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CORRECTIONS TO THE CASIMIR FORCE BETWEEN PLATES WITH STOCHASTIC SURFACES

by

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Abstract

A perturbative approach is formulated for the calculation of corrections to the Casimir force between plates with small stochastic deviations from the plane parallel geometry. These corrections are calculated up to the fourth order inclusive with respect to the relative dispersions of the deviations. There are considered deviations described by stationary stochastic functions and by non-stationary ones. It is shown that the corrections do not depend on the correlation radius.

1 Introduction

The Casimir effect has been paid attention in the last years in different areas in physics ranging from problems in elementary particle physics and cosmology (see, e.g., the reviews [1,2] and the monograph [3]) over thin films [4] and wetting processes on alkali metals [5] up to the possibility of obtaining stronger restrictions on the constants of hypothetical long-range interactions [6,7]. The Casimir effect attracts much attention from the experimental point of view [8]. It is one of the rare vacuum quantum effects having macroscopic consequences.

The corrections to the Casimir force due to small deviations from the basic geometry are of importance in different applications. Some papers (see, e.g., [9-12]) are devoted to the calculation of corrections to the Casimir force for configurations with small deviations from two plane parallel plates. The deviations considered there were periodic and non-periodic but described by definite functions of two coordinates.

The surfaces of real plates are covered by small occasional inhomogeneities the concrete form of which is unknown. They can be described by stochastic functions. So we have the problem of a configuration of parallel plates with small stochastic deviations from the plane parallel geometry. Thereby the smallness of the deviations means the smallness of their dispersions in comparison with the distance between the plates. Note, that corrections to the van der Waals and Casimir forces between parallel plates due to stochastic deviations of their surfaces were investigated in [13-15]. However, the results of that papers are in some disagreement to each other.

In the present paper we construct a perturbative approach to the calculation of the Casimir force between plates whose surfaces may be described by stochastic functions. The dispersions of these functions are assumed to be much smaller than the distance between the plates. The paper is organized as follows. In Sec. 2 we outline the main features of an approximative method for the calculation of the Casimir force. In the Sec. 3 the perturbative approach is formulated and the corrections to the Casimir force are obtained up to the fourth order in the relative dispersion for stationary stochastic functions. Our results disagree with those obtained for these cases in the papers [15]. We give a criticism of the results obtained in [15]. The fourth section is devoted to some particular non-stationary stochastic functions for which the corresponding corrections to the Casimir force are calculated.

Throughout the paper we use units with $\hbar = c = 1$.

2 The approximative method for the calculation of the Casimir force

The exact calculation of the Casimir force is possible only for configurations with a high symmetry (see the Refs. [1-3]). In all other cases one is left with approximative methods. The most standard one is the perturbation theory for the Green function with respect to a small parameter. In the case of deviations of the surfaces from the plane geometry this method turns out to be quite complicated and not very effective.

We use in this paper a simpler method, developed in Refs. [11,12,16], which allows for simpler calculations retaining a high accuracy.

Let us recall the exact result for the Casimir force per unit area acting between two plane parallel plates [17]

$$F = -\Psi(\epsilon_1, \epsilon_2) \frac{\pi}{10d^4}, \quad (1)$$

where d is the distance between the plates, $\epsilon_{1,2}$ are the dielectric permeabilities of the plates and we note

$$\begin{aligned} \Psi(\epsilon_1, \epsilon_2) = \frac{5}{16\pi^3} \int_0^\infty \int_1^\infty \frac{x^3}{p^2} & \left\{ \left[\frac{(s_1 + p)(s_2 + p)}{(s_1 - p)(s_2 - p)} e^x - 1 \right]^{-1} \right. \\ & \left. + \left[\frac{(s_1 + p\epsilon_1)(s_2 + p\epsilon_2)}{(s_1 - p\epsilon_1)(s_2 - p\epsilon_2)} e^x - 1 \right]^{-1} \right\} dp dx. \end{aligned} \quad (2)$$

Here, $s_{1,2} \equiv (\epsilon_{1,2} - 1 + p^2)^{1/2}$. For example, for perfect conductors we have $\epsilon_1 = \epsilon_2 \rightarrow \infty$ and $\Psi(\epsilon_1, \epsilon_2) \rightarrow \pi/24$. More information, a graphical representation for instance, concerning this function can be found in [11, 17].

According to the method of Refs. [11, 12, 16], the potential of the Casimir force acting between two test bodies can be obtained by summation of the interatomic potentials over all atoms of the test bodies with a subsequent multiplicative renormalization:

$$U_R(d) = -\frac{CN_1N_2}{K} \int_{V_1} d\vec{r}_1 \int_{V_2} d\vec{r}_2 |\vec{r}_1 - \vec{r}_2|^{-7}. \quad (3)$$

In Eq. (3) the integrations run over the volumes V_1 resp. V_2 of the test bodies, N_1 resp. N_2 are the numbers of atoms per unit volume, C is the constant of the retarded van der Waals interatomic potential, K is a special renormalization constant and d is the distance between the test bodies. Note that even the simple summation of the interatomic potentials (i.e., Eq. (3) without the correction factor K^{-1}) gives the proper dependence of U_R on d for three dimensional configurations (see, e.g., Ref. [18]). But the values of the coefficients in such dependencies come out to be larger than their true values due to the screening effects.

The renormalization procedure allows us to take into account approximately the effects of screening of the farther layers of the test bodies matter by the nearer ones. The value of the constant K is determined in Ref. [14] as the ratio of the Casimir force potentials between two infinite plane parallel plates obtained by the summation method (see Eq. (3) without the correction factor K) and by the exact solution (Eq. (1)). The result is

$$K = \frac{CN_1N_2}{\Psi(\epsilon_1, \epsilon_2)} > 1, \quad (4)$$

where the function $\Psi(\epsilon_1, \epsilon_2)$ was defined in (2).

The Eqs. (3) and (4) can be rewritten in the form:

$$U_R(d) = -\Psi(\epsilon_1, \epsilon_2) \int_{V_1} d\vec{r}_1 \int_{V_2} d\vec{r}_2 |\vec{r}_1 - \vec{r}_2|^{-7}. \quad (5)$$

As it was shown in Ref. [16], the relative error of the potential (5) is less than 3.8% for the configuration of an arbitrary shaped body over a plane. In Ref. [12] it was shown that the relative error of the potential (5) does not exceed 0.01% for the configuration of two plates with small deviations from the plane parallel geometry. This estimation of the accuracy was achieved by a comparison of the results obtained from Eq. (5) with that resulting from the exact Green function calculation (as far as such calculation can be carried out). Therefore we can apply the formula (5) in our case of plates with small stochastic deviations.

3 The perturbative approach for stationary stochastic functions

As zero order contribution to our perturbation method we consider the Casimir force between two square parallel plates P_1 resp. P_2 with sidelength $2L$ and thickness D and distance d between them (see Fig. 1, where also the coordinate system is shown). We assume that the surfaces of both plates possess deviations from the plane parallel geometry which are described by two stochastic functions $\{\delta_i f_i(x_i, y_i)\}$, $i = 1, 2$ with dispersion δ_i and mean values

$$\langle \delta_i f_i(x_i, y_i) \rangle_i = 0. \quad (6)$$

Here, $\langle \dots \rangle_i$ denotes the averaging over the ensembles of all particular realizations $\delta_i f_i(x_i, y_i)$ of the corresponding stochastic functions. The factor δ_i is written in front of f_i to have the dispersion of the functions $\{f_i(x_i, y_i)\}$ be equal to unity. So, the surfaces of the plates under consideration are given by the functions

$$z_1^{(s)} = \delta_1 f_1(x_1, y_1), \quad z_2^{(s)} = d + \delta_2 f_2(x_2, y_2). \quad (7)$$

For test bodies with the surfaces (7), the potential U_R in Eq. (5) has to be substituted by $\langle\langle U_R \rangle_1 \rangle_2$. Then the Casimir force is given by

$$F = -\frac{1}{(2L)^2} \frac{\partial}{\partial d} \langle\langle U_R \rangle_1 \rangle_2. \quad (8)$$

In our perturbative treatment we assume $\delta_i \ll d$. At the same time, in all real situations we have d/D , $d/L \ll \delta_i/d$, so that we have to look for a perturbative expansion with respect to powers of δ_i/d , taken in zeroth order with respect to d/D and d/L .

The non-renormalized potential of one atom at the distance z_2 over the plate P_1 is given by

$$U_A(x_2, y_2, z_2) = -CN_1 \int_{-L}^L dx_1 \int_{-L}^L dy_1 \int_{-D}^{\delta_1 f_1(x_1, y_1)} dz_1 \times \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{-7/2}. \quad (9)$$

Let us calculate this expression as a series with respect to the parameter δ_1/z_2 which is small due to $z_2 > d \gg \delta_1$. Carrying out these calculations one can neglect the corrections of the order of z_2/L and z_2/D , i.e., assume that the thickness and the length of the sides of the first plate are infinitely large.

The result of the expansion up to the fourth order with respect to the small parameter δ_1/z_2 may be written in the form

$$U_A(x_2, y_2, z_2) = -CN_1 \left\{ \frac{\pi}{10z_2^4} + \int_{-L}^L dx_1 \int_{-L}^L dy_1 \left[\frac{z_2 f_1(x_1, y_1)}{X^{7/2}} \left(\frac{\delta_1}{z_2} \right) + \frac{7z_2^3 f_1^2(x_1, y_1)}{2X^{9/2}} \left(\frac{\delta_1}{z_2} \right)^2 + \frac{7z_2^3}{6X^{9/2}} \left(\frac{9z_2^2}{X} - 1 \right) f_1^3(x_1, y_1) \left(\frac{\delta_1}{z_2} \right)^3 + \frac{21z_2^5}{8X^{11/2}} \left(\frac{11z_2^2}{X} - 3 \right) f_1^4(x_1, y_1) \left(\frac{\delta_1}{z_2} \right)^4 \right] \right\} \quad (10)$$

with $X = (x_1 - x_2)^2 + (y_1 - y_2)^2 + z_2^2$.

In accordance with Eq. (8) it is necessary to calculate the mean value over all realizations of the stochastic functions $\{f_{1,2}\}$. For the first function it is suitable to do this directly in Eq. (10). Using the normal distribution in every point of the surfaces and the corresponding mean values

$$\langle f_i \rangle_i = \langle f_i^3 \rangle_i = 0, \quad \langle f_i^2 \rangle_i = 1, \quad \langle f_i^4 \rangle_i = 3 \quad (11)$$

we obtain from (10)

$$\langle U_A(x_2, y_2, z_2) \rangle_1 = -CN_1 \left[\frac{\pi}{10z_2^4} + \int_{-L}^L dx_1 \int_{-L}^L dy_1 \left[\frac{7z_2^3}{2X^{9/2}} \left(\frac{\delta_1}{z_2} \right)^2 + \frac{63z_2^5}{8X^{11/2}} \left(\frac{11z_2^2}{X} - 3 \right) \left(\frac{\delta_1}{z_2} \right)^4 \right] \right] \quad (12)$$

If the deviations of the surfaces are described by stationary stochastic functions with $\delta_i = \text{const}$, we obtain in the limit $L \rightarrow \infty$ the result

$$\langle U_A(x_2, y_2, z_2) \rangle_1 = -CN_1 \frac{\pi}{10z_2^4} \left[1 + 10 \left(\frac{\delta_1}{z_2} \right)^2 + 105 \left(\frac{\delta_1}{z_2} \right)^4 \right] \quad (13)$$

In order to obtain the renormalized potential of the Casimir force between the plates we integrate the expression of Eq. (13) over the volume V_2 of the second plate with the boundary function $z_2^{(s)}$ given by Eq. (7). Then we calculate the mean value over all realizations of the stochastic function $\{f_2\}$ and divide the result by the renormalization constant K given by Eq. (4):

$$\langle U_R \rangle_1 = \frac{N_2}{K} \left\langle \int_{-L}^L dx_2 \int_{-L}^L dy_2 \int_{d+\delta_2 f_2(x_2, y_2)}^{d+D} dz_2 \langle U_A(x_2, y_2, z_2) \rangle_1 \right\rangle_2 \quad (14)$$

Substituting (14) into (8) we obtain for the Casimir force

$$F = \frac{1}{(2L)^2} \frac{N_2}{K} \int_{-L}^L dx_2 \int_{-L}^L dy_2 \langle U_A(x_2, y_2, d + \delta_2 f_2(x_2, y_2)) \rangle_1 \rangle_2 . \quad (15)$$

Let us now expand the quantity $\langle U_A \rangle_1$ defined by (13) as a series with respect to the small parameter δ_2/d . Then we substitute this series into Eq. (15). Using the Eqs. (11) and the condition $\delta_2 = \text{const}$ for the stationary stochastic functions we obtain the final result

$$F = F_0 \left\{ 1 + 10 \left[\left(\frac{\delta_1}{d} \right)^2 + \left(\frac{\delta_2}{d} \right)^2 \right] + 105 \left[\left(\frac{\delta_1}{d} \right)^2 + \left(\frac{\delta_2}{d} \right)^2 \right]^2 \right\} . \quad (16)$$

It is seen that the correction to the Casimir force depends on the sum $\delta_1^2 + \delta_2^2$ only and does not depend on the correlation radiuses $\rho_{1,2}$ of the stochastic functions describing the distortions. This result coincides with the result of Ref. [13] where the corresponding correction to the van der Waals and Casimir forces were calculated (up to the second order in the dispersions only). For a typical value of $\delta_{1,2} \approx 0.1$ the correction given by Eq. (16) is 24% to F_0 where 4% results from the fourth order.

The same problem as in this section has been considered in the papers [15] for the particular case $\delta \ll \rho \ll d$ (i.e. for the case of smooth, extremely small distortions on one plate). The correction to the Casimir force obtained in [15] has the form $\sim \delta^2/(\rho d)$ and is in disagreement with our result (16). This corrections cannot be true by the following reasons. As it is well known the main contribution to the Casimir energy density results from the zero fluctuations of the electromagnetic field with frequencies $\omega_0 \sim 1/d$ [3,17]. Then, only scattering of minor frequencies $\omega \sim \rho^{-1}$ occurs on the distortions with $\rho \ll d$ which are under consideration in [15] (the same situation takes place for $\rho \gg d$). In these situations there are no physical reasons to expect an essential dependence of the corrections on ρ . For $\rho \sim d$ the dependence of the corrections on ρ may take place but only in interference terms which take into account the mutual correlations of distortions on different plates [12]. For stochastic distortions such correlations are absent naturally. Note also that for smooth, extremely small distortions the method of calculation used here is nearly exact because the configurations are very close to that of plane parallel plates. Further, there is a formal reason which have to be raised against the result of Ref. [15]. The correlation function used there reads:

$$\langle f_1(\vec{r}_1) f_2(\vec{r}_2) \rangle_1 = \delta^2 \exp \left(-\frac{|\vec{r}_1 - \vec{r}_2|^2}{\rho^2} \right) ,$$

i.e., does not depend on the sign of ρ . At the same time the correction to the Casimir force obtained in [15], presented above, depends on the sign of ρ . By the same arguments some of the results obtained in [14] are doubtful. There, besides corrections of the form $\sim \delta^2/d^2$, also contributions proportional to $\delta^2/(\rho d)$ have been obtained (in the case of van der Waals forces).

4 Some results for non-stationary stochastic functions

In this section we assume that the deviations from plane plates are described by non-stationary stochastic functions whose mean values do not depend on x_i, y_i so that Eq. (6) may be applied. The dispersions of such functions are some definite functions of the coordinates: $\delta_i = \delta_i(x_i, y_i)$. The calculation of the Casimir force is in this case a more complicated problem than that in the previous section. All calculations which led to Eq. (12) are still valid, however the quantity δ_1 in (12) depends on x_1, y_1 now. Therefore, in order to obtain the Casimir force, one has to expand Eq. (12) with $z_2 = d + \delta_2 f_2(x_2, y_2)$ in a series with respect to the small parameter δ_2/d (where δ_2 now also may depend on x_2, y_2) and to substitute this series into Eq. (15). After that, using Eq. (11), it is possible to calculate the mean value $\langle \dots \rangle_2$ in Eq. (15). To obtain the Casimir force one has finally to integrate over x_1, y_1 and x_2, y_2 and arrives to

$$F = F_0 \left\{ 1 + 10 \left(\frac{\bar{\delta}_1^2}{d^2} + \frac{\bar{\delta}_2^2}{d^2} \right) + 105 \left(\frac{\bar{\delta}_1^4}{d^4} + \frac{\bar{\delta}_2^4}{d^4} \right) + \right. \\ \left. + \frac{315d^9}{2\pi(2L)^2} \int_{-L}^L dx_2 \int_{-L}^L dy_2 \int_{-L}^L dx_1 \int_{-L}^L dy_1 \frac{\delta_1^2(x_1, y_1) \delta_2^2(x_2, y_2)}{[(x_1 - x_2)^2 + (y_1 - y_2)^2 + d^2]^{11/2}} \right. \\ \left. \times \left[\frac{11d^2}{(x_1 - x_2)^2 + (y_1 - y_2)^2 + d^2} - 3 \right] \right\}. \quad (17)$$

Here, the notation

$$\bar{\delta}_i^n = \frac{1}{(2L)^2} \int_{-L}^L dx_i \int_{-L}^L dy_i \delta_i^n(x_i, y_i)$$

has been used.

The result of the integration in the last term in Eq. (17) depends on the particular form of the functions $\delta_1(x_1, y_1)$ and $\delta_2(x_2, y_2)$. In the lowest order in the parameter d/L it is possible to change the variables of integration by means of $x_2 = x + x_1, y_2 = y + y_1$ and integrate over x, y in infinite limits.

Let us illustrate this procedure by a particular example. We consider the configuration of two plane parallel plates. Then, let every plate turn around the y axis by small angles $\alpha_{1,2}$ which are considered as random variables with zero mean values and with dispersions Δ_i . Clearly, the angle $\alpha \equiv \alpha_1 - \alpha_2$ is a random variable too. Its dispersion reads

$$\Delta^2 = \Delta_1^2 + \Delta_2^2. \quad (18)$$

The dispersions of the stochastic functions describing the surfaces of the perturbed plates are

$$\delta_i(x_i) = \Delta_i x_i. \quad (19)$$

Substituting Eq. (19) into Eq. (17) and integrating in accordance with the above remark we obtain

$$F = F_0 \left[1 + \frac{10}{3} \frac{(\Delta_1^2 + \Delta_2^2)L^2}{d^2} + 21 \frac{(\Delta_1^2 + \Delta_2^2)^2 L^4}{d^4} \right]. \quad (20)$$

This result corresponds to that of Eq. (16) in the case of stationary stochastic functions.

The Casimir force (20) may be compared with the exact result obtained by the Green functions method. Using the Casimir energy density inside the dihedral angle [19, 20] and integrating it over the space between the two plates with a small angle α between them ($\alpha L \ll d$) one obtains the result

$$F_\alpha = F_0 \left[1 + \frac{10}{3} \left(\frac{\alpha L}{d} \right)^2 + 7 \left(\frac{\alpha L}{d} \right)^4 \right]. \quad (21)$$

By considering α in (21) as a random variable with normal distribution and with the dispersion (18) one can calculate the mean value of (21)

$$F = F_0 \left[1 + \frac{10}{3} \left(\frac{\Delta L}{d} \right)^2 + 21 \left(\frac{\Delta L}{d} \right)^4 \right], \quad (22)$$

which coincides with (20). Substituting into (22) the typical value $\Delta L/d \approx 0.1$ we conclude that the contribution of the second order correction is about 3.3% and that of the fourth order about 0.2%. So the approximative method used here gives the result which coincides with the exact one within a relative error of about 0.01%.

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Fig. 1. The coordinate system for the configuration of two plates with stochastic deviations from the plane parallel geometry.