AD
UNIVERSITÄT LEIPZIG

Naturwissenschaftlich-Theoretisches
Zentrum

Construction of Grand Unified Models
by Dimensional Reduction

by

G. Rudolph¹ and I.P. Volobuev²

¹ University of Leipzig
Institute for Theoretical Physics
Augustusplatz 10/11
04109 Leipzig, Germany

² Institute of Nuclear Physics
Moscow State University
119 899 Moscow, Russia

SW 9447

Abstract

We discuss the problem of building grand unified models in the framework of the dimensional reduction method. For that purpose, we develop a new technique for constructing bases of generators adapted to reductions of representations of Lie algebras to subalgebras. In particular, we derive the bosonic sector of the flipped $SU(5)$ grand unified model.

1 Introduction

An important feature of gauge theories with space-time symmetries is the existence of symmetric fields, which are invariant, up to a gauge transformation, under the action of the symmetry group. It turns out that the K -symmetric sector of a gauge theory in a space of the form $E = M \times K/H$ can be consistently reduced to a gauge theory in physical space-time M . This procedure is known as dimensional reduction of symmetric gauge fields, or coset space dimensional reduction (CSDR), see [1], [2], [5] and further references therein. If one starts with a pure gauge theory, then the reduced theory contains also scalar fields with a quartic self-interaction, the potential often being of the Higgs-type and leading to spontaneous symmetry breaking in the reduced theory. Such reduced theories contain only a small number of parameters, and, therefore, it is tempting to try to construct with this method realistic models of particle interactions. In particular, there were numerous attempts to construct models of grand unification type, see e.g. [7], [8] and [9]. It turned out that it is quite easy to get the correct Higgs multiplets, but – until now – nobody succeeded in getting Higgs potentials leading to correct spontaneous symmetry breaking (including hierarchies of masses). There were attempts to circumvent this problem by inventing additional mechanisms, namely Wilson flux line symmetry breaking – see [6] and [9]. In our opinion, this is a serious deviation from the original idea. Here we present a solution to the above problem following rather the original philosophy. However, there is an additional Z_2 -symmetry, which we have to impose and which – at the moment – we are not able to interpret geometrically.

For that purpose we develop a special technique of constructing bases of generators adapted to reductions of representations of Lie algebras to subalgebras. This method is based on tensor product representations for classical Lie algebras, which are generalizations of the canonical matrix representations.

Our paper is organized as follows: In section 2 we give a short review of the dimensional reduction method including a discussion of its application to the construction of models. In section 3 we present the construction of bases of generators in Lie algebras adapted to the reduction to a subalgebra. In section 4 we use this technique to construct a realistic model of grand unification type with gauge group $SU(5) \times U(1)$.

2 Dimensional Reduction and Construction of Models

We consider a pure gauge theory with compact gauge group G on a multidimensional universe

$$E = M \times K/H, \quad (2.1)$$

where M is four-dimensional space-time and K a symmetry group (compact Lie group) acting on E with stabilizer H . This action is given by the canonical left action of K on K/H ; we denote it by

$$\delta : K \times E \mapsto E. \quad (2.2)$$

We assume that E is equipped with a pseudo-Riemannian metric

$$g = \eta \oplus \gamma, \quad (2.3)$$

$\text{sign}(g) = (-, +, \dots, +)$, with η being a metric on M and γ being a K -invariant metric on K/H . Thus,

$$\delta_k^* g = g. \quad (2.4)$$

For E being of the form (2.1) it is easy to classify principal bundles $P(E, G)$ admitting lifts of the action (2.2) to automorphisms of P . Such bundles are in one-to-one correspondence, see [3], with pairs $(\lambda, \hat{P})$, where

$$\lambda : H \mapsto G \quad (2.5)$$

is a group homomorphism and $\hat{P}$ is a principal fibre bundle over M with reduced gauge group $C = C(\lambda(H))$ (centralizer of $\lambda(H)$ in G). We denote the natural lift of (2.2) to a bundle of the above type by

$$\sigma : K \times P \mapsto P. \quad (2.6)$$

A K -invariant gauge potential is a connection form ω on P satisfying

$$\sigma_k^* \omega = \omega. \quad (2.7)$$

Denoting the Lie algebras of G , K and H by $\mathfrak{G}$, $\mathfrak{K}$ and $\mathfrak{H}$ respectively, and fixing an Ad -invariant scalar product on $\mathfrak{K}$ we have the orthogonal reductive decomposition

$$\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{M}. \quad (2.8)$$

One can prove, see [4], that a K -invariant connection form is in one-to-one correspondence with a pair $(\hat{\omega}, \phi)$, where $\hat{\omega}$ is a connection form on $\hat{P}$ and ϕ is a mapping

$$\phi : \hat{P} \mapsto \mathfrak{M}^* \otimes \mathfrak{G}, \quad (2.9)$$

satisfying

$$Ad \lambda(h) \cdot \phi(\hat{p}) = \phi(\hat{p}) \cdot Ad h, \quad (2.10)$$

with $h \in H$. Here $Ad \lambda(\cdot)$ denotes the restriction $Ad G \downarrow \lambda(H)$ and $Ad(\cdot)$ means the restriction $Ad K \downarrow_H (\mathfrak{M})$. The infinitesimal version of this equation reads

$$ad \tau(h) \cdot \phi(\hat{p}) = \phi(\hat{p}) \cdot ad h, \quad (2.11)$$

with $h \in \mathfrak{H}$ and $\tau : \mathfrak{H} \mapsto \mathfrak{G}$ denoting the Lie algebra homomorphism induced by λ , $\tau = d\lambda$. The pure Yang-Mills action on P reduces due to K -invariance to the following action on $\hat{P}$, see [4],

$$\mathfrak{S} = \frac{1}{\hat{g}^2} \int_{\hat{P}} \left(\langle \hat{\Omega}, \hat{\Omega} \rangle_{(1)} + \frac{1}{2} \langle D\phi, D\phi \rangle_{(2)} + V(\phi) \right) dv_{\hat{P}}, \quad (2.12)$$

where $\hat{\Omega}$ is the curvature of $\hat{\omega}$, $D\phi$ the covariant derivative of ϕ with respect to $\hat{\omega}$ and

$$V(\phi) = -\langle \mathfrak{P}(\phi), \mathfrak{P}(\phi) \rangle_{(3)} \quad (2.13)$$

with

$$\mathfrak{P}(\phi) = \frac{1}{2} ([\phi, \phi] - \phi \circ [\cdot, \cdot] \downarrow \mathfrak{M} - \tau \circ [\cdot, \cdot] \downarrow \mathfrak{H}). \quad (2.14)$$

Moreover, $\langle \cdot, \cdot \rangle_{(i)}$ denote the scalar products in the spaces of horizontal forms on $\hat{P}$ with values in $\mathfrak{G}$, $\mathfrak{M}^* \otimes \mathfrak{G}$ and $\Lambda^2 \mathfrak{M}^* \otimes \mathfrak{G}$ respectively. Let (e_k) be a basis of $\mathfrak{M}$, orthonormal with respect to γ . Then

$$V(\phi) = - \sum_{k,l} \langle F_{kl}, F_{kl} \rangle, \quad (2.15)$$

with

$$F_{kl} = [\phi(e_k), \phi(e_l)] - \phi([e_k, e_l] \downarrow \mathfrak{M}) - \tau([e_k, e_l] \downarrow \mathfrak{H}) \quad (2.16)$$

and $\langle \cdot, \cdot \rangle$ being the Ad -invariant scalar product on $\mathfrak{G}$.

We see from this formula that, formally, the potential V is of fourth order in ϕ . But ϕ is subject to the constraint (2.11) and, therefore, in order to

find V explicitly one has to solve this equation. In group-theoretical terms this equation means that ϕ is an operator intertwining the representations $Ad G \downarrow \lambda(H)$ and $Ad K \downarrow_H (\mathfrak{M}) = Ad(H)\mathfrak{M}$. Technically, it is convenient to complexify the above Lie algebras, which automatically gives the correct intertwining number. A general method for solving the constraint equations based on this technique (including a rather general group theoretical analysis of possible cases) was discussed in [5]. It turns out that in many cases the potential V is of Higgs type leading to spontaneous symmetry breaking in the reduced theory. This fact was used in many papers for constructing realistic models; for a quite exhausting list of references we refer to [5]. Most attention was paid to the construction of the Weinberg-Salam model and of models of grand unification type. The latter models could be of particular interest for present day physics, because they contain only a small number of parameters (the original coupling constant and a few parameters of purely geometrical origin). Therefore, they could give interesting predictions in the energy range of accelerators of the new generation. Interesting attempts to construct such models were made in [7], [8] and [9], we refer for further references to these papers. Nevertheless, from the viewpoint of phenomenology, until now no satisfactory solution has been found. It is quite easy to find theories with the correct Higgs and gauge field multiplets, but it is highly nontrivial to obtain the correct symmetry breaking pattern.

Let us, for example, take the standard $SU(5)$ -model, which was discussed in [7]. If one restricts oneself to regular subalgebras, then this model can be derived only from a multidimensional gauge theory with gauge group E_8 . But it turns out that the potential of the reduced theory does not contain a quartic self-interaction term for the Higgs field in the adjoint representation of $SU(5)$ necessary for the correct symmetry breaking. In view of these problems the authors of [9] invented an additional symmetry breaking mechanism based on Wilson flux lines. This, however, seems to be a serious deviation from the original idea.

Instead of it, we found it interesting to try to construct within the original CSDR-scheme the flipped $SU(5)$ -model, which is the only grand unified model not employing Higgs field in the adjoint representation of the gauge group. This model is believed to be compatible with the existing experimental data and is widely discussed in physical literature [10], [11]. We succeeded in constructing the bosonic sector of this model with correct field multiplets and self-interaction terms, provided we impose an additional Z_2 -

symmetry on the reduced theory. It turns out that the corresponding Higgs potential does lead to the correct spontaneous symmetry breaking and can explain the hierarchy of masses.

We recall that the flipped $SU(5)$ grand unified model has the gauge group $SU(5) \times U(1)$, the Higgs fields carrying its representations $\underline{10}(-1)$ and $\underline{5}(2)$ [10], [11]. The field $\underline{10}(-1)$ is the heavy Higgs field, which breaks the grand unification symmetry, whereas the field $\underline{5}(2)$ is responsible for breaking the electroweak symmetry. We found three possibilities to obtain these multiplets by the dimensional reduction method:

1. $G = E_6, \quad K/H = SO(5)/SU(3) \times U(1), \quad \dim K/H = 6$
2. $G = E_7, \quad K/H = SO(7)/SU(3) \times U(1), \quad \dim K/H = 12$
3. $G = E_8, \quad K/H = SO(9)/SU(4) \times U(1), \quad \dim K/H = 20$

It turned out that the most interesting case is the second one. To construct the corresponding model explicitly, we have to find the intertwining operator ϕ , insert it into the reduced action (2.12) and calculate the kinetic term and the potential of scalar fields. These are rather complicated calculations, for which bases of generators in $\mathfrak{K}$ and $\mathfrak{G}$ are needed, adapted to the decompositions of the representations $ad(\mathfrak{H})\mathfrak{M}$ and $Ad \mathfrak{G} \downarrow \tau(\mathfrak{H})$ into irreducible ones. We have developed a special technique for this purpose, which allowed us to construct such bases rather easily and simplified the calculations essentially. We will discuss this technique in the next section.

3 Construction of Bases Adapted to the Reduction to a Subalgebra

It is common knowledge that a semisimple subalgebra of a simple Lie algebra is defined, up to equivalence, by the decomposition into irreducible representations of the restriction to this subalgebra of the defining representation [12]. For classical Lie algebras, the defining representation coincides with the fundamental one. It is also known that the adjoint representations of classical Lie algebras can be expressed in terms of the fundamental representations by the following formulae [12]:

$$ad \, sl(n) = \underline{n}^* \otimes \underline{n}$$

$$\begin{aligned}
ad\ so(n) &= \underline{n} \wedge \underline{n} \\
ad\ sp(n) &= (\underline{2n}) \overset{s}{\otimes} (\underline{2n}),
\end{aligned} \tag{3.1}$$

where tilde means dropping out one-dimensional trivial representations, $\underline{n}^*$ denotes the contragradient representation, $t(x)^* = -t(x)^T$, and $\overset{s}{\otimes}$ denotes the symmetrized tensor product. Thus, if we know the decomposition of the defining representation, we can find the decomposition of the adjoint representation. In fact, it means that we can find the corresponding embedding explicitly. Indeed, the formulae (3.1) imply that the linear spaces $sl(n)$, $so(n)$ and $sp(n)$ can be expressed as tensor products of the representation space $R_{\underline{n}}$ of $\underline{n}$ by similar formulae. In what follows we denote the representation space of $\underline{n}^*$ by $R_{\underline{n}}^*$, which is by definition of $\underline{n}^*$ the dual of $R_{\underline{n}}$. It is possible to endow these tensor product spaces with Lie algebra structures, which makes them isomorphic to the corresponding Lie algebras. Thus, we get a tensor product representation for classical Lie algebras. If we decompose the representation space of the fundamental representation in accordance with its decomposition into irreducible representations of the subalgebra, we can construct explicitly the corresponding embedding of the subalgebra. We will show examples of this technique below. But first we discuss the tensor product representations.

Starting with $sl(n)$, we denote the pairing of elements $u \in R_{\underline{n}} \cong \mathbb{C}^n$ and $v^* \in R_{\underline{n}}^* \cong \mathbb{C}^n$ by $(v^*, u) = v^T u$. This pairing canonically induces a bilinear form on the tensor product $R_{\underline{n}}^* \otimes R_{\underline{n}}$:

$$(u^* \otimes v, w^* \otimes x) = (u^*, x)(w^*, v). \tag{3.2}$$

A Lie algebra structure in $R_{\underline{n}}^* \otimes R_{\underline{n}}$ can be introduced as follows:

$$[u^* \otimes v, w^* \otimes x] = (w^*, v) u^* \otimes x - (u^*, x) w^* \otimes v. \tag{3.3}$$

It is easy to verify that the space $R_{\underline{n}}^* \otimes R_{\underline{n}}$ with this commutator is isomorphic to the Lie algebra $gl(n)$.

Taking a basis $\{e_a\}$ in $R_{\underline{n}}$ and the dual $\{e^a\}$ in $R_{\underline{n}}^*$, we get the canonical product basis $\{e^a \otimes e_b\}$ in $R_{\underline{n}}^* \otimes R_{\underline{n}}$. Denoting $D = \sum_{a=1}^n e^a \otimes e_a$, we define the following generators of the Lie algebra $sl(n)$:

$$h_a^b = e^b \otimes e_a - \frac{D}{n} \delta_a^b, \tag{3.4}$$

with $h_a^a = 0$ (here and further on we use the Einstein summation convention). The commutation relations for them immediately follow from (3.3)

$$[h_a^b, h_c^d] = \delta_a^d h_c^b - \delta_c^b h_a^d, \quad (3.5)$$

and their normalization is given by

$$\langle h_a^b, h_c^d \rangle = \delta_a^d \delta_c^b - \frac{1}{n} \delta_c^d \delta_a^b. \quad (3.6)$$

It is clear that these elements form - after removing one diagonal generator - a non-orthogonal basis of $sl(n)$. The elements h_a^b , with $a \neq b$, are root vectors of $sl(n)$, whereas $h_a^a - h_b^b$ (without summation) are elements of the Cartan subalgebra. We also note that $\langle \cdot, \cdot \rangle$ is the canonically normalized bilinear form, because the squares of all roots are equal to 2.

Similar procedures can be carried out for orthogonal and symplectic Lie algebras. In the case of orthogonal Lie algebra $so(n)$, the defining representation $\underline{n}$ is self-conjugate (fundamental and its contragradient representation coincide) and the space $R_{\underline{n}}$ possesses a symmetric bilinear form $(\cdot, \cdot)$. If we choose a basis $\{e_a\}$ in $R_{\underline{n}}$, the generators $\{e_a \wedge e_b\}$ form a basis in $R_{\underline{n}} \wedge R_{\underline{n}} \cong so(n)$ with commutation relations

$$\begin{aligned} [e_a \wedge e_b, e_c \wedge e_d] = & \frac{1}{2} ((e_b, e_c) e_a \wedge e_d - (e_a, e_c) e_b \wedge e_d + \\ & + (e_b, e_d) e_c \wedge e_a - (e_a, e_d) e_c \wedge e_b). \end{aligned} \quad (3.7)$$

If the basis $\{e_a\}$ is orthonormal, one recovers the standard commutation relations for the generators of $so(n)$. But we emphasize that in principle it can be chosen arbitrarily, which is useful when one considers subalgebras of $so(n)$. The basis $\{e_a \wedge e_b\}$ is normalized as follows:

$$\langle e_a \wedge e_b, e_c \wedge e_d \rangle = \frac{1}{4} ((e_b, e_c) (e_a, e_d) - (e_a, e_c) (e_b, e_d)), \quad (3.8)$$

with $\langle \cdot, \cdot \rangle$ being the canonically normalized bilinear form on $so(n)$. The decomposition of the basis vectors into root vectors and Cartan subalgebra elements is not evident, but can be achieved by an appropriate choice of the basis $\{e_a\}$.

For $sp(n)$, the fundamental representation $\underline{2n}$ is self-conjugate, and the representation space $R_{\underline{2n}}$ possesses an invariant antisymmetric bilinear form

$(\cdot, \cdot)$. A basis $\{e_a\}$ in R_{2n} gives a basis $\{e_a \overset{s}{\otimes} e_b\}$ in $R_{2n} \overset{s}{\otimes} R_{2n}$, which has the commutation relations

$$\begin{aligned} [e_a \overset{s}{\otimes} e_b, e_c \overset{s}{\otimes} e_d] &= \frac{1}{2}((e_b, e_c) e_a \overset{s}{\otimes} e_d + (e_a, e_c) e_b \overset{s}{\otimes} e_d + \\ &+ (e_b, e_d) e_a \overset{s}{\otimes} e_c + (e_a, e_d) e_b \overset{s}{\otimes} e_c). \end{aligned} \quad (3.9)$$

and is normalized as follows:

$$\langle e_a \overset{s}{\otimes} e_b, e_c \overset{s}{\otimes} e_d \rangle = \frac{1}{2}((e_b, e_c)(e_d, e_a) + (e_a, e_c)(e_d, e_b)). \quad (3.10)$$

As previously, we denote by $\langle \cdot, \cdot \rangle$ the invariant bilinear form on $sp(n)$ induced by $(\cdot, \cdot)$. A decomposition of basis vectors into root vectors and Cartan subalgebra elements again can be achieved by an appropriate choice of the basis $\{e_a\}$.

The above defined commutators for all three series of algebras are obviously antisymmetric and one can easily check that they fulfil the Jacobi identity.

The representations discussed here are, of course, equivalent to the standard matrix representations for classical Lie algebras. But they are much more convenient for treating subalgebras, because the latter are defined by decompositions of the fundamental representations, which enter the tensor products.

Here we discuss in detail two examples, which will be used in the next section for constructing a model with gauge group $SU(5) \times U(1)$. As the coset space we will take $SO(7)/(SU(3) \times U(1))$. As mentioned in section 2, we need the decomposition of the isotropy representation. We define the embedding of $su(3) \oplus u(1)$ into $so(7)$ by the decomposition of the defining representation $\underline{7}$ of $so(7)$:

$$\underline{7} \rightarrow \underline{3}(-1) \oplus \underline{1}(0) \oplus \underline{3}^*(1). \quad (3.11)$$

Taking a basis $\{e_m\}$ in $R_{\underline{7}}$ adapted to this decomposition, we get

$$\begin{aligned} \{e_m\} &= \{e_a, e_0, e^b, \}, \quad a, b = 1, 2, 3 \\ (e_a, e^b) &= \delta_a^b, \quad (e_0, e_0) = 1. \end{aligned} \quad (3.12)$$

A basis of $so(7)$ adapted to the above decomposition is given by:

$$\begin{aligned} h_a^b &= 2(e^b \wedge e_a - \tfrac{1}{3}\delta_a^b e^c \wedge e_c), & h &= 2e_a \wedge e^a \\ Q_a &= 2e_0 \wedge e_a, & Q^a &= 2e^a \wedge e_0 \\ P_a &= -\epsilon_{abc} e^b \wedge e^c, & P^a &= \epsilon^{abc} e_b \wedge e_c. \end{aligned} \quad (3.13)$$

From (3.8) and (3.12) we get the following normalization conditions:

$$\begin{aligned} \langle h, h \rangle &= 3, & \langle h_a^b, h_c^d \rangle &= \delta_a^d \delta_c^b - \tfrac{1}{3}\delta_a^b \delta_c^d \\ \langle Q_a, Q^b \rangle &= \delta_a^b, & \langle P_a, P^b \rangle &= \delta_a^b. \end{aligned} \quad (3.14)$$

All other scalar products vanish. Using (3.7) and (3.13) we obtain the commutation relations for the adapted basis:

$$\begin{aligned} [h_a^b, h_c^d] &= \delta_a^d h_c^b - \delta_c^b h_a^d \\ [h, Q_a] &= Q_a, & [h, Q^a] &= -Q^a \\ [h, P_a] &= -2P_a, & [h, P^a] &= 2P^a \\ [h_a^b, Q_c] &= -\delta_c^b Q_a + \tfrac{1}{3}\delta_a^b Q_c, & [h_a^b, Q^c] &= \delta_a^c Q^b - \tfrac{1}{3}\delta_a^b Q^c \\ [h_a^b, P_c] &= -\delta_c^b P_a + \tfrac{1}{3}\delta_a^b P_c, & [h_a^b, P^c] &= \delta_a^c P^b - \tfrac{1}{3}\delta_a^b P^c \\ [Q_a, Q_b] &= -\epsilon_{abc} P^c, & [Q^a, Q^b] &= \epsilon^{abc} P_c \\ [Q_a, P_b] &= -\epsilon_{abc} Q^c, & [Q^a, P^b] &= \epsilon^{abc} Q_c \\ [Q^a, Q_b] &= h_b^a - \tfrac{1}{3}\delta_b^a h, & [P^a, P_b] &= h_b^a + \tfrac{2}{3}\delta_b^a h. \end{aligned} \quad (3.15)$$

All other commutators vanish.

In the next section we shall consider a model obtained by the dimensional reduction of a pure gauge theory on the multidimensional universe $M \times SO(7)/(SU(3) \times U(1))$ with gauge group E_7 . In order to construct - by the dimensional reduction method - a model on physical space-time M with gauge group $SU(5) \times U(1)$ we have to embed $SU(3) \times U(1)$ into E_7 in such a way that the centralizer of this group is $SU(5) \times U(1)$. Unfortunately, for exceptional Lie groups the adjoint representation either coincides with the defining one or cannot be expressed so easily in terms of the latter. Therefore, we have no tensor product representations of the type discussed in section 2. But in the case under consideration we can embed the $su(3) \oplus u(1)$ subalgebra into the regular $su(3) \oplus su(6)$ subalgebra of e_7 . Thus, we first have to take the standard embedding of $su(3) \oplus su(6)$ into e_7 and next to decompose $su(6)$

with respect to a regular embedding of the algebra $su(5) \oplus u(1)$. This way we get the embedding of $su(3) \oplus u(1)$ into e_7 with the required centralizer.

The standard decomposition of $ade_7 \downarrow su(3) \oplus su(6)$ can be found in [12] and looks as follows:

$$ade_7 \downarrow su(3) \oplus su(6) = (\underline{8}, \underline{1}) \oplus (\underline{3}, \underline{15}^*) \oplus (\underline{3}^*, \underline{15}) \oplus (\underline{1}, \underline{35}).$$

Bases in the spaces of the adjoint representations $\underline{8}$ of $su(3)$ and $\underline{35}$ of $su(6)$ can be constructed by the above tensor product technique. Observe that $\underline{15} \cong \underline{6} \wedge \underline{6}$. Thus, a basis in this space can also be constructed from a basis in $\underline{6}$. Denoting the bases in $\underline{3}$ of $su(3)$ and $\underline{6}$ of $su(6)$ by $\{e^a\}$, $(e_a, e^b) = \delta_a^b$, $a, b = 1, 2, 3$ and $\{e_i\}$, $(e_i, e^k) = \delta_i^k$, $i, k = 0, 1, \dots, 5$, we get a representation for the generators of e_7 . The canonically normalized bases of $\underline{8}$ and $\underline{35}$ are denoted by H_a^b and H_i^k , see (3.4), and have the commutation relations (3.6). The basis in the representation space of $(\underline{3}, \underline{15}^*)$ is given by

$$P_{ik}^a = e^a \otimes e_i \wedge e_k. \quad (3.16)$$

The dual basis of the representation $(\underline{3}^*, \underline{15})$ is of course

$$P_a^{ik} = e_a \otimes e^i \wedge e^k, \quad (3.17)$$

with the normalization condition

$$\langle P_{ik}^a, P_b^{lm} \rangle = \delta_b^a \delta_{ik}^{lm}, \quad (3.18)$$

where $\delta_{ik}^{lm} = \delta_i^l \delta_k^m - \delta_i^m \delta_k^l$. We get the following commutation relations for these generators:

$$\begin{aligned} [H_a^b, P_{ik}^c] &= \delta_a^c P_{ik}^b - \frac{1}{3} \delta_a^b P_{ik}^c, & [H_a^b, P_c^{ik}] &= -\delta_c^b P_a^{ik} + \frac{1}{3} \delta_a^b P_c^{ik} \\ [H_i^k, P_{lm}^a] &= -\delta_i^l P_{lm}^a - \delta_m^k P_{li}^a + \frac{1}{3} \delta_i^k P_{lm}^a \\ [H_i^k, P_a^{lm}] &= \delta_i^l P_a^{km} + \delta_i^m P_a^{lk} - \frac{1}{3} \delta_i^k P_a^{lm} \\ [P_{ik}^a, P_b^{lm}] &= H_b^a \delta_{ik}^{lm} - \delta_b^a (\delta_i^l H_k^m - \delta_i^m H_k^l + \delta_k^m H_i^l - \delta_k^l H_i^m) \\ [P_{ik}^a, P_{lm}^b] &= \frac{1}{2} \epsilon^{abc} \epsilon_{iklmnp} P_c^{np}, & [P_a^{ik}, P_b^{lm}] &= -\frac{1}{2} \epsilon_{abc} \epsilon^{iklmnp} P_{np}^c. \end{aligned} \quad (3.19)$$

Next we embed $su(5) \oplus u(1)$ into $su(6) \subset e_7$ in accordance with the decomposition of the defining representation

$$\underline{6} \longrightarrow \underline{5} \left(\frac{1}{2} \right) \oplus \underline{1} \left(-\frac{5}{2} \right)$$

for which the centralizer of $u(1)$ in $su(6)$ is $su(5) \oplus u(1)$. Thus, the required decomposition of ade_7 is

$$\begin{aligned} ade_7 \downarrow su(3) \oplus su(5) \oplus u(1) &= (\underline{8}, \underline{1})(0) \oplus \\ &\oplus (\underline{1}, \underline{24})(0) \oplus (\underline{1}, \underline{1})(0) \oplus (\underline{3}^*, \underline{10})(1) \oplus \\ &\oplus (\underline{3}, \underline{10}^*)(-1) \oplus (\underline{3}^*, \underline{5})(-2) \oplus (\underline{3}, \underline{5}^*)(2). \end{aligned} \quad (3.20)$$

Correspondingly we split the basis $\{e_i\}$ into $\{e_0, e_p\}$, $p = 1, \dots, 5$. In the same way we split the generators $\{H_i^k\}$ of $su(6)$ into $\{C_p^q, E_r, E^s, H\}$, $p, q, r, s = 1, \dots, 5$. From now on latin indices run from 1 to 5. Here

$$\begin{aligned} H &= -3H_0^0, \quad C_p^q = H_p^q - \frac{1}{15}\delta_p^q H \\ E_r &= H_r^0, \quad E^s = H_0^s. \end{aligned} \quad (3.21)$$

The generators $\{P_{ik}^a, P_b^{lm}\}$ also split into $\{P_\tau^a, P_b^s, Q_{rs}^a, Q_b^{rs}\}$, where

$$\begin{aligned} P_\tau^a &= P_{0\tau}^a, \quad P_b^s = P_b^{0s} \\ Q_{rs}^a &= P_{rs}^a, \quad Q_b^{rs} = P_b^{rs}. \end{aligned} \quad (3.22)$$

Thus, we have got a set of generators $\{H, H_a^b, C_p^q, E_r, E^s, P_\tau^a, P_b^s, Q_{rs}^a, Q_b^{rs}\}$ adapted to the decomposition (3.20). They are normalized as

$$\begin{aligned} \langle H, H \rangle &= \frac{15}{2}, \quad \langle H_a^b, H_c^d \rangle = \delta_a^d \delta_c^b - \frac{1}{3} \delta_a^b \delta_c^d \\ \langle C_i^k, C_l^m \rangle &= \delta_i^m \delta_l^k - \frac{1}{5} \delta_i^k \delta_l^m, \quad \langle E_i, E^k \rangle = \delta_i^k \\ \langle P_i^a, P_b^k \rangle &= \delta_b^a \delta_i^k, \quad \langle Q_{ik}^a, Q_b^{lm} \rangle = \delta_b^a \delta_{ik}^{lm}, \end{aligned} \quad (3.23)$$

and their commutation relations are

$$\begin{aligned} [H, E_i] &= -3E_i, \quad [H, E^i] = 3E^i \\ [H, P_i^a] &= 2P_i^a, \quad [H, P_a^i] = -2P_a^i \\ [H, Q_{ik}^a] &= Q_{ik}^a, \quad [H, Q_{ik}^a] = -Q_{ik}^a \\ [H_a^b, P_i^c] &= \delta_a^c P_i^b - \frac{1}{3} \delta_a^b P_i^c, \quad [H_a^b, P_c^i] = -\delta_c^b P_a^i + \frac{1}{3} \delta_a^b P_c^i \\ [H_a^b, Q_{ik}^c] &= \delta_a^c Q_{ik}^b - \frac{1}{3} \delta_a^b Q_{ik}^c, \quad [H_a^b, Q_c^{ik}] = -\delta_c^b Q_a^{ik} + \frac{1}{3} \delta_a^b Q_c^{ik} \\ [C_i^k, P_l^a] &= -\delta_l^k P_i^a + \frac{1}{5} \delta_i^k P_l^a, \quad [C_i^k, P_a^l] = \delta_i^l P_a^k - \frac{1}{5} \delta_i^k P_a^l \\ [C_i^k, Q_{lm}^a] &= -\delta_l^k Q_{im}^a - \delta_m^k Q_{li}^a + \frac{2}{5} \delta_i^k Q_{lm}^a \end{aligned}$$

$$\begin{aligned}
[C_i^k, Q_a^{lm}] &= \delta_i^l Q_a^{km} + \delta_i^m Q_a^{lk} - \frac{2}{5} \delta_i^k Q_a^{lm} \\
[C_i^k, E_l] &= -\delta_l^k E_i, \quad [C_i^k, E^l] = \delta_i^l E^k \\
[E^k, E_l] &= \frac{1}{3} \delta_l^k H, \quad [E^k, P_a^l] = Q_a^{kl}, \quad [E_k, P_l^a] = -Q_{kl}^a \\
[E^k, Q_{lm}^a] &= -\delta_l^k P_m^a + \delta_m^k P_l^a, \quad [E_k, Q_a^{lm}] = \delta_k^l P_a^m - \delta_k^m P_a^l \\
[P_i^a, P_b^k] &= H_b^a \delta_i^k - \delta_b^k C_i^a + \frac{4}{15} \delta_i^k \delta_b^a H \\
[P_i^a, Q_{kl}^b] &= \frac{1}{2} \epsilon^{abc} \epsilon_{iklmn} Q_c^{mn}, \quad [P_a^i, Q_b^{kl}] = -\frac{1}{2} \epsilon_{abc} \epsilon^{iklmn} Q_{mn}^c \\
[Q_{ik}^a, Q_{lm}^b] &= \epsilon^{abc} \epsilon_{iklmn} P_c^n, \quad [Q_a^{ik}, Q_b^{lm}] = -\epsilon_{abc} \epsilon^{iklmn} P_n^c \\
[Q_{ik}^a, Q_b^{lm}] &= H_b^a \delta_{ik}^{lm} - \frac{2}{15} \delta_b^a \delta_{ik}^{lm} H - \delta_b^a (\delta_i^l C_k^m - \delta_i^m C_k^l + \delta_k^m C_i^l - \delta_k^l C_i^m) \\
[Q_{ik}^a, P_b^l] &= \delta_b^a (\delta_i^l E_k - \delta_k^l E_i), \quad [Q_a^{ik}, P_l^b] = -\delta_a^b (\delta_i^l E_k - \delta_k^l E_i). \quad (3.24)
\end{aligned}$$

The commutation relations for H_a^b and C_i^k are, of course, of the form (3.7). All other commutators vanish.

4 Flipped $SU(5)$ model

Now that we have adapted bases of generators in $so(7)$ and e_7 we define the homomorphism τ , see (2.11), explicitly as

$$\tau(h) = H, \quad \tau(h_b^a) = H_b^a. \quad (4.1)$$

One can easily check that the intertwining operator ϕ , see (2.10), for this homomorphism is non-trivial. Therefore, we can introduce basic intertwining operators by the relations

$$\begin{aligned}
I^{ik}(Q_a) &= Q_a^{ik}, & I_{ik}(Q^a) &= Q_{ik}^a \\
I^i(P_a) &= P_a^i, & I_i(P^a) &= P_i^a.
\end{aligned} \quad (4.2)$$

Then the intertwining operator ϕ (2.10) can be expressed as

$$\phi = \varphi^i(x) I_i + \frac{1}{2} \beta^{ik}(x) I_{ik} + c.c., \quad (4.3)$$

$\varphi^i(x)$ and $\beta^{ik}(x) = -\beta^{ki}(x)$ being unconstrained scalar fields.

To calculate the potential V (2.15) explicitly, we need an $SO(7)$ -invariant metric on $SO(7)/(SU(3) \times U(1))$. This metric is two-parametric. We denote by M the parameter corresponding to the subspace $\mathfrak{M}_1$ of the representation

$\underline{3}^*(1) \oplus \underline{3}(-1)$ in $\mathfrak{M}$, and by $M\kappa$ the parameter corresponding to the subspace $\mathfrak{M}_2$ of the representation $\underline{3}(2) + \underline{3}^*(-2)$ in $\mathfrak{M}$. The parameter M has the dimension of mass, whereas the parameter κ is dimensionless. Then the invariant metric at the origin of $SO(7)/(SU(3) \times U(1))$ is given by

$$\gamma(a, b) = -\frac{1}{M^2} (\langle a_1, b_1 \rangle + \frac{1}{\kappa^2} \langle a_2, b_2 \rangle), \quad (4.4)$$

with $a_i, b_i \in \mathfrak{M}_i$. A non-compact basis of $\mathfrak{M}$ orthonormal with respect to γ is

$$\{MQ_a, MQ^b, M\kappa P_a, M\kappa P^b\}. \quad (4.5)$$

If we substitute the operator ϕ , see (4.3), and this basis into formula (2.15), we can calculate the potential explicitly. A straightforward calculation gives the following result:

$$\begin{aligned} V = & 36M^4(\beta^2)^2 - 6M^4 Tr \beta^4 - 12M^4(3 - 2\kappa^2)\beta^2 + \\ & + 24M^4\kappa^4(\varphi^2)^2 - 12M^4(4\kappa^4 - 1)\varphi^2 + 24M^4\kappa^2\beta^2\varphi^2 - \\ & - 12M^4\kappa^2(\beta\varphi)^2 - 6M^4(1 + 2\kappa^2)Re\varphi(\beta \times \beta) + 3M^4(7 + 12\kappa^4) \end{aligned} \quad (4.6)$$

where

$$\begin{aligned} \beta^2 &= \frac{1}{2}\beta_{ik}\beta^{ik}, \quad Tr \beta^4 = \beta_{pr}\beta_{qs}\beta^{pq}\beta^{rs}, \quad \varphi^2 = \varphi_n\varphi^n, \\ \varphi(\beta \times \beta) &= \beta_{ik}\beta_{lm}\varphi_n\epsilon^{iklmn}, \quad (\beta\varphi)^2 = \beta_{li}\beta^{lk}\varphi_k\varphi^i. \end{aligned} \quad (4.7)$$

This is the most general potential in the sense that it includes all the possible invariants up to the fourth order in φ and β . A peculiar feature of it is the presence of the cubic term $\varphi(\beta \times \beta)$. Standard Higgs potentials of grand unified models contain only even powers of the Higgs fields, because it is customary to assume in grand unified model building that the Higgs potential is invariant under the change of sign of the Higgs fields [13]. We also make this assumption and introduce this additional Z_2 -symmetry into the reduced theory. Unfortunately, at the moment we are not able to implement this symmetry geometrically. This problem deserves a special study and will be discussed elsewhere.

The additional symmetry having been imposed, the term $\varphi(\beta \times \beta)$ in (4.6) drops out, and we get a canonical Higgs potential

$$\begin{aligned} V = & 36M^4(\beta^2)^2 - 6M^4 Tr \beta^4 - 12M^4(3 - 2\kappa^2)\beta^2 + \\ & + 24M^4\kappa^4(\varphi^2)^2 - 12M^4(4\kappa^4 - 1)\varphi^2 + 24M^4\kappa^2\beta^2\varphi^2 - \\ & - 12M^4\kappa^2(\beta\varphi)^2 + 3M^4(7 + 12\kappa^4). \end{aligned} \quad (4.8)$$

It is worth mentioning at this point that the potentials for the gauge groups E_6 and E_8 differ from this one in several aspects. First of all, they have different factors in front of the invariants. Moreover, in the case of the group E_6 , the term with $Tr\beta^4$ is absent, whereas in the case of the group E_8 there is no $(\beta\varphi)^2$ term. It turns out that these terms are important for spontaneous symmetry breaking.

To investigate the extrema of the potential, it is necessary to find a convenient gauge. One can show that there exists a gauge in which

$$\begin{aligned} \beta^{12} = -\beta^{21} = a, \quad \beta^{34} = -\beta^{43} = b, \quad \text{all other components vanish,} \\ \varphi^1 = R, \quad \varphi^3 = r, \quad \varphi^5 = \alpha, \quad \varphi^2 = \varphi^4 = 0, \quad \alpha, a, b, r, R \in \mathbb{R}. \end{aligned} \quad (4.9)$$

The values of the invariants in this gauge are

$$\begin{aligned} \beta^2 = a^2 + b^2, \quad Tr\beta^4 = 2(a^4 + b^4), \\ \varphi^2 = R^2 + r^2 + \alpha^2, \quad (\beta\varphi)^2 = a^2 R^2 + b^2 r^2. \end{aligned} \quad (4.10)$$

If we insert these values into the potential, we get a function of five variables, the extrema of which we have to find. In so doing we are interested in the extremum for which

$$\alpha = b = r = 0, \quad a^2 \neq 0, \quad R^2 \neq 0, \quad (4.11)$$

because it gives the group $SU(3) \times U(1)$ of unbroken symmetry. There are two methods to look for the extrema of the potential: a simplified one, which corresponds to the physical picture of two stages of symmetry breaking, and a rigorous one.

The simplified physical prescription for finding the extrema is the following: first we neglect the light Higgs field φ and look for the extrema of the potential of the heavy Higgs field β . Then we look for the extrema of the potential of the field φ when the field β is in the Higgs vacuum. This procedure gives the absolute minimum of the potential (4.8) with the required group of the unbroken symmetry for $\frac{1}{2} < \kappa^2 < \frac{3}{2}$, whereas in the cases of the gauge groups E_6 and E_8 it shows that there are no minima with the required group of the unbroken symmetry. This is due to the fact that two crucial terms are missing in the corresponding potentials.

A rigorous analysis gives a more accurate result. It shows that for

$$\frac{1 + \sqrt{33}}{12} < \kappa^2 < \frac{3 + \sqrt{12}}{6} \quad (4.12)$$

the extremum with the $SU(3) \times U(1)$ group of unbroken symmetry is the absolute minimum of the potential (4.8).

To characterize the obtained theory precisely, we have to transform the reduced action (2.12) to the canonical form. To this end we split the $SU(5) \times U(1)$ gauge field into non-abelian and abelian components A_μ and A'_μ , redefine the coupling constants

$$g^2 = \frac{5\hat{g}^2}{4}, \quad g_A^2 = \frac{\hat{g}^2}{30} \quad (4.13)$$

and rescale the Higgs fields according to the formulae

$$\frac{M}{g}\sqrt{15}\beta \mapsto \beta, \quad \frac{M\kappa}{g}\sqrt{15}\varphi \mapsto \varphi. \quad (4.14)$$

The reduced Lagrangian takes the form

$$\mathcal{L} = \frac{1}{8}Tr F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - \overline{D_\mu\beta}D^\mu\beta - \overline{D_\mu\varphi}D^\mu\varphi - V(\beta, \varphi), \quad (4.15)$$

where

$$\begin{aligned} (D^{m\mu}\beta)^{ik} &= \partial_\mu\beta^{ik} + ig_A A'_\mu\beta^{ik} + gA_\mu^a T(a)_{lm}^{ik}\beta^{lm} \\ (D^{m\mu}\varphi)^k &= \partial_\mu\varphi^k - 2ig_A A'_\mu\varphi^k + gA_\mu^a T(a)_i^k\varphi^i, \end{aligned} \quad (4.16)$$

$T(a)_{lm}^{ik}$ and $T(a)_i^k$, $a = 1, \dots, 24$, being the generators of $SU(5)$ in the representations 10 and 5. The ratio of the coupling constants is given by

$$\frac{g_A^2}{g^2} = \frac{2}{75} \quad (4.17)$$

and the potential is

$$\begin{aligned} V &= \frac{g^2}{5}(\beta^2)^2 - \frac{g^2}{30}Tr\beta^4 - M^2(3 - 2\kappa^2)\beta^2 + \frac{2g^2}{15}(\varphi^2)^2 - \\ &- M^2\frac{4\kappa^4-1}{\kappa^2}\varphi^2 + \frac{2g^2}{15}\beta^2\varphi^2 - \frac{g^2}{15}(\beta\varphi)^2 + \frac{15}{4g^2}M^4(7 + 12\kappa^4). \end{aligned} \quad (4.18)$$

It is worth mentioning that the constructed bosonic sector, in contrast to the models built by hand, has only three free parameters which can be

chosen to fit the experimental data. For example, we get the following values for the masses of the Higgs fields:

$$\begin{aligned} m_\beta^2 &= 16M^2 \frac{12\kappa^2 - 12\kappa^4 + 1}{\kappa^2}, \\ m_\varphi^2 &= 16M^2 \frac{18\kappa^4 - 3\kappa^2 - 4}{\kappa^2}. \end{aligned} \quad (4.19)$$

It is convenient to rewrite these formulae, introducing the ratio of masses $\mu^2 = \frac{m_\varphi^2}{m_\beta^2}$. Since $\mu^2 \ll 1$, we get

$$\kappa^2 = \frac{1 + \sqrt{33}}{12} + \frac{5(\sqrt{33} - 1)}{18\sqrt{33}} \mu^2.$$

Substituting this into (4.19), we get the following expressions for the masses

$$\begin{aligned} m_\beta^2 &= 10M^2(17 - \sqrt{33}), \\ m_\varphi^2 &= 10M^2(17 - \sqrt{33})\mu^2. \end{aligned}$$

Thus, we see that the model is capable of fitting any light Higgs mass m_φ and allows the hierarchy of masses, if $\kappa^2 \simeq \frac{1+\sqrt{33}}{12}$. We would like to emphasize that this fact is not trivial in the dimensional reduction method, because the particle masses are fixed by the geometry. There would have been no hierarchy of masses, if the polynomial in the expression for m_φ^2 , see (4.19), had had no real roots.

The standard group theoretical technique easily leads to expressions in the tree approximation for g in terms of α and for the Weinberg angle. They are

$$\frac{g^2}{4\pi} = \frac{25}{6}\alpha, \quad \sin^2\theta_w = \frac{3}{5}.$$

The latter is only a little bit larger than the value of the Weinberg angle in the standard $SU(5)$ model $\sin^2\theta_w = \frac{3}{8}$. This value is believed to be an asymptotic one, the observed low energy value being much smaller.

We cannot fail to make one more remark. As far as we know, our model is the only grand unified model built by the dimensional reduction method, which leads to the correct spontaneous symmetry breaking and allows the hierarchy of masses. Unfortunately, it turns out that it is impossible to obtain chiral fermions by the standard dimensional reduction of fermion fields of the

gauge theory with the gauge group E_7 in the space $M \times SO(7)/(SU(3) \times U(1))$ [5]. We hope that the new Z_2 -symmetry introduced in this paper can improve the situation, but this problem deserves a separate investigation.

Acknowledgments

The authors are indebted to O. Richter for checking their calculations using a computer algebra programme. One of the authors (I.V.) is grateful to the Center of Natural Sciences of the University of Leipzig for the warm hospitality extended to him during his stay in Leipzig.

References

- [1] P. Forgacs and N. Manton, Commun. Math. Phys. 72 (1980) 15
- [2] J. Harnad, S. Shnider and J. Tafel, Lett. Math. Phys. 4 (1980) 107
- [3] G. Rudolph, Lett. Math. Phys. 14 (1987) 133
- [4] G. Rudolph and I. P. Volobuev, Teor. Mat. Fiz. 62 (1985) 388
- [5] Yu. A. Kubyshin, J. M. Mourão, G. Rudolph and I. P. Volobuev, Dimensional Reduction of Gauge Theories, Spontaneous Compactification and Model Building, Springer Lecture Notes in Physics 349 (1989)
- [6] D. Kapetanakis and G. Zoupanos, Phys. Lett. B232 (1989), 104
- [7] G. Chapline, N. S. Manton, Nucl. Phys. B184 (1981) 39
- [8] F. A. Bais, K. J. Barnes, P. Forgacs, G. Zoupanos, Nucl. Phys. B263 (1986) 557
- [9] D. Kapetanakis, G. Zoupanos, Unification in Higher Dimensions, preprint CERN-TH 5641/90
- [10] S.M. Barr, Phys. Lett. B112 (1982) 219
- [11] I. Antoniadis, J. Ellis, J. Hagelin, D.V. Nanopoulos, Phys. Lett. B208 (1988) 209; B231 (1989) 65

- [12] E. B. Dynkin, Trudy Mosk. Matem. Obschestva (1952), Vol.1, 39
- [13] P.Langacker, Phys. Rep.(Phys. Lett. Sect. C) v.72 (1981) 185 - 385