The Invariant Charges of the Nambu-Goto Theory:
Non-additive Composition Laws

by

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Abstract: With continuity as the essential ingredient, insight is gained into the existence, number and structure of algebraic relations among the invariant charges corresponding to the three branches of a string vertex in the classical Nambu Goto theory. C - number and algebraic aspects of these relations are discussed.

In previous publications [1,2,3] it has been shown that the classical Nambu-Goto theory of closed bosonic strings moving in $(d-1)+1$ dimensional Minkowski space-time possesses infinitely many algebraically independent, reparametrization invariant charges $\mathcal{L}_{\mu_1 \dots \mu_n}^{+(K)}$ which are piece-wise conserved, i.e. conserved for each "tube-shaped" branch of the string trajectory homeomorphic to a two - dimensional cylinder. These "invariant charges" serve as infinitesimal generators of active (observable) symmetry transformations. Their (non-linear) Poisson algebra exhibits a truly remarkable structure [4].

The question of completeness of the invariant charges of the classical Nambu-Goto theory in its Euclidean version was addressed in ref.[5]. There the following fact was established: With respect to a given (Cartesian) frame of reference, the corresponding Euclidean string "trajectories" (i.e. complete minimal surfaces of class $\mathcal{S}$ [6] and their isometric immersions) can be reconstructed from the knowledge of the numerical values of a complete set of the invariant charges evaluated on one of its tube-shaped branches - apart from translations in some well-defined, charge dependent direction. The reconstruction exploits the conformal structure of minimal surfaces.

An immediate consequence of the above result is the follow-

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ing statement: Let any two tube-shaped branches of the same complete minimal surface be given. Then each invariant charge evaluated on one of these two branches is a function of the invariant charges evaluated on the other branch. However, in general these functions will be transcendental.

Next, consider two tube-shaped branches of the same complete minimal surface merging into a single third such branch. The "energy-momentum vectors" of the two merging branches, respectively, when added yield the "energy-momentum vector" of the merged third branch. The same goes for the corresponding "angular momentum tensors". Naturally the following questions arise: Besides the above familiar additive composition laws, do there exist algebraic relations involving the additional internal invariant charges of the three branches under discussion? If yes, how many independent such relations do there exist and what is the structure of these relations? The answer to these questions does not seem to be known.

Returning to the classical Nambu-Goto theory in Minkowski space-time, a straight-forward adaptation of the arguments of ref.[5] shows that with respect to a given (Lorentzian) frame of reference each tube-shaped branch of the string trajectory can be reconstructed from the knowledge of the corresponding numerical values of a complete set of invariant charges - apart from translations in the direction of the corresponding energy-momentum vector. The reconstruction exploits the freedom of choosing - as long as there is no change in topology - locally isothermal coordinates with the help of which within the respective charts the equations of motion of the Nambu-Goto string can be solved explicitly in terms of the initial data. Of course, in Minkowski space the complete string trajectory is not determined by the information encoded in the numerical values of a complete set of invariant charges evaluated on just one of its tube-shaped branches. However, a string vertex consisting of two tube-shaped branches of the same string trajectory merging in one distinguished space-time point $x^{(0)}$ into a single third such branch is essentially determined by the information encoded in the numerical values of a complete set of invariant charges evaluated on two of its three branches separately. It is a well-known fact that the energy-momentum vectors of the two merging branches, respectively, upon addition yield the energy-momentum vector of the merged third branch and that the same goes for the corresponding infinitesimal generators of the restricted homogeneous Lorentz transformations. As

above, questions arise concerning the existence, number and structure of algebraic relations involving the additional internal invariant charges evaluated separately on the three branches under discussion. In the sequel, a partial answer to these questions will be given providing at the same time a partial answer to the analogous questions for the Euclidean situation.

The algebra $\mathfrak{h}_\varphi$ of (internal) invariant charges corresponding to any given tube-shaped branch of the complete string trajectory decomposes into a direct sum of two algebras $\mathfrak{h}_\varphi^+$ and $\mathfrak{h}_\varphi^-$ for the left- and right movers with the components of the energy-momentum vector being central elements and as such being treated as c-numbers, respectively. In all cases, the field of coefficients is the field of complex numbers. The algebras are graded, respectively, both under

- i) the Poisson bracket operation, graded w.r.t. a certain "degree" ℓ ranging over the integers from 0 to ∞ , and
- ii) the commutative ordinary multiplication, graded w.r.t. the "grade" $k = (\ell + 1)$.

The algebras of internal invariant charges for the left- and right-movers, respectively, with the ordinary multiplication as the composition law, are isomorphic. As Poisson algebras, i.e. as Lie algebras with the Poisson bracket operation as the composition law, they differ by a global factor -1 for the structure constants. Thus, it suffices to analyze the algebra for the left-movers.

For the case under discussion, two tube-shaped branches of the complete string trajectory merging into a third such branch, on each branch the corresponding energy-momentum vector takes values on a fixed "positive" mass shell. Due to Lorentz invariance, without loss of generality, on each such branch the algebra of invariant charges will be given w.r.t. the corresponding momentum rest frame: $\mathfrak{h}_{(m,0,\dots,0)}^+ \doteq \mathfrak{h}_\varphi$. The respective algebras are not finitely generated. However, in terms of eigenvalues of the monodromy matrix corresponding to a certain multi-parameter Lax pair, the generators are explicitly given in references [2,4].

For purely technical reasons (simplicity of the irreducible representations of the "little" group of the Lorentz group) I shall restrict the subsequent analysis to the Nambu-Goto theory (of left-movers) in 2+1 dimensional Minkowski space. For this case and for each tube-shaped branch considered separately, the generators of the algebra of internal invariant charges in the corresponding momentum rest frame are:

- i) the generator of rotations in the (1,2)-plane (for the left-movers on the tube-shaped branch considered):

$$Q = -\mathcal{L}_{012}^{+(2)} / (2m),$$

the degree of Q being set equal to zero,

- ii) four internal invariant charges of degree one and "spin" s , $s = \pm 1, \pm 2 : L_s$

$$L_{\pm 1} \doteq -\frac{1}{2} \left[\mathcal{L}_{0212}^{+(2)} \pm i \mathcal{L}_{0121}^{+(2)} \right],$$

$$L_{\pm 2} \doteq -\frac{1}{4} \left[\mathcal{L}_{0101}^{+(2)} - \mathcal{L}_{0202}^{+(2)} \pm 2i \mathcal{L}_{0102}^{+(2)} \right],$$

and

- iii) infinitely many real scalar spinless elements $L_{\textcircled{2}}$ carrying degree ℓ , $\ell = 1, 3, 5, \dots$ which are associated with elements in involution [4]: ($L_{\textcircled{1}} \doteq L_0$)

$$L_{\textcircled{1}} \doteq \frac{1!}{4} \sum_{i=1}^2 \mathcal{L}_{0i0i}^{+(2)}, L_{\textcircled{3}} \doteq \frac{3!}{4} \sum_{i=1}^2 \mathcal{L}_{000i000i}^{+(4)}, L_{\textcircled{5}} \doteq \frac{5!}{4} \sum_{i=1}^2 \mathcal{L}_{00000i00000i}^{+(6)}, \dots$$

It is the aim of a separate line of investigations [4] to present this algebra in terms of defining relations generating it from the symmetric algebra over the free Poisson algebra with the above generators. Here we try to collect additional information in order to restrict the possible quantum deformations of the algebra.

As there are always two essentially different aspects to any dynamical quantity, its numerical value and its role as a generator of symplectic transformations, I shall address first the c-number aspect of the relations in question and then proceed to discuss the algebraic aspect.

According to references [1,2] and [3], the invariant charges on each one of the three branches - and so, in particular, on the merged branch - can be extracted from the trace (of integer powers of the logarithm) of the corresponding monodromy matrices $\Phi_{(A)}^{(\gamma(\tau))}(\sigma)$, $\Phi_{(A)}^{(\gamma_1(\tau))}(\sigma)$, $\Phi_{(A)}^{(\gamma_2(\tau))}(\sigma)$ associated with the parameter dependent linear differential equation with periodic coefficients

$$\frac{\partial}{\partial \sigma} \varphi_{(A)}(\tau, \sigma) = u_{\mu}(\tau, \sigma) A^{\mu\alpha} T_{\alpha} \varphi_{(A)}(\tau, \sigma)$$

where

σ parametrizes the closed curves $\gamma(\tau)$, $\gamma_1(\tau)$, and $\gamma_2(\tau)$ of intersection of the merged branch and the merging branches, respectively, with leafs labelled by the real parameter τ of a foliation of the Minkowski

space-time by space-like hypersurfaces, the orientation of the curves being such that each one of them together with the corresponding energy-momentum vector forms a right-handed screw;

$$u_{\mu}(\tau, \sigma) = p_{\mu}(\tau, \sigma) + \frac{\partial}{\partial \sigma} x_{\mu}(\tau, \sigma),$$

$x_{\mu}(\tau, \sigma)$ and $p_{\mu}(\tau, \sigma)$ being the canonical string variables, the inverse string tension α' having been set equal to $(2\pi)^{-1}$;

T_{α} $\alpha = 1, 2, \dots$ are the infinitesimal generators of $\mathcal{M}(\infty)$

$A^{\mu\alpha}$ $\mu=0, 1, 2$; $\alpha = 1, 2, 3, \dots$ are arbitrary real parameters.

Since the invariant charges are piece-wise conserved, they can be evaluated in the limit that the space-like hypersurface labelled by τ within the given foliation approaches the space-like hypersurface containing the point of mergence $x^{(0)}$ and being labelled by the parameter τ_0 . In this limit the closed double-point free curve $\gamma(\tau)$ approaches the closed curve $\gamma_0 = \gamma(\tau_0)$ with a single double point $x^{(0)}$ separating the two closed double-point free parts $\gamma_{10} = \gamma_1(\tau_0)$ and $\gamma_{20} = \gamma_2(\tau_0)$ of the curve γ_0 from each other. In terms of the curve parameter σ , let the periods of circulation of the curves γ_0 , γ_{10} , and γ_{20} be $\omega_0 = \omega(\tau_0)$, $\omega_{10} = \omega_1(\tau_0)$, and $\omega_{20} = \omega_2(\tau_0)$, respectively: $\omega_0 = \omega_{10} + \omega_{20}$, and let the start parameters for the circulation of the curves γ_0 , γ_{10} , and γ_{20} be σ_0 / $\sigma_0 + \omega_{10}$, σ_0 , and $\sigma_0 + \omega_{10}$, respectively. In this limit, the monodromy matrices $\Phi_{(A)}^{(\gamma_0)}(\sigma_0)$ and $\Phi_{(A)}^{(\gamma_0)}(\sigma_0 + \omega_{10})$ factorize as follows:

$$\begin{aligned}\Phi_{(A)}^{(\gamma_0)}(\sigma_0) &= \Phi_{(A)}^{(\gamma_{20})}(\sigma_0 + \omega_{10}) \cdot \Phi_{(A)}^{(\gamma_{10})}(\sigma_0), \\ \Phi_{(A)}^{(\gamma_0)}(\sigma_0 + \omega_{10}) &= \Phi_{(A)}^{(\gamma_{20})}(\sigma_0) \cdot \Phi_{(A)}^{(\gamma_{10})}(\sigma_0 + \omega_{10}).\end{aligned}$$

By exploiting the arbitrariness of the parameters $A^{\mu\alpha}$ (cf. ref.[3]), it is possible to derive relations among the corresponding path-ordered integrals:

$$R_{\mu_1 \dots \mu_N}^{(\gamma_0)}(\sigma_0) = \sum_{0 \leq a \leq N} R_{\mu_1 \dots \mu_a}^{(\gamma_{20})}(\sigma_0 + \omega_{10}) \cdot R_{\mu_{(a+1)} \dots \mu_N}^{(\gamma_{10})}(\sigma_0),$$

$$R_{\mu_1 \dots \mu_N}^{(\gamma_0)}(\sigma_0 + \omega_{10}) = \sum_{0 \leq a \leq N} R_{\mu_1 \dots \mu_a}^{(\gamma_{10})}(\sigma_0) \cdot R_{\mu_{(a+1)} \dots \mu_N}^{(\gamma_{20})}(\sigma_0 + \omega_{10}),$$

$$R_{\mu_1 \dots \mu_N}^{(\gamma_0)}(\sigma_0) = \int_{\sigma_0}^{\sigma_0 + \omega_0} d\sigma_1 \int_{\sigma_0}^{\sigma_1} d\sigma_2 \dots \int_{\sigma_0}^{\sigma_{N-1}} d\sigma_N \prod_{j=1}^N u_{\mu_j}(\tau_0, \sigma_j)$$

and analogous definitions for the remaining $R^{(\dots)}(\dots)$ -symbols.

Inversion of the following equations (cf. Proposition 3 of ref.[2]):

$$R_{\mu_1 \dots \mu_N}^{(\gamma)}(\sigma) = \sum_{K=1}^N R_{\mu_1 \dots \mu_N}^{(\gamma)(K)}(\sigma),$$

$$R_{\mu_1 \dots \mu_N}^{(\gamma)(K)}(\sigma) = \frac{1}{K!} \sum_{0 < a_1 < \dots < a_{K-1} < N} R_{\mu_1 \dots \mu_{a_1}}^{(\gamma)t}(\sigma) R_{\mu_{a_1+1} \dots \mu_{a_2}}^{(\gamma)t}(\sigma) \dots R_{\mu_{a_{K-1}+1} \dots \mu_N}^{(\gamma)t}(\sigma)$$

finally yields relations among the "truncated" path-ordered integrals $R^{(\dots)t}(\dots)$ corresponding to the curves γ_0 , γ_{10} , and γ_{20} with the start parameters σ_0 / $\sigma_0 + \omega_{10}$, σ_0 , and $\sigma_0 + \omega_{10}$, respectively.

By taking powers of the logarithm of the above factorization equation for the monodromy matrices and by exploiting the arbitrariness of the parameters $A^{\mu\mu}$ as before, I obtain relations between the basic invariant charges of a given degree, say ℓ , on the merged branch on the one hand, and certain polynomials in the truncated path-ordered integrals $R^{(\gamma_0)t}(\sigma_0)$ and $R^{(\gamma_{20})t}(\sigma_0 + \omega_{10})$ on the other hand.

At this point I invoke an alternative representation of the invariant charges of each one of the three branches, viz. the unambiguous representation (cf. ref.[2]) of the invariant charges as homogeneous polynomials in the truncated path-ordered integrals corresponding to the relevant monodromy matrices with given start parameters (here: σ_0 / $\sigma_0 + \omega_{10}$, σ_0 , and $\sigma_0 + \omega_{10}$, respectively), e.g.

$$\mathcal{L}_{\mu_1 \dots \mu_N}^{[\gamma](K)} = \mathcal{L}_{\mu_1 \dots \mu_N}^{+(\gamma(\tau))(K)} = Z_N R_{\mu_1 \dots \mu_N}^{(\gamma(\tau))(K)}(\sigma)$$

where the symbol $[\gamma]$ denotes the homotopy class of the curve $\gamma(\tau)$ on the trajectory surface and the symbol Z_N denotes cyclic symmetrization in the indices $\mu_1, \mu_2, \dots, \mu_N$.

Here:

$$\mathcal{L}_{\mu_1 \dots \mu_N}^{[\gamma_0](K)} = \mathcal{L}_{\mu_1 \dots \mu_N}^{+(\gamma_0)(K)} = \mathcal{L}_{\mu_1 \dots \mu_N}^{+(\gamma_0)(K)} = Z_N R_{\mu_1 \dots \mu_N}^{(\gamma_0)(K)}(\sigma_0)$$

$$= Z_N R_{\mu_1 \dots \mu_N}^{(\gamma_0)(K)}(\sigma_0 + \omega_{10}),$$

$$\mathcal{L}_{\mu_1 \dots \mu_N}^{[\gamma_{10}](K)} = \mathcal{L}_{\mu_1 \dots \mu_N}^{+(\gamma_{10})(K)} = \mathcal{L}_{\mu_1 \dots \mu_N}^{+(\gamma_{10})(K)} = Z_N R_{\mu_1 \dots \mu_N}^{(\gamma_{10})(K)}(\sigma_0),$$

$$\mathcal{L}_{\mu_1 \dots \mu_N}^{[\gamma_{20}](K)} = \mathcal{L}_{\mu_1 \dots \mu_N}^{+(\gamma_{20})(K)} = \mathcal{L}_{\mu_1 \dots \mu_N}^{+(\gamma_{20})(K)} = Z_N R_{\mu_1 \dots \mu_N}^{(\gamma_{20})(K)}(\sigma_0 + \omega_{10}).$$

Next, I determine those linear combinations (with energy-momentum dependent coefficients) of the basic invariant charges on the merged branch of degree ℓ in the rest frame of the merged branch such that the complete "leading"

linear part (cf. ref. [2]) of the corresponding polynomial (with energy-momentum dependent coefficients) in the $R^{(\gamma_{10})^t}(\sigma_0)$'s and in the $R^{(\gamma_{20})^t}(\sigma_0 + \omega_{10})$'s with two indices or more can be replaced (by means of the alternative representation of the invariant charges and at the expense of the appearance of additional higher monomials in the $R^{(\gamma_{10})^t}(\sigma_0)$'s and $R^{(\gamma_{20})^t}(\sigma_0 + \omega_{10})$'s with two indices or more) by invariant charges of degree ℓ on the merging branches. Among the total of $n(3, \ell+2) - n(3, \ell+1)$ linearly independent linear combinations there are at least $n(3, \ell+2) - 2 \times n(3, \ell+1)$ ones for which this complete replacement of the leading linear part is possible. Here for a given natural number N , the symbol $n(3, N)$ is defined by

$$n(3, N) = \frac{1}{N} \sum_{D|N} \mu(D) \cdot 3^{N/D}$$

where the sum extends over all divisors D of N (including 1 and N) and where $\mu(D)$ stands for the Moebius function. For such linear combinations after the replacement, the leading part of the corresponding remaining polynomial is of second or higher order in the $R^{(\gamma_{10})^t}(\sigma_0)$'s and $R^{(\gamma_{20})^t}(\sigma_0 + \omega_{10})$'s with two indices or more. Now I attempt to replace completely that higher order leading part (by means of the alternative representation of the invariant charges and at the expense of the appearance of higher monomials in the relevant R^t 's) by polynomials in the invariant charges of degree less than ℓ on all three branches of the string vertex. For those linear combinations of the basic invariant charges on the merged branch of degree ℓ for which this still turns out to be possible, I continue this procedure until there is no remaining polynomial in the $R^{(\gamma_{10})^t}(\sigma_0)$'s and $R^{(\gamma_{20})^t}(\sigma_0 + \omega_{10})$'s with two indices or more left or until all such linear combinations are exhausted.

In the cases $\ell = 0, 1, 2$ studied in detail so far, beyond the restrictions for the feasibility of the first step no further restrictions turned up for the feasibility of the remaining steps. This suggests that there are at least $n(3, \ell+2) - 2 \times n(3, \ell+1)$ independent linear combinations of the basic invariant charges of degree ℓ on the merged branch, the numerical values of which are determined by the basic invariant charges of degree ℓ or less and of degree less than ℓ evaluated on the merging branches and on the merged branch, respectively.

These relations can be cast into relativistically invariant

forms. Once this has been achieved, I transcribe certain parts of a given relation which are separately relativistically invariant into the rest frame of an appropriate branch of the string vertex. In this fashion I obtain the desired relations involving the elements of the three rest frame algebras of invariant charges corresponding to the three branches of the string vertex.

I define the degree of a monomial in the elements of the three algebras to be given by the sum of the degrees of the individual factors of the three algebras plus the number of factors minus one. (It does not make any difference whether one includes the energy-momentum factors into the counting - assigning the degree -1 to each one of them - or whether one does not count such factors at all.)

As a consequence of the above construction procedure, each one of the desired relations is obtained in a form in which all its monomial terms have the common degree ℓ .

I define an operation for any given branch of the string vertex, called crossing, which consists of the replacement of the corresponding ζ, σ - family of monodromy matrices by their respective inverses, implying in particular the reversal of the corresponding energy-momentum vector and the reversal of orientation of the curves of intersection with the leafs of the Minkowski space-time foliation.

Again, as a consequence of the above construction procedure, the set of the desired relations of a given degree ℓ is invariant under crossing.

To conclude the discussion of the numerical aspects of the relations under consideration, I shall quote the results for the cases $\ell = 0$ and $\ell = 1$. To this end, I introduce the following notation: Let the index ν : $\nu = 0, 1, 2$ label the three branches of the string vertex, the index value 0 standing for the merged branch, the index values 1 and 2 standing for the merging branches.

$\mathcal{P}_{\mu\nu}^{[r]}$, $\mathcal{P}_{\mu\nu}^{[x]}$ and $\mathcal{P}_{\mu\nu}^{[x]}$ denote the μ -components of the energy-momentum vectors $\mathcal{P}_{[r]}$, $\mathcal{P}_{[x]}$ and $\mathcal{P}_{[x]}$ of the merged branch and the merging branches, respectively, in the rest frame of the branch ν . The complex combinations $\mathcal{P}_{1\nu}^{[r]} + i \mathcal{P}_{2\nu}^{[r]}$, $\mathcal{P}_{1\nu}^{[x]} + i \mathcal{P}_{2\nu}^{[x]}$, and $\mathcal{P}_{1\nu}^{[x]} + i \mathcal{P}_{2\nu}^{[x]}$ are denoted by $\mathcal{P}_{\pm\nu}^{[r]}$, $\mathcal{P}_{\pm\nu}^{[x]}$, and $\mathcal{P}_{\pm\nu}^{[x]}$, respectively. The symbols m , $m^{(1)}$ and $m^{(2)}$ stand for the respective masses.

The case $\ell = 0$:

There does not exist a single relation of the desired type corresponding to the degree 0.

The case $\ell = 1$:

There exist exactly two independent relations of the desired type corresponding to the degree 1, viz.

$$\begin{aligned}
 & m \left[\left(\mathcal{P}_{-10}^{[\mathcal{X}_1]} - \mathcal{P}_{-10}^{[\mathcal{X}_2]} \right) \cdot \mathcal{L}_{+1}^{[\mathcal{X}]} + \left(\mathcal{P}_{+10}^{[\mathcal{X}_1]} - \mathcal{P}_{+10}^{[\mathcal{X}_2]} \right) \cdot \mathcal{L}_{-1}^{[\mathcal{X}]} \right] \\
 &= \left[\left(\mathcal{P}_{-12}^{[\mathcal{X}_1]} \right)^2 \cdot \mathcal{L}_{+2}^{[\mathcal{X}_2]} + 2 \left(\mathcal{P}_{-12}^{[\mathcal{X}_1]} \cdot \mathcal{P}_{+12}^{[\mathcal{X}_1]} \right) \cdot \mathcal{L}_0^{[\mathcal{X}_2]} + \left(\mathcal{P}_{+12}^{[\mathcal{X}_1]} \right)^2 \cdot \mathcal{L}_{-2}^{[\mathcal{X}_2]} \right. \\
 &\quad + 2 \mathcal{P}_{012}^{[\mathcal{X}]} \cdot \left(\mathcal{P}_{-12}^{[\mathcal{X}_1]} \cdot \mathcal{L}_{+1}^{[\mathcal{X}_2]} + \mathcal{P}_{+12}^{[\mathcal{X}_1]} \cdot \mathcal{L}_{-1}^{[\mathcal{X}_2]} \right) + 8 \left(\mathcal{P}_{012}^{[\mathcal{X}]} \right)^2 \cdot \left(Q^{[\mathcal{X}_2]} \right)^2 \\
 &\quad \left. + 8m Q^{[\mathcal{X}]} \cdot \left(\mathcal{P}_{010}^{[\mathcal{X}_2]} \cdot Q^{[\mathcal{X}]} - 2m^{(2)} \cdot Q^{[\mathcal{X}_2]} \right) \right] - [\text{branch} \\
 &\quad \text{index 1} \leftrightarrow \text{branch index 2}], \\
 & m \left[\left(\mathcal{P}_{-10}^{[\mathcal{X}_1]} \cdot \mathcal{P}_{-10}^{[\mathcal{X}_2]} \right) \cdot \mathcal{L}_{+2}^{[\mathcal{X}]} + \left(\mathcal{P}_{-10}^{[\mathcal{X}_1]} \cdot \mathcal{P}_{+10}^{[\mathcal{X}_2]} + \mathcal{P}_{+10}^{[\mathcal{X}_1]} \cdot \mathcal{P}_{-10}^{[\mathcal{X}_2]} \right) \cdot \mathcal{L}_0^{[\mathcal{X}]} + \left(\mathcal{P}_{+10}^{[\mathcal{X}_1]} \cdot \mathcal{P}_{+10}^{[\mathcal{X}_2]} \right) \cdot \mathcal{L}_{-2}^{[\mathcal{X}]} \right] \\
 &= \left[-\mathcal{P}_{010}^{[\mathcal{X}_2]} \cdot \left(\left(\mathcal{P}_{-12}^{[\mathcal{X}_1]} \right)^2 \cdot \mathcal{L}_{+2}^{[\mathcal{X}_2]} + 2 \mathcal{P}_{-12}^{[\mathcal{X}_1]} \cdot \mathcal{P}_{+12}^{[\mathcal{X}_1]} \cdot \mathcal{L}_0^{[\mathcal{X}_2]} + \left(\mathcal{P}_{+12}^{[\mathcal{X}_1]} \right)^2 \cdot \mathcal{L}_{-2}^{[\mathcal{X}_2]} \right) \right. \\
 &\quad + 2 \left(\mathcal{P}_{012}^{[\mathcal{X}]} \cdot \mathcal{P}_{010}^{[\mathcal{X}_1]} - \mathcal{P}_{010}^{[\mathcal{X}]} \cdot \mathcal{P}_{012}^{[\mathcal{X}_1]} \right) \cdot \left(\mathcal{P}_{-12}^{[\mathcal{X}_1]} \cdot \mathcal{L}_{+1}^{[\mathcal{X}_2]} + \mathcal{P}_{+12}^{[\mathcal{X}_1]} \cdot \mathcal{L}_{-1}^{[\mathcal{X}_2]} \right) \\
 &\quad + 8 \left(\left(\mathcal{P}_{012}^{[\mathcal{X}]} \right)^2 \cdot \mathcal{P}_{010}^{[\mathcal{X}_1]} - \mathcal{P}_{010}^{[\mathcal{X}]} \cdot \left(\mathcal{P}_{012}^{[\mathcal{X}_1]} \right)^2 \right) \cdot \left(Q^{[\mathcal{X}_2]} \right)^2 + 8m m^{(1)} Q^{[\mathcal{X}_1]} \cdot m^{(2)} Q^{[\mathcal{X}_2]} \\
 &\quad \left. + 4m Q^{[\mathcal{X}]} \cdot \mathcal{P}_{010}^{[\mathcal{X}_1]} \cdot \left(\mathcal{P}_{010}^{[\mathcal{X}_2]} \cdot Q^{[\mathcal{X}]} - 4m^{(2)} \cdot Q^{[\mathcal{X}_2]} \right) \right] + [\text{branch} \\
 &\quad \text{index 1} \leftrightarrow \text{branch index 2}].
 \end{aligned}$$

Now, I turn to the algebraic aspect of the relations under discussion. Obviously, the three non-linear rest frame Poisson algebras of invariant charges $\mathfrak{h}^{(0)}$, $\mathfrak{h}^{(1)}$, and $\mathfrak{h}^{(2)}$ corresponding to the merged branch and the merging branches of the string vertex, respectively, must be embedded in a larger Poisson algebra of reparametrization invariant ele-

ments. I shall concentrate on the minimal embedding Poisson algebra generated by the conserved invariant charges of the algebras $\mathfrak{h}^{(0)}$, $\mathfrak{h}^{(1)}$, and $\mathfrak{h}^{(2)}$. However, due to the presence of the distinguished point of mergence $x^{(0)}$, it must be anticipated that the reparametrization invariant elements of the minimal embedding Poisson algebra are not necessarily piece-wise conserved. In other words, a reference to this point of mergence (e.g. in the form of the non-averaged starting point for certain path-ordered integrals) will be contained in some of the elements of the minimal embedding algebra. As to the embedding of the algebras $\mathfrak{h}^{(1)}$ and $\mathfrak{h}^{(2)}$ corresponding to the merging branches, locality and piece-wise conservation of their respective elements imply the vanishing of the Poisson brackets between any given element of $\mathfrak{h}^{(1)}$ and any given element of $\mathfrak{h}^{(2)}$. Further, the Poisson brackets between the components of the energy-momentum vector $\mathcal{P}^{[r]}$ and all invariant charges of all three branches vanish. This is evident from the fact that the components of the energy-momentum vector $\mathcal{P}^{[r]}$ generate rigid translations in the three coordinate directions whereas each and every invariant charge of the three algebras corresponding to the three branches involves position differences only. In fact it involves the canonical variables in the form

$$u_\mu(\tau, \sigma) = p_\mu(\tau, \sigma) + \frac{\partial}{\partial \sigma} x_\mu(\tau, \sigma) ,$$

only. This is the reason why also the elements of the minimal embedding algebra must be expected to involve the canonical variables in the same restricted form only, in particular to depend on the position differences only and to produce vanishing Poisson brackets with the components of $\mathcal{P}^{[r]}$. Hence, the components of the energy-momentum vector $\mathcal{P}^{[r]}$ are central elements for the minimal embedding algebra and will be treated as c-numbers. Since these c-numbers lie on the forward mass-shell corresponding to the mass m , Lorentz invariance of the Nambu-Goto theory ensures that no generality is lost when I specify the minimal embedding Poisson algebra in the momentum rest frame of the merged branch.

In contrast to their vector addition, the components of the energy-momentum vectors corresponding to the merging branches cannot possibly be central elements of the embedding Poisson algebra. While they still have vanishing Poisson brackets with the invariant charges of the algebras $\mathfrak{h}^{(1)}$ and $\mathfrak{h}^{(2)}$, they fail to produce vanishing Poisson brackets with the invariant charges of the algebra $\mathfrak{h}^{(0)}$, e.g.

$$\{Q^{[r]}, P_{\pm 10}^{[r_i]}\} = \mp i P_{\pm 10}^{[r_i]} (, \{Q^{[r]}, P_{010}^{[r_i]}\} = 0) \quad j = 1, 2 .$$

These Poisson bracket relations are implied by the fact

that $Q^{[r]}$ generates rigid rotations in the plane orthogonal to the vector $\mathcal{P}^{[r]}$. By the same line of reasoning, the Poisson brackets of $Q^{[r]}$ on the one hand, and the Lorentz invariant quantities $Q^{[x_i]}, Q^{[x_i]}$ and $(m), m^{(1)}, m^{(2)}$ on the other hand, vanish. In any case, the components of the energy-momentum vectors $\mathcal{P}^{[r]}$ and $\mathcal{P}^{[x_i]}$ must be included in the embedding Poisson algebra with the degree -1 assigned to them.

In order to proceed beyond these few scattered remarks to a systematic analysis of the structure of the minimal embedding Poisson algebra, I choose the limiting leaf of the Minkowski space-time foliation containing the point of emergence $x^{(0)}$ and being labelled by the parameter τ_0 for the comparison and embedding of the three algebras. Starting from the separate clear-cut situations before the limits $\tau \rightarrow \tau_0$ are taken, I have no alternative compatible with the above remarks but to postulate the following Poisson brackets

$$\begin{aligned} \{u_\mu(\tau_0, \sigma), u_\nu(\tau_0, \sigma')\} &= -2 g_{\mu\nu} \delta'_{\omega_0}(\sigma - \sigma') \text{ for } \sigma, \sigma' \in [\sigma_0, \sigma_0 + \omega_{10}), \\ \{u_\mu(\tau_0, \sigma), u_\nu(\tau_0, \sigma')\} &= -2 g_{\mu\nu} \delta'_{\omega_{20}}(\sigma - \sigma') \text{ for } \sigma, \sigma' \in [\sigma_0 + \omega_{10}, \sigma_0 + \omega_0), \\ \{u_\mu(\tau_0, \sigma), u_\nu(\tau_0, \sigma')\} &= -2 g_{\mu\nu} \delta'_{\omega_0}(\sigma - \sigma') \text{ for } \sigma, \sigma' \in [\sigma_0, \sigma_0 + \omega_0) \end{aligned}$$

for the composition of the invariant charges of the algebras $\mathfrak{h}^{(1)}, \mathfrak{h}^{(2)}$, and $\mathfrak{h}^{(0)}$, separately, and

$$\{u_\mu(\tau_0, \sigma), u_\nu(\tau_0, \sigma')\} = -2 g_{\mu\nu} \begin{cases} 0 & \text{for } \sigma \in [\sigma_0, \sigma_0 + \omega_{10}), \sigma' \in [\sigma_0 + \omega_{10}, \sigma_0 + \omega_0), \\ \delta'_{\omega_0}(\sigma - \sigma') & \text{for } \sigma \in [\sigma_0, \sigma_0 + \omega_0), \sigma' \in [\sigma_0, \sigma_0 + \omega_{10}), \\ \delta'_{\omega_0}(\sigma - \sigma') & \text{for } \sigma \in [\sigma_0, \sigma_0 + \omega_0), \sigma' \in [\sigma_0 + \omega_{10}, \sigma_0 + \omega_0) \end{cases}$$

for the composition of the invariant charges of the algebras $\mathfrak{h}^{(1)}$, and $\mathfrak{h}^{(0)}$ with the invariant charges of the algebras $\mathfrak{h}^{(2)}$, and $\mathfrak{h}^{(0)}/\mathfrak{h}^{(2)}$, respectively.

Here the symbols $\delta'_{\omega_{10}}(\sigma - \sigma')$, $\delta'_{\omega_{20}}(\sigma - \sigma')$, and $\delta'_{\omega_0}(\sigma - \sigma')$ stand for the σ -derivatives of the periodic delta functions with periods ω_{10} , ω_{20} , and ω_0 , respectively.

If the two quoted relations for the case $\ell = 1$ are consistent with the Poisson bracket operation, the Poisson bracket between the left hand sides of the two relations for one thing and the right hand sides of the two relations for another necessarily must produce a valid relation of the desired type for the invariant charges of degree $\ell = 2$.

In order to compute this new relation, beyond the separate information about the algebras $\mathfrak{h}^{(1)}$, $\mathfrak{h}^{(2)}$, and $\mathfrak{h}^{(0)}$ and beyond the above scattered pieces of information about the minimal embedding algebra, I need to know the results of the Poisson bracket operation between the invariant charges $L_s^{[r]}$ $s = 0, +1, +2$ and the components of the energy-momentum vectors $P_{\pm j}^{[r]}$, $j = 1, 2$ in the rest frame of the merged algebra. Lorentz invariance takes care of the Poisson brackets involving the invariant charge $Q^{[r]}$. The required results (in particular by themselves consistent with the two quoted relations of degree $\ell=1$) in obvious notation are

$$\{L_{\pm 2}^{[r]}, P_{\pm 10}^{[r_i]}\} = 0, \{L_{\pm 2}^{[r]}, P_{\mp 10}^{[r_i]}\} = -2 \{L_0^{[r]}, P_{\pm 10}^{[r_i]}\},$$

$$\{L_{\pm 2}^{[r]}, P_{010}^{[r_i]}\} = \{L_{\pm 1}^{[r]}, P_{\pm 10}^{[r_i]}\};$$

$$\{L_{\pm 1}^{[r]}, P_{\pm 10}^{[r_i]}\} = -P_{\pm 10}^{[r_i]} \cdot (R_{0\pm 10}^{(r_i)t}(\sigma_0) + R_{0\pm 10}^{(r_i)t}(\sigma_0 + \omega_{10})),$$

$$\{L_{\pm 1}^{[r]}, P_{\mp 10}^{[r_i]}\} = P_{\pm 10}^{[r_i]} \cdot (R_{0\mp 10}^{(r_i)t}(\sigma_0) + R_{0\mp 10}^{(r_i)t}(\sigma_0 + \omega_{10})) + \\ \pm 8i \left[(m^{(i)} \cdot Q^{[r_i]}) - (m^{(\mp i)} \cdot Q^{[r_{\mp i}]}) - \left(\frac{m^{(i)2} - m^{(\mp i)2}}{m^2} \right) (m \cdot Q^{[r]}) \right],$$

$$\{L_{\pm 1}^{[r]}, P_{010}^{[r_i]}\} = \mp 4i P_{\pm 10}^{[r_i]} \cdot Q^{[r]};$$

$$\{L_0^{[r]}, P_{010}^{[r_i]}\} = \frac{1}{2} P_{-10}^{[r_i]} \cdot (R_{0+10}^{(r_i)t}(\sigma_0) + R_{0+10}^{(r_i)t}(\sigma_0 + \omega_{10})) + \\ + \frac{1}{2} P_{+10}^{[r_i]} \cdot (R_{0-10}^{(r_i)t}(\sigma_0) + R_{0-10}^{(r_i)t}(\sigma_0 + \omega_{10})),$$

$$P_{+10}^{[r_i]} \cdot \{L_0^{[r]}, P_{-10}^{[r_j]}\} - P_{-10}^{[r_j]} \cdot \{L_0^{[r]}, P_{+10}^{[r_i]}\} = 8i \left[P_{010}^{[r_i]} \cdot \right. \\ \left. \cdot (m^{(\mp j)} \cdot Q^{[r_{\mp j}]}) + P_{010}^{[r_{\mp j}]} \cdot (m^{(j)} \cdot Q^{[r_j]}) - P_{010}^{[r_i]} \cdot P_{010}^{[r_{\mp j}]} \cdot Q^{[r]} \right],$$

$$P_{+10}^{[r_j]} \cdot \{L_0^{[r]}, P_{-10}^{[r_i]}\} + P_{-10}^{[r_i]} \cdot \{L_0^{[r]}, P_{+10}^{[r_j]}\} =$$

$$= \oint_{\gamma_j} d\sigma u_0(\sigma) \cdot \left(\mathcal{P}_{+10}^{[\gamma_j]} R_{0-10}^{(\gamma_j)t}(\sigma) + \mathcal{P}_{-10}^{[\gamma_j]} R_{0+10}^{(\gamma_j)t}(\sigma) \right); \quad j, \neq j = 1, 2.$$

In the notation of references [2] and [4], the resulting induced relation of degree $\ell = 2$ is:

$$\begin{aligned} & -\frac{m}{8} \left[\left(\mathcal{P}_{-10}^{[\gamma_1]} - \mathcal{P}_{-10}^{[\gamma_2]} \right)^3 \cdot \mathcal{L}_{(1,2)}^{[\gamma]} + \left(\mathcal{P}_{-10}^{[\gamma_1]} - \mathcal{P}_{-10}^{[\gamma_2]} \right)^2 \cdot \left(\mathcal{P}_{+10}^{[\gamma_1]} - \mathcal{P}_{+10}^{[\gamma_2]} \right) \cdot \mathcal{L}_{(-1,2)}^{[\gamma]} \right. \\ & + \left. \left(\mathcal{P}_{-10}^{[\gamma_1]} - \mathcal{P}_{-10}^{[\gamma_2]} \right) \cdot \left(\mathcal{P}_{+10}^{[\gamma_1]} - \mathcal{P}_{+10}^{[\gamma_2]} \right)^2 \cdot \mathcal{L}_{(1,2)}^{[\gamma]} + \left(\mathcal{P}_{+10}^{[\gamma_1]} - \mathcal{P}_{+10}^{[\gamma_2]} \right)^3 \cdot \mathcal{L}_{(-1,2)}^{[\gamma]} \right] \\ & - \left(\frac{m^2 - m^{(1)2} - m^{(2)2}}{4m} \right) \cdot \left[\left(\mathcal{P}_{-10}^{[\gamma_1]} - \mathcal{P}_{-10}^{[\gamma_2]} \right) \cdot \mathcal{L}_{(1,0)}^{[\gamma]} + \left(\mathcal{P}_{+10}^{[\gamma_1]} - \mathcal{P}_{+10}^{[\gamma_2]} \right) \cdot \mathcal{L}_{(-1,0)}^{[\gamma]} \right] \\ & = \left(m^{(1)} \cdot \left[\left(\mathcal{P}_{-11}^{[\gamma_2]} \right)^3 \cdot \mathcal{L}_{(1,2)}^{[\gamma_1]} + \left(\mathcal{P}_{-11}^{[\gamma_2]} \right)^2 \cdot \mathcal{P}_{+11}^{[\gamma_2]} \cdot \mathcal{L}_{(-1,2)}^{[\gamma_1]} + \mathcal{P}_{-11}^{[\gamma_2]} \cdot \left(\mathcal{P}_{+11}^{[\gamma_2]} \right)^2 \cdot \mathcal{L}_{(1,2)}^{[\gamma_1]} + \left(\mathcal{P}_{+11}^{[\gamma_2]} \right)^3 \cdot \mathcal{L}_{(-1,2)}^{[\gamma_1]} \right] \right. \\ & + 2m \left(\mathcal{P}_{010}^{[\gamma_1]} \cdot \mathcal{P}_{011}^{[\gamma_2]} - \mathcal{P}_{011}^{[\gamma_1]} \cdot \mathcal{P}_{010}^{[\gamma_2]} \right) \cdot \left[\mathcal{P}_{-11}^{[\gamma_2]} \cdot \mathcal{L}_{(1,0)}^{[\gamma_1]} + \mathcal{P}_{+11}^{[\gamma_2]} \cdot \mathcal{L}_{(-1,0)}^{[\gamma_1]} \right] \\ & + 4im \left[\left(m^{(1)} \cdot Q^{[\gamma_1]} \right) - \mathcal{P}_{010}^{[\gamma_1]} \cdot Q^{[\gamma]} \right] \cdot \left[\left(\mathcal{P}_{-10}^{[\gamma_2]} - \mathcal{P}_{-10}^{[\gamma_2]} \right)^2 \cdot \mathcal{L}_{+2}^{[\gamma]} - \left(\mathcal{P}_{+10}^{[\gamma_2]} - \mathcal{P}_{+10}^{[\gamma_2]} \right)^2 \cdot \mathcal{L}_{-2}^{[\gamma]} \right] \\ & - 8im \left[\mathcal{P}_{010}^{[\gamma_2]} \cdot \left(m^{(1)} \cdot Q^{[\gamma_1]} \right) + \mathcal{P}_{010}^{[\gamma_1]} \cdot \left(m^{(2)} \cdot Q^{[\gamma_2]} \right) - \mathcal{P}_{010}^{[\gamma_1]} \cdot \mathcal{P}_{010}^{[\gamma_2]} \cdot Q^{[\gamma]} \right] \\ & \cdot \left[\mathcal{P}_{-10}^{[\gamma_1]} \cdot \mathcal{L}_{+1}^{[\gamma]} - \mathcal{P}_{+10}^{[\gamma_1]} \cdot \mathcal{L}_{-1}^{[\gamma]} \right] \\ & - 16i \left[\left(m^2 - m^{(2)2} \right) \cdot Q^{[\gamma_1]} + m^{(1)} \cdot \left(m^{(2)} \cdot Q^{[\gamma_2]} \right) - m^{(1)} \cdot \left(m \cdot Q^{[\gamma]} \right) \right] \cdot \left[\left(\mathcal{P}_{-11}^{[\gamma_2]} \right)^2 \cdot \mathcal{L}_{+2}^{[\gamma_1]} - \left(\mathcal{P}_{+11}^{[\gamma_2]} \right)^2 \cdot \mathcal{L}_{-2}^{[\gamma_1]} \right] \\ & - 8i \left[\left(m^2 + m^{(1)2} - m^{(2)2} \right) \cdot \mathcal{P}_{011}^{[\gamma_2]} \cdot Q^{[\gamma_1]} + \left(m^2 + m^{(1)2} - m^{(2)2} \right) \cdot \left(m^{(2)} \cdot Q^{[\gamma_2]} \right) - \left(m^2 - m^{(1)2} - m^{(2)2} \right) \cdot \left(m \cdot Q^{[\gamma]} \right) \right] \\ & \cdot \left[\mathcal{P}_{-11}^{[\gamma_2]} \cdot \mathcal{L}_{+1}^{[\gamma_1]} - \mathcal{P}_{+11}^{[\gamma_2]} \cdot \mathcal{L}_{-1}^{[\gamma_1]} \right] \\ & - \left(\text{branch index } 1 \leftrightarrow \text{branch index } 2 \right). \end{aligned}$$

For the degree $\ell = 2$, there are exactly two more relations of the desired type which will be quoted here:

$$\begin{aligned}
 & -im \left[\mathcal{P}_{-10}^{[\delta_1]} \mathcal{P}_{-10}^{[\delta_2]} \mathcal{L}_{(2,0)}^{[\delta]} + (\mathcal{P}_{-10}^{[\delta_1]} \mathcal{P}_{+10}^{[\delta_2]} + \mathcal{P}_{+10}^{[\delta_1]} \mathcal{P}_{-10}^{[\delta_2]}) \mathcal{L}_{(1,-1)}^{[\delta]} - \mathcal{P}_{+10}^{[\delta_1]} \mathcal{P}_{+10}^{[\delta_2]} \mathcal{L}_{(-2,0)}^{[\delta]} \right] \\
 & = \left(-\frac{i}{3} \left[(\mathcal{P}_{-12}^{[\delta_1]})^3 \mathcal{L}_{(1,2)}^{[\delta_2]} + 3(\mathcal{P}_{-12}^{[\delta_1]})^2 \mathcal{P}_{+12}^{[\delta_1]} \mathcal{L}_{(-1,2)}^{[\delta_2]} - 3\mathcal{P}_{-12}^{[\delta_1]} (\mathcal{P}_{+12}^{[\delta_1]})^2 \mathcal{L}_{(1,-2)}^{[\delta_2]} - (\mathcal{P}_{+12}^{[\delta_1]})^3 \mathcal{L}_{(-1,-2)}^{[\delta_2]} \right] \right. \\
 & \quad + i \mathcal{P}_{012}^{[\delta]} \left[(\mathcal{P}_{-12}^{[\delta_1]})^2 \mathcal{L}_{(2,0)}^{[\delta_2]} - (\mathcal{P}_{+12}^{[\delta_1]})^2 \mathcal{L}_{(-2,0)}^{[\delta_2]} \right] \\
 & \quad + 2i \mathcal{P}_{-12}^{[\delta_1]} \mathcal{P}_{+12}^{[\delta_1]} \left[\mathcal{P}_{-12}^{[\delta_1]} \mathcal{L}_{(1,0)}^{[\delta_2]} + \mathcal{P}_{012}^{[\delta]} \mathcal{L}_{(1,-1)}^{[\delta_2]} - \mathcal{P}_{+12}^{[\delta_1]} \mathcal{L}_{(-1,0)}^{[\delta_2]} \right] \\
 & \quad - 8m Q^{[\delta]} \mathcal{P}_{-10}^{[\delta_1]} \mathcal{P}_{+10}^{[\delta_2]} \mathcal{L}_0^{[\delta]} + 8 \left[\mathcal{P}_{010}^{[\delta]} Q^{[\delta]} - m^{(1)} Q^{[\delta_1]} \right] \cdot \\
 & \quad \cdot \left[(\mathcal{P}_{-12}^{[\delta_1]})^2 \mathcal{L}_{+2}^{[\delta_2]} + 2\mathcal{P}_{-12}^{[\delta_1]} \mathcal{P}_{+12}^{[\delta_1]} \mathcal{L}_0^{[\delta_2]} + (\mathcal{P}_{+12}^{[\delta_1]})^2 \mathcal{L}_{-2}^{[\delta_2]} \right] \\
 & \quad - 8 \mathcal{P}_{012}^{[\delta]} Q^{[\delta_2]} \left[(\mathcal{P}_{-12}^{[\delta_1]})^2 \mathcal{L}_{+2}^{[\delta_2]} + (\mathcal{P}_{+12}^{[\delta_1]})^2 \mathcal{L}_{-2}^{[\delta_2]} \right] + 16 m^{(2)} Q^{[\delta_2]} \mathcal{P}_{-12}^{[\delta_1]} \mathcal{P}_{+12}^{[\delta_1]} \mathcal{L}_0^{[\delta_2]} \\
 & \quad + 8 \left[\mathcal{P}_{012}^{[\delta]} (\mathcal{P}_{010}^{[\delta]} Q^{[\delta]} - m^{(1)} Q^{[\delta_1]}) - (m^2 + m^{(1)2} - m^{(2)2}) Q^{[\delta_2]} \right] \left[\mathcal{P}_{-12}^{[\delta_1]} \mathcal{L}_{+1}^{[\delta_2]} + \mathcal{P}_{+12}^{[\delta_1]} \mathcal{L}_{-1}^{[\delta_2]} \right] \\
 & \quad + 16 \left[-\frac{2}{3} + \frac{m^{(1)2} + m^{(2)2}}{m^2} - \left(\frac{m^{(1)2} - m^{(2)2}}{m^2} \right)^2 \right] \cdot (m Q^{[\delta]})^3 \\
 & \quad + 64 (m Q^{[\delta]}) \cdot (m^{(2)} Q^{[\delta_2]}) \cdot \left[(m^{(1)} Q^{[\delta_1]}) - (m^{(2)} Q^{[\delta_2]}) - (\mathcal{P}_{010}^{[\delta]} - \mathcal{P}_{010}^{[\delta_2]}) Q^{[\delta]} \right] \\
 & \quad - 64 (\mathcal{P}_{012}^{[\delta]} Q^{[\delta_2]})^2 \cdot \left[(m^{(1)} Q^{[\delta_1]}) - \frac{1}{3} (\mathcal{P}_{012}^{[\delta]} Q^{[\delta_2]}) - (\mathcal{P}_{010}^{[\delta]} Q^{[\delta]}) \right] \\
 & \quad - 32 (m^2 + m^{(1)2} - m^{(2)2}) \cdot \mathcal{P}_{012}^{[\delta]} \cdot (Q^{[\delta_2]})^3 \\
 & \quad + (\text{branch index } 1 \leftrightarrow \text{branch index } 2) ,
 \end{aligned}$$

$$\begin{aligned}
 & i \frac{m}{8} \left[(\mathcal{P}_{-10}^{[\delta_1]} - \mathcal{P}_{-10}^{[\delta_2]})^3 \mathcal{L}_{(1,2)}^{[\delta]} + 3(\mathcal{P}_{-10}^{[\delta_1]} - \mathcal{P}_{-10}^{[\delta_2]})^2 (\mathcal{P}_{+10}^{[\delta_1]} - \mathcal{P}_{+10}^{[\delta_2]}) \mathcal{L}_{(-1,2)}^{[\delta]} \right. \\
 & \quad \left. - 3(\mathcal{P}_{-10}^{[\delta_1]} - \mathcal{P}_{-10}^{[\delta_2]}) (\mathcal{P}_{+10}^{[\delta_1]} - \mathcal{P}_{+10}^{[\delta_2]})^2 \mathcal{L}_{(1,-2)}^{[\delta]} - (\mathcal{P}_{+10}^{[\delta_1]} - \mathcal{P}_{+10}^{[\delta_2]})^3 \mathcal{L}_{(-1,-2)}^{[\delta]} \right] \\
 & \quad + \frac{3}{2} i m (\mathcal{P}_{-10}^{[\delta_1]} \mathcal{P}_{+10}^{[\delta_2]} + \mathcal{P}_{+10}^{[\delta_1]} \mathcal{P}_{-10}^{[\delta_2]}) \cdot \left[(\mathcal{P}_{-10}^{[\delta_1]} - \mathcal{P}_{-10}^{[\delta_2]}) \mathcal{L}_{(1,0)}^{[\delta]} - (\mathcal{P}_{+10}^{[\delta_1]} - \mathcal{P}_{+10}^{[\delta_2]}) \mathcal{L}_{(-1,0)}^{[\delta]} \right]
 \end{aligned}$$

$$\begin{aligned}
&= \left(i \mathcal{P}_{010}^{[\delta_2]} \left[\mathcal{P}_{-12}^{[\delta_1]^3} \mathcal{L}_{(1,2)}^{[\delta_2]} + 3 \mathcal{P}_{-12}^{[\delta_1]^2} \mathcal{P}_{+12}^{[\delta_1]} \mathcal{L}_{(-1,2)}^{[\delta_2]} - 3 \mathcal{P}_{-12}^{[\delta_1]} (\mathcal{P}_{+12}^{[\delta_1]^2} \mathcal{L}_{(1,-2)}^{[\delta_2]} - (\mathcal{P}_{+12}^{[\delta_1]^3} \mathcal{L}_{(-1,-2)}^{[\delta_2]}) \right] \right. \\
&\quad - 3i \frac{m^{(2)}}{m} \mathcal{P}_{-12}^{[\delta_1]} \mathcal{P}_{+12}^{[\delta_1]} \left[(\mathcal{P}_{-12}^{[\delta_1]^2} \mathcal{L}_{(2,0)}^{[\delta_2]} + 2 \mathcal{P}_{-12}^{[\delta_1]} \mathcal{P}_{+12}^{[\delta_1]} \mathcal{L}_{(1,-1)}^{[\delta_2]} + (\mathcal{P}_{+12}^{[\delta_1]^2} \mathcal{L}_{(-2,0)}^{[\delta_2]}) \right] \\
&\quad - 6i \mathcal{P}_{010}^{[\delta_2]} \mathcal{P}_{-12}^{[\delta_1]} \mathcal{P}_{+12}^{[\delta_1]} \left[\mathcal{P}_{-12}^{[\delta_1]} \mathcal{L}_{(1,0)}^{[\delta_2]} - \mathcal{P}_{+12}^{[\delta_1]} \mathcal{L}_{(-1,0)}^{[\delta_2]} \right] \\
&\quad + 12 \left[2 \mathcal{P}_{010}^{[\delta_2]} \left((m^{(1)} Q^{[\delta_1]}) + (\mathcal{P}_{012}^{[\delta_1]} Q^{[\delta_2]}) \right) + \left(\frac{m^{(1)^2} - m^{(2)^2}}{m} - m^{(1)} - m^{(2)^2} \right) Q^{[\delta]} \right] \\
&\quad \cdot \left[(\mathcal{P}_{-12}^{[\delta_1]^2} \mathcal{L}_{+2}^{[\delta_2]} + 2 \mathcal{P}_{-12}^{[\delta_1]} \mathcal{P}_{+12}^{[\delta_1]} \mathcal{L}_0^{[\delta_2]} + (\mathcal{P}_{+12}^{[\delta_1]^2} \mathcal{L}_{-2}^{[\delta_2]}) \right] \\
&\quad - \frac{48}{m} (\mathcal{P}_{-12}^{[\delta_1]} \mathcal{P}_{+12}^{[\delta_1]})^2 (m^{(2)} Q^{[\delta_2]}) \mathcal{L}_0^{[\delta_2]} \\
&\quad + 48 \frac{m^{(2)}}{m} \mathcal{P}_{-12}^{[\delta_1]} \mathcal{P}_{+12}^{[\delta_1]} \left[(m^{(1)} Q^{[\delta_1]}) - (m^{(2)} Q^{[\delta_2]}) - (\mathcal{P}_{010}^{[\delta_1]} - \mathcal{P}_{010}^{[\delta_2]}) Q^{[\delta]} \right] \\
&\quad \cdot \left[\mathcal{P}_{-12}^{[\delta_1]} \mathcal{L}_{+1}^{[\delta_2]} + \mathcal{P}_{+12}^{[\delta_1]} \mathcal{L}_{-1}^{[\delta_2]} \right] + 64 m \mathcal{P}_{010}^{[\delta_2]} \left[4 (\mathcal{P}_{010}^{[\delta_1]^2}) - 3 m^{(1)^2} \right] (Q^{[\delta]})^3 \\
&\quad - 192 \left[(m^{(1)^2} - m^{(2)^2}) \mathcal{P}_{010}^{[\delta_1]} + 2 m \mathcal{P}_{-10}^{[\delta_1]} \mathcal{P}_{+10}^{[\delta_1]} \right] (Q^{[\delta]})^2 (m^{(2)} Q^{[\delta_2]}) \\
&\quad - 384 \mathcal{P}_{010}^{[\delta_2]} (m Q^{[\delta]}) (m^{(1)} Q^{[\delta_1]}) (m^{(2)} Q^{[\delta_2]}) \\
&\quad + 192 \left[\mathcal{P}_{010}^{[\delta_2]} (\mathcal{P}_{-12}^{[\delta_1]} \mathcal{P}_{+12}^{[\delta_1]} - m^2) + 2 m m^{(2)^2} \right] (Q^{[\delta_2]})^2 \left[(m^{(1)} Q^{[\delta_1]}) - \mathcal{P}_{012}^{[\delta_1]} Q^{[\delta_2]} \right. \\
&\quad \left. - \mathcal{P}_{010}^{[\delta_1]} Q^{[\delta]} \right] + 64 \left[2 (\mathcal{P}_{010}^{[\delta_2]^2}) Q^{[\delta_2]} + 3 \mathcal{P}_{-10}^{[\delta_1]} \mathcal{P}_{+10}^{[\delta_1]} Q^{[\delta]} \right] (\mathcal{P}_{012}^{[\delta_2]} Q^{[\delta_2]})^2 \\
&\quad + 192 \left[m^2 (\mathcal{P}_{-12}^{[\delta_1]} \mathcal{P}_{+12}^{[\delta_1]} - \mathcal{P}_{010}^{[\delta_2]} \mathcal{P}_{012}^{[\delta_2]} - (\mathcal{P}_{010}^{[\delta_2]^2})) + m (4 m^{(2)^2} \mathcal{P}_{012}^{[\delta_2]} - (\mathcal{P}_{010}^{[\delta_2]^3})) \right] \\
&\quad \cdot (Q^{[\delta_2]})^3 - 128 m (m^{(2)} Q^{[\delta_2]})^3 \Bigg) \\
&\quad - (\text{branch index } 1 \leftrightarrow \text{branch index } 2) .
\end{aligned}$$

For the degree $\ell = 3$, it has been checked that all six relations induced from the relations of lower degree remain independent and that there are at most six genuinely new relations of the desired type.

In the present communication I have given a few examples of algebraic relations among the invariant charges corresponding to the three branches of a string vertex. In addition, I have discussed c-number and rudimentary algebraic aspects of these relations. A more systematic discussion of the algebraic aspect of these non-additive composition laws and a more complete analysis of the minimal algebra embedding the three separate algebras of invariant charges corresponding to the three branches will be given elsewhere.

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