in Refs. 16, 17. But only the first non-zero correction to the Casimir force was obtained.

Some results concerning configurations of the second type were obtained in Ref. 14, where the specific cases of a paraboloid and of a hyperboloid over a plane were investigated.

Much more attention was paid in literature to the configurations of the third type. In the Refs. 12, 13 the first order corrections to the Casimir force with respect to the relative amplitude of the distortions were obtained for the configuration of a parallelepiped. However, these results cannot be applied to large parallel plates because the first order corrections vanish in that limiting case. In Ref. 15 the case of periodic distortions only was examined using the approximative method of Ref. 14.

In the present paper we formulate the general approach to the calculation of the corrections to the Casimir force for configurations with small deviations from the plane parallel geometry of arbitrary type. This approach is based on the approximative method developed in Refs. 14, 15 and yields the Casimir force as a perturbation expansion with respect to the parameters A_1/d and A_2/d , where A_1 resp. A_2 are the maximal deviation of the first resp. second plate from the plane parallel geometry and d is the distance between the two plates. All calculations are done for zero temperature because corrections due to finite temperature are shown to be negligible in comparison with the corrections under consideration.

The paper is organized as follows. In the Sec. 2 we outline the main features of our approximative method, developed in Refs. 14, 15, and derive the general perturbation framework. The corrections to the Casimir force (1) are obtained up to the fourth order in the parameters $A_{1,2}/d$. In Sec. 3 the developed formalism is applied to the case of non-parallel plates. The results obtained in our approach are compared with those obtained from the Green functions method. The significant role of mixed terms containing products of the type $A_1^n A_2^m$ has been paid special attention. Sec. 4 is devoted to the consideration of plates with a large-scale unevenness and Sec. 5 to short-scale distortions, which can be both periodical and non-periodical. Sec. 6 contains the discussion and conclusions. In the Appendix the details of some calculations are presented.

2 The general perturbation formulation for the calculation of the Casimir force

At first let us outline the main features of the approximative method which may be applied for the calculation of the Casimir force for configurations of arbitrary form^{14,15}. According to this method the potential of the Casimir force acting between two test bodies can be obtained by summation of the interatomic potentials over all atoms of the test bodies with a subsequent multiplicative renormalization:

$$U_R(d) = -\frac{CN_1N_2}{K} \int_{V_1} d\vec{r}_1 \int_{V_2} d\vec{r}_2 |\vec{r}_1 - \vec{r}_2|^{-7}. \quad (3)$$

In Eq. (3) the integrations run over the volumes V_1 resp. V_2 of the test bodies, N_1 resp. N_2 are the numbers of atoms per unit volume, C is the constant of the retarded van der Waals interatomic potential, K is a special renormalization constant and d is the distance between the test bodies. Note that even the simple summation of the interatomic potentials (i.e., Eq. (3) without the correction factor K^{-1}) gives the proper dependence of U_R on d for three dimensional configurations (see, e.g., Refs. 18-20). But the values of the coefficients in such dependencies come out to be larger than their true values due to the screening effects.

The renormalization procedure allows us to take into account approximately the effects of screening of the farther layers of the test bodies matter by the nearer ones. The value of the constant K is determined in Ref. 14 as the ratio of the Casimir force potentials between two infinite plane parallel plates obtained by the summation method (see Eq. (3) without the correction factor K) and by the exact solution (Eq. (1)). The result is

$$K = \frac{CN_1N_2}{\Psi(\epsilon_1, \epsilon_2)} > 1, \quad (4)$$

where the function $\Psi(\epsilon_1, \epsilon_2)$ was defined in (2).

The Eqs. (3) and (4) can be rewritten in the form:

$$U_R(d) = -\Psi(\epsilon_1, \epsilon_2) \int_{V_1} d\vec{r}_1 \int_{V_2} d\vec{r}_2 |\vec{r}_1 - \vec{r}_2|^{-7}. \quad (5)$$

As it was shown in Ref. 14, the relative error of the potential (5) is less than 3.8% for configurations of an arbitrary body over a plane plate. This estimation of the relative error was obtained in the following way. For the case of two infinite plane parallel plates our method gives the exact result by construction. The relative error of Eq. (5) has been shown¹⁴ to reach its largest value of 3.8%, among the configurations of arbitrary shaped bodies over a plane plate, for a small ball over a plane. Therefore, for other bodies over a plane plate the relative error of (5) is less than this value. For the configuration of two plates with small deviations from planes it is even much smaller. As it is shown in Sec. 3 for such configurations the relative error of the method used does not exceed $10^{-2}\%$.

It should be noted that it is possible – at last in principle – to apply the perturbation method for Green functions (e.g., that of Ref. 11) in order to obtain the corresponding

results for $U_R(d)$ in a more fundamental way. This will be a task of enormous efforts. But as long as the maximal relative error is not required to be less than 3.8% it would only repeat the results of the much simpler formula (5).

As the zero approximation to our perturbation theory we consider the Casimir force between the square plane plate P_1 with the sidelength $2L$ and the thickness D and the other plane plate P_2 which is parallel to P_1 and has the same length and thickness (see Fig. 1). Our aim is to calculate the Casimir force between plates which surfaces possess some small deviations from the plane geometry. Let us describe the surface of the first plate by the equation $z_1^{(s)} = A_1 f_1(x_1, y_1)$ and the surface of the second plate by the equation $z_2^{(s)} = d + A_2 f_2(x_2, y_2)$, where d is the mean value of the distance between the plates (the coordinate system is shown in Fig. 1). The values of the amplitudes are chosen in such way that $\max |f_i(x_i, y_i)| = 1$. It is suitable to choose the zero point in the z -axis so that

$$\begin{aligned} \langle z_1^{(s)} \rangle &\equiv A_1 \langle f_1(x_1, y_1) \rangle \equiv \frac{A_1}{(2L)^2} \int_{-L}^L dx_1 \int_{-L}^L dy_1 f_1(x_1, y_1) = 0, \\ \langle z_2^{(s)} \rangle &\equiv d + A_2 \langle f_2(x_2, y_2) \rangle = d. \end{aligned} \quad (6)$$

We suggest in our perturbation expansion that $A_i \ll d$, $d \ll D$ and $d \ll L$. At the same time, in all real situations, we have $d/D, d/L \ll A_i/d$, so that we are looking for the perturbation expansion in the powers of A_i/d and in the zeroth orders in d/D and d/L .

The non-renormalized potential of one atom at the distance z_2 over the plate P_1 is given by

$$\begin{aligned} U_A(x_2, y_2, z_2) &= -CN_1 \int_{-L}^L dx_1 \int_{-L}^L dy_1 \int_{-D}^{A_1 f_1(x_1, y_1)} dz_1 \\ &\quad \times \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{-7/2}. \end{aligned} \quad (7)$$

Let us calculate this expression as a series with respect to the parameter A_1/z_2 which is small due to $z_2 > d \gg A_1$. Carrying out these calculations one can neglect the corrections of the order of z_2/L and z_2/D , i.e., assume that the thickness and the length of the sides of the first plate are infinitely large.

The result of the expansion up to the fourth order with respect to the small parameter A_1/z_2 may be written in the form

$$\begin{aligned}
U_A(x_2, y_2, z_2) = & -CN_1 \left\{ \frac{\pi}{10z_2^4} \right. \\
& + \int_{-L}^L dx_1 \int_{-L}^L dy_1 \left[\frac{z_2 f_1(x_1, y_1)}{X^{7/2}} \left(\frac{A_1}{z_2} \right) + \frac{7z_2^3 f_1^2(x_1, y_1)}{2X^{9/2}} \left(\frac{A_1}{z_2} \right)^2 \right. \\
& \quad \left. + \frac{7z_2^3}{6X^{9/2}} \left(\frac{9z_2^2}{X} - 1 \right) f_1^3(x_1, y_1) \left(\frac{A_1}{z_2} \right)^3 \right. \\
& \quad \left. \left. + \frac{21z_2^5}{8X^{11/2}} \left(\frac{11z_2^2}{X} - 3 \right) f_1^4(x_1, y_1) \left(\frac{A_1}{z_2} \right)^4 \right] \right\}, \quad (8)
\end{aligned}$$

with $X = (x_1 - x_2)^2 + (y_1 - y_2)^2 + z_2^2$. Here, the limiting transition $L \rightarrow \infty$ is performed in the first item which describes the perfect plates without deviations from planes.

The renormalized potential of the Casimir force between the plates may be obtained by the integration of Eq. (8) over the volume V_2 of the second plate using the boundary function $A_2 f_2(x_2, y_2)$ and by division of the obtained result by the renormalization constant K from Eq. (4):

$$U_R = \frac{N_2}{K} \int_{-L}^L dx_2 \int_{-L}^L dy_2 \int_{d+A_2 f_2(x_2, y_2)}^{d+D} dz_2 U_A(x_2, y_2, z_2). \quad (9)$$

The Casimir force between the plates per unit area is given by

$$F = -\frac{1}{(2L)^2} \frac{\partial U_R}{\partial d}. \quad (10)$$

Substituting (9) into (10) we can write

$$F = \frac{1}{(2L)^2} \frac{N_2}{K} \int_{-L}^L dx_2 \int_{-L}^L dy_2 U_A(x_2, y_2, d + A_2 f_2(x_2, y_2)). \quad (11)$$

Let us now represent the quantity U_A defined in Eq. (8) as a series up to the fourth order in the small parameter A_2/d . Then we substitute this series into Eq. (11). The result may be written in the form:

$$F = F_0 \sum_{k=0}^4 \sum_{l=0}^{4-k} c_{kl} \left(\frac{A_1}{d} \right)^k \left(\frac{A_2}{d} \right)^l, \quad (12)$$

where F_0 is defined in (1). For the coefficients in (12) we note

$$c_{00} = 1, \quad c_{01} = c_{10} = 0. \quad (13)$$

The last two equalities follow from our choice (6).

The coefficients whose first index is zero are

$$c_{02} = 10 \langle f_2^2 \rangle, \quad c_{03} = -20 \langle f_2^3 \rangle, \quad c_{04} = 35 \langle f_2^4 \rangle, \quad (14)$$

where the notations for the averaged values are used like in Eq. (6).

The remaining coefficients c_{kl} in Eq. (12) are more complicated. They read:

$$\begin{aligned}
c_{20} &= \frac{35}{\pi} d^7 \langle\langle f_1^2 Y^{-9} \rangle\rangle, & c_{30} &= \frac{35}{\pi} d^7 \langle\langle f_1^3 \varphi_1(Y) \rangle\rangle, \\
c_{40} &= \frac{105}{4\pi} d^9 \langle\langle f_1^4 \varphi_2(Y) \rangle\rangle, & c_{11} &= -\frac{70}{\pi} d^7 \langle\langle f_1 f_2 Y^{-9} \rangle\rangle, \\
c_{12} &= \frac{35}{\pi} d^7 \langle\langle f_1 f_2^2 \varphi_1(Y) \rangle\rangle, & c_{21} &= -\frac{35}{\pi} d^7 \langle\langle f_1^2 f_2 \varphi_1(Y) \rangle\rangle, \\
c_{13} &= -\frac{105}{\pi} d^9 \langle\langle f_1 f_2^3 \varphi_2(Y) \rangle\rangle, & c_{31} &= -\frac{105}{\pi} d^9 \langle\langle f_1^3 f_2 \varphi_2(Y) \rangle\rangle, \\
c_{22} &= \frac{315}{\pi} d^9 \langle\langle f_1^2 f_2^2 \varphi_2(Y) \rangle\rangle.
\end{aligned} \tag{15}$$

In (15) we used the notations

$$\begin{aligned}
Y &= [(x_1 - x_2)^2 + (y_1 - y_2)^2 + d^2]^{1/2}, \\
\varphi_1(Y) &= 9d^2 Y^{-11} - Y^{-9}, & \varphi_2(Y) &= 11d^2 Y^{-13} - 3Y^{-11}
\end{aligned} \tag{16}$$

and the following average procedure for a function depending on 4 variables:

$$\langle\langle \Phi(x_1, y_1; x_2, y_2) \rangle\rangle = \frac{1}{(2L)^2} \int_{-L}^L dx_2 \int_{-L}^L dy_2 \int_{-L}^L dx_1 \int_{-L}^L dy_1 \Phi(x_1, y_1; x_2, y_2). \tag{17}$$

Besides, we used in Eq. (14) the notations $f_1 \equiv f_1(x_1, y_1)$, $f_2 \equiv f_2(x_2, y_2)$.

To obtain the general result for corrections to the Casimir force we need to calculate the values of the coefficients c_{lk} defined in (15). We start with the coefficients c_{l0} which depend on one deviation function only. Integrating over x_2 and y_2 according to (17) from $-\infty$ to ∞ we obtain in zeroth order in d/L :

$$c_{20} = 10 \langle f_1^2 \rangle, \quad c_{30} = 20 \langle f_1^3 \rangle, \quad c_{40} = 35 \langle f_1^4 \rangle. \tag{18}$$

These results are in agreement with Eq. (13) and may be obtained also from symmetry considerations.

The calculation of the mixed coefficients (depending on the deviation functions of both plates) is quite complicated. It is performed in the App. The results read:

$$\begin{aligned}
c_{11} &= -\frac{4}{3} \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} G_{mn}^{(1,1)} z_{mn}^{7/2} K_{7/2}(z_{mn}), \\
c_{12} &= \frac{2}{3} \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} G_{mn}^{(1,2)} z_{mn}^{7/2} [z_{mn} K_{9/2}(z_{mn}) - K_{7/2}(z_{mn})], \\
c_{13} &= -\frac{2}{9} \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} G_{mn}^{(1,3)} z_{mn}^{9/2} [z_{mn} K_{11/2}(z_{mn}) - K_{9/2}(z_{mn})], \\
c_{22} &= 210 g_{00}^{(2)} h_{00}^{(2)} + \frac{1}{3} \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} G_{mn}^{(2,2)} [z_{mn} K_{11/2}(z_{mn}) - K_{9/2}(z_{mn})].
\end{aligned} \tag{19}$$

Let us explain the notations in Eqs. (19). The functions $K_\nu(z)$ are the modified Bessel functions,

$$z_{mn} \equiv \pi \frac{d}{L} \sqrt{n^2 + m^2}. \quad (20)$$

The quantities $G_{mn}^{(i,k)}$ are given by

$$G_{mn}^{(i,k)} = \frac{1}{4} (1 + \delta_{m0}) \sum_{l=1}^4 g_{l,mn}^{(i)} h_{l,mn}^{(k)}, \quad (21)$$

where δ_{m0} is the Kronecker symbol and $g_{l,mn}^{(i)}$ and $h_{l,mn}^{(k)}$ are the Fourier coefficients of the functions f_1^i and f_2^k , considered as periodic functions with the period $2L$ (for details see the App.). The quantities $g_{00}^{(2)}$ and $h_{00}^{(2)}$ are the zeroth terms of the Fourier expansion of the functions f_1^2 and f_2^2 , respectively. The coefficient c_{21} differs from c_{12} by its sign and by the sequence of upper indices of G . For obtaining c_{31} it is enough to change the sequence of the upper indices of G in c_{13} only.

So, the developed perturbation formalism allows to obtain the Casimir force for the configurations with deviations from plane parallel geometry in the form of the series, given by Eq. (12), with coefficients defined in Eqs. (13), (14), (18) and (19). These equations for the coefficients may be simplified significantly for particular classes of deviations which were discussed in the introduction. This simplification is the subject of the next sections. Also, there will be given concrete examples.

3 The Casimir force between non-parallel plates

Let us consider the Casimir force for the configuration of two plane plates with angle α between them. This angle is assumed to be small, so that $\alpha L \ll d$ holds. This configuration is a particular example for the deviations from plane parallel geometry. That can be described by non-periodic functions $f_{1,2}$. In this case the expressions (19) for the mixed coefficients c_{ik} can be simplified considerably.

We use the explicit expressions of the modified Bessel functions²¹ in (19) and obtain:

$$\begin{aligned} c_{11} &= -20 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} G_{mn}^{(1,1)} e^{-z_{mn}} \left(1 + z_{mn} + \frac{2}{5} z_{mn}^2 + \frac{1}{15} z_{mn}^3 \right), \\ c_{12} &= 60 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} G_{mn}^{(1,2)} e^{-z_{mn}} \left(1 + z_{mn} + \frac{13}{30} z_{mn}^2 + \frac{1}{10} z_{mn}^3 + \frac{1}{90} z_{mn}^4 \right), \\ c_{13} &= -140 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} G_{mn}^{(1,3)} e^{-z_{mn}} [1 + z_{mn} \Pi(z_{mn})], \\ c_{22} &= 210 \left\{ g_{00}^{(2)} h_{00}^{(2)} + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} G_{mn}^{(2,2)} e^{-z_{mn}} [1 + z_{mn} \Pi(z_{mn})] \right\}, \end{aligned} \quad (22)$$

where $\Pi(z) = 1 + \frac{19}{42}z + \frac{5}{42}z^2 + \frac{2}{105}z^3 + \frac{1}{630}z^4$.

For the deviation functions under consideration the Fourier coefficients of the expansions (A.2), (A.3) decrease quickly with the number of harmonics. Due to Eq. (21) the quantities $G_{mn}^{(i,k)}$ also decrease with the increasing of m and n . The quantity z_{mn} , defined in Eq. (20), is of order of $d/L \ll 1$ for all harmonics which give a significant contribution to the coefficients c_{ik} . So, we can put in Eq. (22) $z_{mn} = 0$ without loss of accuracy:

$$\begin{aligned} c_{11} &= -20 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} G_{mn}^{(1,1)} & c_{12} &= 60 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} G_{mn}^{(1,2)} \\ c_{13} &= -140 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} G_{mn}^{(1,3)} & c_{22} &= 210 \left(g_{00}^{(2)} h_{00}^{(2)} + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} G_{mn}^{(2,2)} \right). \end{aligned} \quad (23)$$

Taking into account Eq. (21) and the elementary properties of Fourier expansions it is possible to rewrite Eqs. (23) in the form

$$\begin{aligned} c_{11} &= -20 \langle f_1 f_2 \rangle, & c_{12} &= 60 \langle f_1 f_2^2 \rangle, \\ c_{13} &= -140 \langle f_1 f_2^3 \rangle, & c_{22} &= 210 \langle f_1^2 f_2^2 \rangle, \\ c_{21} &= -60 \langle f_1^2 f_2 \rangle, & c_{31} &= -140 \langle f_1^3 f_2 \rangle \end{aligned} \quad (24)$$

(in the last line the correlations between c_{ik} and c_{ki} have been used – see Sec. 2).

Using Eqs. (13), (14), (18) and (24) the final result for the Casimir force from Eq. (12) can be represented in the form:

$$\begin{aligned} F &= F_0 \left\{ 1 + 10 \left[\langle f_1^2 \rangle \left(\frac{A_1}{d} \right)^2 - 2 \langle f_1 f_2 \rangle \left(\frac{A_1}{d} \right) \left(\frac{A_2}{d} \right) + \langle f_2^2 \rangle \left(\frac{A_2}{d} \right)^2 \right] \right. \\ &\quad + 20 \left[\langle f_1^3 \rangle \left(\frac{A_1}{d} \right)^3 - 3 \langle f_1^2 f_2 \rangle \left(\frac{A_1}{d} \right)^2 \left(\frac{A_2}{d} \right) \right. \\ &\quad \left. + 3 \langle f_1 f_2^2 \rangle \left(\frac{A_1}{d} \right) \left(\frac{A_2}{d} \right)^2 - \langle f_2^3 \rangle \left(\frac{A_2}{d} \right)^3 \right] \\ &\quad + 35 \left[\langle f_1^4 \rangle \left(\frac{A_1}{d} \right)^4 - 4 \langle f_1^3 f_2 \rangle \left(\frac{A_1}{d} \right)^3 \left(\frac{A_2}{d} \right) \right. \\ &\quad + 6 \langle f_1^2 f_2^2 \rangle \left(\frac{A_1}{d} \right)^2 \left(\frac{A_2}{d} \right)^2 \\ &\quad \left. \left. - 4 \langle f_1 f_2^3 \rangle \left(\frac{A_1}{d} \right) \left(\frac{A_2}{d} \right)^3 + \langle f_2^4 \rangle \left(\frac{A_2}{d} \right)^4 \right] \right\}. \end{aligned} \quad (25)$$

As it can be seen from Eq. (25), the mixed terms have an evident interference character. For example, in the particular case $f_2 = \mp f_1$ we have

$$\begin{aligned} F &= F_0 \left[1 + 10 \langle f_1^2 \rangle \left(\frac{A_1 \pm A_2}{d} \right)^2 + 20 \langle f_1^3 \rangle \left(\frac{A_1 \pm A_2}{d} \right)^3 \right. \\ &\quad \left. + 35 \langle f_1^4 \rangle \left(\frac{A_1 \pm A_2}{d} \right)^4 \right]. \end{aligned} \quad (26)$$

Now, let us apply the results (25), (26) to the case of two plane plates with a small angle α between them. This configuration can be realized in three ways. The corresponding disposition of the plates is shown in Fig. 2 (a) - (c). In Fig. 2 (a) the

upper plane is non-perturbed and the lower plates deviation from the parallel position is described by the function

$$f_1(x_1, y_1) = \frac{x_1}{L} \quad (27)$$

with the amplitude $A_1 = \alpha L$. Substituting (27) into expression (25) for the Casimir force and putting $A_2 = 0$ we obtain:

$$F = F_0 \left[1 + 10 \langle f_1^2 \rangle + \left(\frac{A_1}{d} \right)^2 + 20 \langle f_1^3 \rangle + \left(\frac{A_1}{d} \right)^3 + 35 \langle f_1^4 \rangle + \left(\frac{A_1}{d} \right)^4 \right]. \quad (28)$$

Using the average values calculated from the function (27) we obtain further

$$F = F_0 \left[1 + \frac{10}{3} \left(\frac{\alpha L}{d} \right)^2 + 5 \left(\frac{\alpha L}{d} \right)^4 \right]. \quad (29)$$

In the Figs. 2 (b) and 2 (c) there are shown the situations where both plates are perturbed with the amplitudes $A_1 = \alpha_1 L$ and $A_2 = \alpha_2 L$. The angle between the plates is $\alpha = \alpha_1 + \alpha_2$ (Fig. 2(b)) or $\alpha = |\alpha_1 - \alpha_2|$ (Fig. 2 (c)). The function f_1 is given by (27) in both cases and the function f_2 is $f_2 = \mp f_1$, respectively.

It is easy to see that the calculation of the Casimir force according to Eq. (26) repeats the result (29). This example is interesting because it gives the possibility to observe the role of the interference terms in (25). Such contribution have necessarily to be taken into account for obtaining the proper results.

The configuration of two non-parallel plates can be also considered by the method of Green functions. For this purpose let us use the exact expression for the Casimir energy density inside a dihedral angle obtained in Refs. 16, 17 (see also Ref. 3):

$$\epsilon_R = -\frac{(n^2 - 1)(n^2 + 11)}{720\pi^2 \rho^4}, \quad (30)$$

where ρ is the distance from the top of the angle and $n = \pi/\alpha$.

The total energy in the space between two square plates of length $2L = \rho_2 - \rho_1$ with angle α between them is

$$U_R = \int_0^\alpha d\varphi \int_0^{2L} dz \int_{\rho_1}^{\rho_2} d\rho \rho \epsilon_R. \quad (31)$$

Substituting Eq. (30) into Eq. (31) and integrating for $\alpha \ll 1$ we obtain the result:

$$U_R = -\frac{\pi^2 L}{720\alpha} \left[\frac{1}{(d - \alpha L)^2} - \frac{1}{(d + \alpha L)^2} \right]. \quad (32)$$

Then the Casimir force can be obtained from Eq. (10):

$$F = -\frac{\pi^2}{1440\alpha L} \left[\frac{1}{(d - \alpha L)^3} - \frac{1}{(d + \alpha L)^3} \right]. \quad (33)$$

Expanding this expression for the force as a series with respect to powers of the small parameter $\alpha L/d$ up to the fourth order we obtain exactly Eq. (29). Note, that the second order contribution corresponding to (29) was obtained in Ref. 13 namely in this way.

So it is shown that at last up to the fourth order inclusive in the parameter $\alpha L/d$ our approach based on the additive summation with a subsequent renormalization yields exactly the same result as the Green functions method. This allows to make an estimation of the real relative error of our approach for configurations with small deviations from the plane parallel geometry. Taking the realistic value of $\alpha L/d \approx 10^{-1}$ we come to the conclusion that the relative error of the result (25) for the Casimir force is much less than $10^{-2}\%$. Therefore, the application of our approach can be expected to give reliable results up to the fourth order inclusive in the parameter $A_{1,2}/d$.

4 Plates with large-scale inhomogeneities

Let us consider first the case of large-scale inhomogeneities which are described by non-periodic functions $f_{1,2}$. In this case the general considerations of the preceding section apply and the Casimir force is given by Eqs. (25) and (26).

As first example we consider a configuration of two plates with steps of height H (see Fig. 3 (a), (b)). Here, the amplitudes of the deviations are $A_1 = A_2 = H/2$ and the function f_1 is given by:

$$f_1(x_1, y_1) = \begin{cases} 1, & -L \leq x_1 < 0, \\ -1, & 0 \leq x_1 \leq L. \end{cases} \quad (34)$$

For the configuration shown in Fig. 3 (a) we have $f_2 = -f_1$. Using Eq. (26) with the upper signs and calculating all mean values using the function f_1 given by Eq. (34) we obtain the Casimir force

$$F = F_0 \left[1 + 10 \left(\frac{H}{d} \right)^2 + 35 \left(\frac{H}{d} \right)^4 \right]. \quad (35)$$

It is interesting to note that this result coincides with a half a sum of the Casimir forces (1) acting between two pairs of plates with the distances $d_{1,2} = d \pm H$ between them.

For the configuration of Fig. 3 (b) we have $f_1 = f_2$. Using Eq. (26) with the lower signs we obtain that all corrections are equal to zero and the Casimir force coincides with those between two plane parallel plates with distance d between them.

The next example is the configuration of a plane plate over a plate whose surface is a paraboloid (see Fig. 4). It is assumed that the height of the paraboloid is small: $H \ll d$. The deviation amplitude is in this case $A_1 = H/2$ and the function f_1 is given by:

$$f_1(x_1, y_1) = 1 - \frac{2(x_1^2 + y_1^2)}{L^2}. \quad (36)$$

In this configuration only one plate has a perturbation and formula (28) applies. Calculating the mean values of f_1^2 , f_1^3 , f_1^4 with the function (36) and substituting them

into Eq. (28) we obtain:

$$F = F_0 \left[1 + \frac{5}{6} \left(\frac{H}{d} \right)^2 + \frac{7}{16} \left(\frac{H}{d} \right)^4 \right]. \quad (37)$$

This result is in agreement with the Casimir force for a paraboloid over a plane plate calculated in Ref. 14 by the exact integration according to (5). To see this it is necessary to simplify the result of Ref. 14 by considering the paraboloid as a small deviation of the one plate surface from the plane geometry.

Above, we considered a convex paraboloid. If we replace it by a concave one described by the equation

$$f_1(x_1, y_1) = -1 + \frac{2(x_1^2 + y_1^2)}{L^2} \quad (38)$$

the result for the Casimir force remains unchanged.

If the split between the two plates is a convex-convex or a concave-concave lens (i.e., when both plates have deviations, $A_1 = A_2$), the Casimir force increases due to the interference terms. The result reads

$$F = F_0 \left[1 + \frac{10}{3} \left(\frac{H}{d} \right)^2 + 7 \left(\frac{H}{d} \right)^4 \right]. \quad (39)$$

In the case of a convex-concave lens with $A_1 = A_2$ the corrections are equal to zero and the Casimir force for such curved split coincides with that for plane parallel plates.

Now we shall deal with a special case of large-scale inhomogeneities when the deviation functions are periodical with periods T_x and T_y in the coordinates x and y , resp. Let m and n be the numbers of periods T_x resp. T_y , fitting into the plate (whose length is $2L$) respectively. For small numbers m and n , only a few first coefficients in the Fourier expansions of Eqs. (A.2), (A.3) work. This situation is completely equivalent to those discussed in Sec. 3 and the result (25) applies.

For $T_x, T_y \ll L$ the Fourier modes with rather large numbers of the order of n and m are significant. The corresponding value of z_{mn} , defined in Eq. (20), is of order of d/T_x or d/T_y . Nevertheless, the characteristic length of the large-scale inhomogeneities have to be much larger than the distance d between the plates. So we have $d/T_x, d/T_y \ll 1$ in this case and the result (25) applies once more.

As an example we consider the longitudinal deviations with amplitudes $A_{1,2}$ described by the functions:

$$f_1(x_1, y_1) = \sin \alpha x_1, \quad f_2(x_2, y_2) = \sin(\alpha x_2 + \delta). \quad (40)$$

In the case when $\alpha^{-1} \ll L$ the mean values of the functions $f_{1,2}^k$ in Eq. (25) may be calculated as the mean values for one period. Substituting these results into Eq. (25)

we obtain:

$$\begin{aligned}
F = F_0 & \left\{ 1 + 5 \left[\left(\frac{A_1}{d} \right)^2 - 2 \cos \delta \left(\frac{A_1}{d} \right) \left(\frac{A_2}{d} \right) + \left(\frac{A_2}{d} \right)^2 \right] \right. \\
& + \frac{105}{8} \left[\left(\frac{A_1}{d} \right)^4 - 4 \cos \delta \left(\frac{A_1}{d} \right)^3 \left(\frac{A_2}{d} \right) + 2(2 + \cos 2\delta) \left(\frac{A_1}{d} \right)^2 \left(\frac{A_2}{d} \right)^2 \right. \\
& \left. \left. - 4 \cos \delta \left(\frac{A_1}{d} \right) \left(\frac{A_2}{d} \right)^3 + \left(\frac{A_2}{d} \right)^4 \right] \right\}. \tag{41}
\end{aligned}$$

It is seen that the result depends significantly on the value of δ .

If the periods of the deviation functions in Eq. (40) are different it is necessary to calculate the mean values in the mixed terms over the whole plate according to Eq. (6). It can be seen that in this case only the contribution of $\langle f_1^2 f_2^2 \rangle = 1/4$ to the mixed terms is not equal to zero. So, the Casimir force is

$$F = F_0 \left\{ 1 + 10 \left[\left(\frac{A_1}{d} \right)^2 + \left(\frac{A_2}{d} \right)^2 \right] + \frac{105}{8} \left[\left(\frac{A_1}{d} \right)^4 + 4 \left(\frac{A_1}{d} \right)^2 \left(\frac{A_2}{d} \right)^2 + \left(\frac{A_2}{d} \right)^4 \right] \right\}. \tag{42}$$

The force from Eq. (42) is evidently larger than the force from Eq. (41) for $\delta = 0$ but smaller than that for $\delta = \pi$.

5 Short-scale distortions

The surfaces of real plates are always covered by some short-scale distortions of different types. It is necessary to take into account the contribution of such distortions in precision experiments on Casimir force measurements. The characteristic longitudinal scale of such distortions is of order of (or less than) the distance d between the plates.

The distortions of the plates surfaces may be periodical and non-periodical. In both cases the general result for the Casimir force (see. Eq. (12) with the coefficients (13), (14), (18) and (22)) acquire one and the same, more simple form than those presented in Eq. (25). As we shall see below most of the mixed terms in Eq. (12) turn into zero for the short-scale distortions. The reasons for such simplifications are different for the cases of periodical and non-periodical distortions. Let us start with the non-periodical case which is the more general one.

If the functions $f_{1,2}$ are non-periodical, the result (25) for the Casimir force applies. Because the characteristic scale of the distortions is of order of $d \ll L$ and by means of Eq. (6) we obtain for the odd numbers (i, k)

$$\langle f_1^i f_2^k \rangle = 0. \tag{43}$$

Using this, Eq. (25) takes the form

$$F = F_0 \left\{ 1 + 10 \left[\langle f_1^2 \rangle \left(\frac{A_1}{d} \right)^2 + \langle f_2^2 \rangle \left(\frac{A_2}{d} \right)^2 \right] + \right. \\ \left. + 35 \left[\langle f_1^4 \rangle \left(\frac{A_1}{d} \right)^4 + 6 \langle f_1^2 f_2^2 \rangle \left(\frac{A_1}{d} \right)^2 \left(\frac{A_2}{d} \right)^2 + \langle f_2^4 \rangle \left(\frac{A_2}{d} \right)^4 \right] \right\}. \quad (44)$$

The same result takes place for large-scale periodic perturbations with different periods.

Now, let the functions $f_{1,2}$ be periodical. If the periods of f_1 and f_2 are different (at last in one coordinate), Eq. (43) is valid once more and we return to expression (44) for the Casimir force.

A different formula appears only in the case when the periods T_x and T_y of f_1 and f_2 are equal in both coordinates respectively. It is evident in our case that $2L \approx mT_x$, $2L \approx nT_y$ holds where $m, n \gg 1$. In the Fourier expansions of the functions f_1^i and f_2^k , represented in (A.2) and (A.3), the coefficients of the modes with the corresponding large numbers are large. So, in Eq. (22) the main contributions are given by the terms containing the coefficients $G_{mn}^{(i,k)}$, which were defined in Eq. (21), with m, n large. The parameter z_{mn} defined in Eq. (20) is of order of $d/T_{x,y}$ in that case.

For extremely short-scale distortions $T_{x,y} \ll d$ we have $e^{-z_{mn}} \rightarrow 0$ and only one coefficient out of the mixed one survives:

$$c_{22} = 210 g_{00}^{(2)} h_{00}^{(2)} \equiv 210 \langle f_1^2 \rangle \langle f_2^2 \rangle. \quad (45)$$

As a consequence the Casimir force for the extremely short-scale distortions has the form:

$$F = F_0 \left\{ 1 + 10 \left[\langle f_1^2 \rangle \left(\frac{A_1}{d} \right)^2 + \langle f_2^2 \rangle \left(\frac{A_2}{d} \right)^2 \right] \right. \\ \left. + 35 \left[\langle f_1^4 \rangle \left(\frac{A_1}{d} \right)^4 + 6 \langle f_1^2 \rangle \langle f_2^2 \rangle \left(\frac{A_1}{d} \right)^2 \left(\frac{A_2}{d} \right)^2 + \langle f_2^4 \rangle \left(\frac{A_2}{d} \right)^4 \right] \right\}. \quad (46)$$

For the case $T_{x,y} \sim d$ we have to calculate the coefficients of Eq. (22) according to non-simplified formulas.

As an example of periodical distortions we shall examine the longitudinal ones described by Eq. (40) with $\alpha^{-1} < d$. Calculating the mixed coefficients according to Eq. (22) and taking into account that only $G_{m,0} = 1$ with $m = \alpha L/\pi$ is not equal to zero we obtain

$$\begin{aligned} c_{11} &= -20 \cos \delta e^{-z_{m0}} \left[1 + z_{m0} + \frac{2}{5}(z_{m0})^2 + \frac{1}{15}(z_{m0})^3 \right], \\ c_{12} &= c_{21} = 0, \\ c_{31} &= c_{31} = -\frac{105}{2} \cos \delta e^{-z_{m0}} [1 + z_{m0} \Pi(z_{m0})], \\ c_{22} &= \frac{105}{2} \left\{ 1 + \cos 2\delta e^{-z_{m0}} [1 + z_{m0} \Pi(z_{m0})] \right\}, \end{aligned} \quad (47)$$

where $\Pi(z)$ was defined in Eq. (22) and $z_{m0} = |\alpha|d$.

If $\alpha d \gg 1$, the terms in Eq. (47) containing the exponential factors can be neglected and we return to the result of Eq. (46) with

$$\langle f_1^2 \rangle = \langle f_2^2 \rangle = \frac{1}{2}, \quad \langle f_1^4 \rangle = \langle f_2^4 \rangle = \frac{3}{8}. \quad (48)$$

It is seen that this result coincides with (42) and does not depend on the phase-shift δ as it has to be expected in this case.

Note that the extremely short-scale distortions of sinusoidal form with the additional suggestion $\alpha A \ll 1$ were treated also in Ref. 22. The authors mentioned correctly that the Casimir force does not depend on δ . But the single correction presented in Ref. 22 is of the order $A^2\alpha/d$, that means it depends on the period. Such dependence seems to be non-physical for the case of extremely small periods and amplitudes. In fact, the principal frequency of fluctuating electromagnetic field which gives the main contribution to the Casimir force F_0 is $\alpha_0 \sim 1/d$.^{2,3,23} It is evident that the effective scattering takes place when the longitudinal scale of inhomogeneities is of the order of the wave length. So, for the deviations from the plane parallel geometry with the characteristic longitudinal length $l \gg d$ or $l \ll d$ the scattering frequencies $\alpha \sim l^{-1}$ differ significantly from the principal frequency α_0 . This means that the corresponding corrections to the Casimir force cannot depend essentially on l (or on T_x, T_y in the periodic case). Such a dependence occurs only in the intermediate region $l \sim d$ for mixed terms which take into account the correlation between the perturbations of the first and the second plates (compare Eq. (41) for $T_x \gg d$ and Eq. (46) for $T_x \ll d$ with the Eq. (47) for the coefficients of Eq. (12) where $T_x \sim d$). Note also that in the case of small period, smooth distortions, as considered in Ref. 22, the calculation method used here is practically exact.

Moreover, the correction of Ref. 22 changes its sign under the transformation $\alpha \rightarrow -\alpha, A \rightarrow -A$, whereas the distortions with $\delta = 0$ are left unchanged. So, the result of Ref. 22 cannot be correct.

For $\alpha^{-1} \sim d$ one ought to make computations. For example, for $z_{m0} = 1$ the results are:

$$c_{11} = -18.08 \cos \delta, \quad c_{13} = -49.98 \cos \delta, \quad c_{22} = \frac{105}{2}(1 + 0.953 \cos 2\delta). \quad (49)$$

The other coefficients coincide with those obtained in Sec. 4 for this example. They are given in Eq. (41) and do not depend on the period.

Substituting all these coefficients into Eq. (12), we obtain the Casimir force which is analogous to Eq. (41) for larger periods. It is seen that under a decreasing of the period the absolute values of the coefficients before the interference terms, i.e., the terms containing δ , decrease.

6 Discussion and conclusions

We have analyzed the corrections to the Casimir force between plates with small deviations from the plane parallel geometry of different types. In fact, all possible deviations

have been investigated for which the surface of the plates can be described by definite functions depending on two variables. The Casimir force between plates with large-scale non-periodical perturbations is given by Eq. (25). For the large- and short-scale perturbations with different periods, as well as for non-periodical short-scale perturbations the Casimir force is represented by Eq. (44). A special investigation has been performed for the case when both plates have periodical perturbations with equal periods $T_{x,y} \ll L$. Here, a continuous transition is observed from the Casimir force for large-scale deviations with $T_{x,y} \gg d$ (Eq. (41)) to short-scale ones with $T_{x,y} \ll d$ (Eq. (46)). It is shown that in both limiting cases the corrections do not depend on the periods. Such dependence takes place only for the fixed phase shift in the intermediate case $T_{x,y} \sim d$.

All these results have been obtained up to the fourth order inclusive in the small parameter $A_{1,2}/d$ and with a relative error much less than $(A_{1,2}/d)^4$.

Let us estimate the magnitude of relative corrections $\delta F = (F - F_0)/F_0$ to the Casimir force F_0 for different configurations. These estimations are made using the realistic value $(a_{1,2}/d) \sim 10^{-1}$. For two planes with an angle α between them we have from Eq. (29) $\delta F = 3.4\%$. When the surface of one of the plates is a paraboloid of the height $H = 0.1 d$, one obtains from Eq. (37) $\delta F = 0.8\%$. For the configurations of convex-convex and concave-concave lenses (Eq. (39)) it holds $\delta F = 3.4\%$. In the case of longitudinal sinusoidal perturbations with different periods on different plates (Eq. (42)) we have $\delta F = 10.8\%$. The corrections to the Casimir force take their maximal values for the appropriate disposition of large-scale periodical deviations of two plates with equal periods. So, for longitudinal sinusoidal large-scale deviations of Eq. (40) with $\delta = -\pi$ we have from Eq. (41) $\delta F = 22\%$. This fact is interesting from the experimental point of view. Note that the temperature corrections for the typical value of $d \sim 0.5 \mu\text{m}$ are of the order of 0.005% . So they can be neglected in comparison with the contribution of the fourth order terms.

The general approach to the calculation of corrections to the Casimir force presented here may be applied to other zero order configurations, e.g., to the configuration of a plane and a spherical lens. It is interesting also to use this approach in order to analyse the deviations from the basic geometry described by stochastic functions. This will be done in a subsequent paper.

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Appendix

In this appendix we derive the Eqs. (19) for the coefficients c_{ik} . Let us consider the functions $f_1(x_1, y_1)$ and $f_2(x_2, y_2)$ as periodic with period $2L$. For the averaging procedure in accordance with Eq. (17) it is convenient to use the notations

$$x = x_1 - x_2, \quad y = y_1 - y_2. \quad (\text{A.1})$$

The Fourier series for the powers of these functions can be written in the form

$$\begin{aligned} f_1^i(x_2 + x, y_2 + y) = & g_{00}^{(i)} + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} [g_{1,mn}^{(i)} \sin \frac{\pi}{L} m(x_2 + x) \sin \frac{\pi}{L} n(y_2 + y) \\ & + g_{2,mn}^{(i)} \sin \frac{\pi}{L} m(x_2 + x) \cos \frac{\pi}{L} n(y_2 + y) \\ & + g_{3,mn}^{(i)} \cos \frac{\pi}{L} m(x_2 + x) \sin \frac{\pi}{L} n(y_2 + y) \\ & + g_{4,mn}^{(i)} \cos \frac{\pi}{L} m(x_2 + x) \cos \frac{\pi}{L} n(y_2 + y)], \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} f_2^k(x_2, y_2) = & h_{00}^{(k)} + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} [h_{1,mn}^{(k)} \sin \frac{\pi}{L} m x_2 \sin \frac{\pi}{L} n y_2 \\ & + h_{2,mn}^{(k)} \sin \frac{\pi}{L} m x_2 \cos \frac{\pi}{L} n y_2 \\ & + h_{3,mn}^{(k)} \cos \frac{\pi}{L} m x_2 \sin \frac{\pi}{L} n y_2 \\ & + h_{4,mn}^{(k)} \cos \frac{\pi}{L} m x_2 \cos \frac{\pi}{L} n y_2]. \end{aligned} \quad (\text{A.3})$$

For the calculation of the mean values in the mixed coefficients (Eqs. (15)) we assume that the length of the first plates sides is infinite. Then we replace the variables x_1, y_1 by x, y (Eq. (A.1)) and change the order of integrations. The quantity Y in Eqs. (16) depends on x and y only. So we have

$$\begin{aligned} \langle\langle f_1^i f_2^k \varphi_l(Y) \rangle\rangle = & \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \varphi_l(Y) \\ & \times \left[\frac{1}{(2L)^2} \int_{-L}^L dx_2 \int_{-L}^L dy_2 f_1^i(x + x_2, y + y_2) f_2^k(x_2, y_2) \right]. \end{aligned} \quad (\text{A.4})$$

Substituting the Fourier series (A.2), (A.3) into Eq. (A.4) and integrating over x_2 and y_2 we obtain

$$\begin{aligned} \langle\langle f_1^i f_2^k \varphi_l(Y) \rangle\rangle = & g_{00}^{(i)} h_{00}^{(k)} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \varphi_l(Y) \\ & + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} G_{mn}^{(i,k)} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \cos \frac{\pi}{L} m x \cos \frac{\pi}{L} n y \varphi_l(Y) \end{aligned} \quad (\text{A.5})$$

with $G_{mn}^{(i,k)}$ defined in Eq. (21).

It is easy to see that for the functions $\varphi_l(Y)$, which were defined in Eq.(16), one has

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \varphi_1(Y) = \frac{12\pi}{7d^7}, \quad \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \varphi_2(Y) = \frac{4\pi}{3d^9}. \quad (\text{A.6})$$

Also, by use of standard formulas (see Ref. 21, for example), one obtains

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{\cos \alpha_1 x \cos \alpha_2 y}{(x^2 + y^2 + d^2)^{k/2}} = \frac{\pi d^{2-k}}{\Gamma(k/2)2^{k/2-2}} \gamma^{k/2-1} K_{k/2-1}(\gamma) \quad (\text{A.7})$$

where $K_\nu(\gamma)$ is the modified Bessel function, $\Gamma(z)$ is the gamma function and

$$\gamma = d\sqrt{\alpha_1^2 + \alpha_2^2}. \quad (\text{A.8})$$

Performing the integrations in Eq. (A.5) with the help of formulas (A.7), (A.8) and substituting the resulting mean values into the corresponding coefficients in Eq. (15) we arrive at Eqs. (19). The parameter z_{mn} (20) in Eq. (19) corresponds to the parameter γ from (A.8) for definite values of m, n .

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Fig. 1. The coordinate system for the configuration of two plates with deviations from the plane parallel geometry.

(a)

(b)

(c)

Fig. 2. The configuration of two plane plates with an angle α between them: (a) $\alpha_1 = \alpha$, $\alpha_2 = 0$, (b) $\alpha = \alpha_1 + \alpha_2$, (c) $\alpha = |\alpha_1 - \alpha_2|$.

Fig. 3. The configuration of two plates with steps: (a) $f_1 = -f_2$, (b) $f_1 = f_2$.

Fig. 4. The configuration with a paraboloidal deviation at one plate.