ABSTRACT

We investigate threshold pion photoproduction in the framework of heavy baryon chiral perturbation theory. We give the expansion of the electric dipole amplitude $E_{0+}$ to three orders in $\mu$, the ratio of the pion to nucleon mass, and show that it is slowly converging. We argue that this observable is not a good testing ground for the chiral dynamics of QCD. In contrast, we exhibit new and fastly converging low-energy theorems in the P-waves which should be used to constrain the data analysis. We also discuss the importance of polarization observables to accurately pin down certain multipoles and give predictions for the reaction $\gamma n \to \pi^0 n$. 
I. INTRODUCTION

Over the last few years, much interest has been focused on pion photoproduction off nucleons. In particular, new accurate data for the processes $\gamma p \to \pi^0 p$ and $\gamma^* p \to \pi^0 p$ close to production threshold have become available [1,2]. These have led to many theoretical investigations. In particular, in refs.[3] baryon chiral perturbation theory was used to give a model-independent description of the pertinent differential cross sections, amplitudes and so on. In these papers, the nucleons were treated as fully relativistic fields which leads to complications in the power counting underlying the effective field theory [4]. These can be overcome by a clever choice of the spin-1/2 fields being velocity-dependent which allows to transform the baryon mass term into a string of $1/m$ (here, $m$ denotes the nucleon mass) suppressed interactions [5,6]. In this formulation, called heavy baryon chiral perturbation theory (HBCHPT), there is a strict correspondence between the loop expansion and the expansion in small momenta and quark masses. Our motivation to come back to the topic of threshold pion photoproduction off protons and neutrons is fivefold. First, the calculation in the relativistic framework indicated that the expansion of the electric dipole amplitude $E_{0+}$ is indeed slowly converging as a function of $\mu = M_\pi/m$, with $M_\pi$ denoting the pion mass. However, in that approach the complete expression of order $\mu^3$ could not be given. In HBCHPT, if one calculates within the one-loop approximation but to next-to-next-to-leading order $q^4$ ($q$ denotes any small momentum or mass), the first three terms in the chiral expansion of $E_{0+}$ can be given. Second, in the relativistic formulation [3] we found that the differential cross sections were not well described due to an essentially energy-independent $E_{0+}$ of $-1.3 \cdot 10^{-3}/M_\pi$. Also, in the $P$-waves the relative strength of the $M_{1+}$ multipole came out too large as compared to the $M_{1+}$. Here, we will show that two of the $P$-waves are severely constrained by novel low-energy theorems and that the third one is completely dominated by the $\Delta(1232)$ resonance. These LETs were implicitly contained in the relativistic calculation but not made explicit. Furthermore, we give here a much better estimation of the pertinent low-energy constants based on the idea of resonance saturation. This in turn leads to a satisfactory description of the existing data in the threshold region. Third, it is obvious that the information gained from the total and differential cross sections is not sufficient to pin down all multipoles uniquely. For doing that, one has to consider polarization observables and we will discuss some of these here. Last, but not least, we also show detailed predictions for the reaction $\gamma n \to \pi^0 n$ which is experimentally very difficult to access but in fact has to be studied for various reasons, one of them being the test of the isospin decomposition based on first order electromagnetism which is usually assumed in the construction of
the invariant amplitudes. Finally, new data for $\gamma p \rightarrow \pi^0 p$ in the threshold region and above taken at MAMI (Mainz) and SAL (Saskatoon) are presently being analyzed.

The manuscript is organized as follows. In section 2, we fix our notation and define the pertinent observables to be discussed later. In section 3, we briefly discuss the effective pion–nucleon Lagrangian underlying our calculation. Sections 4 and 5 contain the main results of this paper. We present the order $q^4$ calculation for the S–wave including some isospin–breaking from the pion mass difference and the corresponding $O(q^3)$ calculation for the P–waves.\(^1\) We then discuss the low–energy theorems (LETs) in the S– and P–waves followed by the presentation and discussion of the numerical results. There are two low–energy constants entering the expression for $E_{0+}(\omega)$ (with $\omega$ the pion cms energy). These can either be fixed by a best fit to the available data or estimated by resonance exchange. Already here we would like to stress that the first method leads to unnaturally large numbers for these coefficients which are a reflection of the importance of higher loops not yet calculated. Consequently, even to this order there remains some appreciable theoretical uncertainty leading to the conclusion that the electric dipole amplitude is in fact not the best testing ground of the chiral dynamics of QCD as was long believed. We also present a two–parameter model for $E_{0+}(\omega)$ which simulates most of the physics in the threshold region. Furthermore, the imaginary part of $E_{0+}$ and its relation to the Watson final state theorem is discussed showing again that to this order the description of the S–wave is not too accurate. However, we also stress that the presently available determinations of $E_{0+}$ close to threshold hinge on a few empirical points because the P–waves quickly dominate the cross section. Therefore, it is imperative to study polarization observables. These allow for a clean separation of certain multipoles and a much more accurate empirical determination of small multipoles like e.g. $E_{1+}$. The summary and outlook is given in section 6 and some lengthy formulae are collected in the appendix.

II. THRESHOLD PION PHOTOPRODUCTION: FORMAL ASPECTS

In this section, we will give the formalism necessary to treat pion photoproduction in the threshold region. We will only be concerned with the kinematics close to threshold and the corresponding multipoles. We also summarize the formulae for the differential and the total cross sections as well as for some polarization observables.

Consider the process $\gamma(k) + N(p_1) \rightarrow \pi^0(q) + N(p_2)$, with $N$ denoting the nucleon (proton or neutron), $\gamma$ a real ($k^2 = 0$) photon and $\pi^0$ the neutral pion. The polarization

\(^1\) Due to the fast convergence in these multipoles, a more accurate calculation does not seem necessary.
vector of the photon is denoted by $\epsilon_\mu$. In the threshold region, the three-momentum $\hat{q}$ of the pion in the $\pi N$ centre-of-mass (cm) frame is small and vanishes at threshold. It is therefore advantageous to perform a multipole decomposition since at threshold only the S-wave survives and close to threshold one can confine oneself to S- and P-waves. The corresponding multipoles are called $(E, M)_{l \pm}$, where $E, M$ stands for electric and magnetic, $l = 0, 1, 2, \ldots$ the pion orbital angular momentum and the $\pm$ refers to the total angular momentum of the pion-nucleon system, $j = l \pm 1/2$. These multipoles parametrize the structure of the nucleon as probed with low energy photons. Consequently, the T-matrix $T$ depends on four multipoles and takes the following form in the cm system

$$
\frac{m}{4\pi\sqrt{s}} T \cdot \epsilon = i \hat{\sigma} \cdot \hat{e}(E_{0+} + \hat{k} \cdot \hat{q}P_1) + i \hat{\sigma} \cdot \hat{k} \epsilon \cdot \hat{q} P_2 + (\hat{q} \times \hat{k}) \cdot \hat{e}P_3
$$

(2.1)

The quantities $P_{1,2,3}$ represent the following combinations of the three $P$-waves, $E_+, M_+$ and $M_-$,

$$
P_1 = 3E_+ + M_+ - M_-
$$

$$
P_2 = 3E_+ - M_+ + M_-
$$

$$
P_3 = 2M_+ + M_-
$$

(2.2)

These four amplitudes are calculable within CHPT. As will become clear later, the particular choice (2.2) of the $P$-waves is best suited for the chiral expansion and the physics related to it.

We now discuss briefly the kinematics for $\gamma p \rightarrow \pi^0 p$. The pion energy in the cm system is given by

$$
\omega = \frac{s - m_p^2 + M_{\pi^0}^2}{2\sqrt{s}} = \frac{E_\gamma + M_{\pi^0}^2/2m_p}{\sqrt{1 + 2E_\gamma/m_p}}
$$

(2.3)

with $s$ the cm energy squared and $E_\gamma = (s - m_p^2)/(2m_p)$ the photon energy in the lab frame. At threshold, $s_{\text{thr}} = (m_p + M_{\pi^0})^2$ and $\omega_{\text{thr}} = \omega_0 = M_{\pi^0}$. The second threshold is related to the opening of the $\pi^+ n$ channel at $\omega_c = 140.11$ MeV (since $s_c = (m_n + M_{\pi^+})^2$).

In what follows, we will take into account the pion mass difference $M_{\pi^+} - M_{\pi^0} = 4.6$ MeV since it subsumes the most important isospin-breaking effects. This is discussed in some detail in ref.[7]. However, we do not differentiate between the proton and the neutron mass in the loops. Consequently, we will use $M_{\pi^\pm} = \omega_c = 140.11$ MeV to account for the proper location of the second threshold. The tiny error induced by this procedure is well within the theoretical uncertainty of our approach.

The differential cross section can be written as

$$
\frac{|\hat{k}|}{|\hat{q}|} \frac{d\sigma}{d\Omega_{\text{cm}}} = A + B \cos \theta + C \cos^2 \theta
$$

(2.4)
in the approximation that only S- and P-waves contribute. \( \theta \) is the cms scattering angle, \( |\vec{k}| = (s - m_p^2)/(2\sqrt{s}) \) and \( |\vec{q}| = \sqrt{\omega^2 - M_{\pi^0}^2} \). The energy-dependent coefficients \( A, B \) and \( C \) are related to the multipoles via

\[
A = |E_{0+}|^2 + \frac{1}{2} |P_2|^2 + \frac{1}{2} |P_3|^2 \\
B = 2 \text{Re}(E_{0+}P_1^*) \\
C = |P_1|^2 - \frac{1}{2} |P_2|^2 - \frac{1}{2} |P_3|^2
\]  

These are real for \( \omega_0 \leq \omega \leq \omega_c \) and complex above \( \omega_c \) (we do not consider here the tiny phase related to the direct \( \pi^0 p \) scattering process [8] because it only shows up at two-loop accuracy). The total cross section follows as

\[
\frac{|\vec{k}|}{|\vec{q}|} \sigma_{\text{tot}} = 4\pi \left( A + \frac{1}{3} C \right) .
\]  

From the nearly forward-backward symmetric angular distributions exhibited by the Mainz data, one can immediately conclude that \( |A|, |C| \gg |B| \) which means that a very accurate knowledge of the P-waves is mandatory to reliably extract the electric dipole amplitude. Therefore, different assumptions on the P-waves can lead to a rather different energy variation of the electric dipole amplitude \( E_{0+}(\omega) \) in the threshold region [9]. Also, in most analysis it is assumed that \( E_{1+} = 0 \). To get a handle on such small multipoles and to allow for a clean separation of the various real and imaginary parts, one has to investigate polarization observables. We will consider here the polarized photon asymmetry \( \Sigma(\theta) \), the polarized target asymmetry \( T(\theta) \) and the recoil polarization \( P(\theta) \). These are given by

\[
\Sigma(\theta) = \Gamma \sin \theta \left( |P_3|^2 - |P_2|^2 \right) \\
T(\theta) = 2\Gamma \text{Im} ((E_{0+} + \cos \theta P_1)(P_3 - P_2)^*) \\
P(\theta) = 2\Gamma \text{Im} ((E_{0+} + \cos \theta P_1)(P_2 + P_3)^*)
\]  

with

\[
\Gamma = \frac{|\vec{q}| \sin \theta}{2|\vec{k}|} \left( \frac{d\sigma}{d\Omega} \right)^{-1}_{\text{cm}} .
\]  

In fact, \( P(\theta) \) and \( T(\theta) \) allow for a direct determination of \( \text{Im} E_{0+} \) above \( \omega_c \) since the P-waves are essentially real in the threshold region (as we will show later). This concludes the necessary formalism.
III. EFFECTIVE LAGRANGIAN

In this section, we will briefly discuss the chiral effective Lagrangian underlying our calculation. To explore in a systematic fashion the consequences of spontaneous and explicit chiral symmetry breaking of QCD, we make use of baryon chiral perturbation theory (in the heavy mass formulation) (HBCCHPT). The nucleon mass is considered large compared to typical momenta in the system. This allows to decompose the nucleon Dirac spinor into "large" (\(H\)) and "small" (\(h\)) components \(\Psi(x) = e^{-imv \cdot x} \{H(x) + h(x)\}\) with \(v\) the nucleon four-velocity, \(v^2 = 1\), and the velocity eigenfields are defined via \(\gamma H = H\) and \(\gamma h = -h\). Eliminating the "small" component field \(h\) (which generates \(1/m^2\) corrections), the leading order chiral \(\pi N\) Lagrangian reads

\[
\mathcal{L}^{(1)}_{\pi N} = \bar{H}(iv \cdot D + g_A S \cdot u)H
\]

(3.1)

where the pions are collected in a SU(2) matrix-valued field \(U(x)\)

\[
U(x) = \frac{1}{F} \left[ \sqrt{F^2 - \vec{\pi}(x)^2} + i \vec{\pi}(x) \right]
\]

(3.2)

with \(F\) the pion decay constant in the chiral limit and the so-called \(\sigma\)-model gauge has been chosen which is of particular convenience for our calculations in the nucleon sector. In eq.(3.1) \(D_\mu = \partial_\mu + \Gamma_\mu\) denotes the nucleon chiral covariant derivative, \(S_\mu\) is a covariant generalization of the Pauli spin vector, \(g_A \approx 1.26\) the nucleon axial vector coupling constant (formally the one in the chiral limit) and \(u_\mu = iu^\dagger \nabla_\mu U u^\dagger\), with \(u = \sqrt{U}\) and \(\nabla_\mu\) the covariant derivative acting on the pion fields. To leading order, \(O(q)\) one has to calculate tree diagrams from

\[
\mathcal{L}^{(1)}_{\pi N} + \frac{F^2}{4} \text{Tr}\left\{ \nabla^\mu U \nabla_\mu U^\dagger + \chi_+ \right\} , \quad \chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u
\]

(3.3)

where the second term is the lowest order mesonic chiral effective Lagrangian, the non-linear \(\sigma\)-model coupled to external sources. The scalar source \(\chi\) is proportional to the quark mass matrix \(M\). Beyond leading order, the effective Lagrangian takes the form

\[
\mathcal{L}_{\text{eff}} = \mathcal{L}^{(1)}_{\pi N} + \mathcal{L}^{(2)}_{\pi N} + \mathcal{L}^{(3)}_{\pi N} + \mathcal{L}^{(4)}_{\pi N} + \mathcal{L}^{(2,4)}_{\pi N}
\]

(3.4)

\(\mathcal{L}^{(2,3,4)}_{\pi N}\) contain \(1/m^2\) corrections and counterterms. The a priori unknown coefficients of these counterterms are the so-called low energy constants. For the calculation to order \(q^4\), one has to consider tree diagrams with insertions from \(\mathcal{L}^{(2,3,4)}_{\pi N}\) as well as one-loop
diagrams with insertions from $\mathcal{L}^{(1,2)}_{\pi N}$. The terms from $\mathcal{L}^{(2)}_{\pi N}$ of relevance to the problem at hand are

$$\mathcal{L}^{(2)}_{\pi N} = \hat{H}\left\{-\frac{1}{2m} D \cdot D + \frac{1}{2m} (v \cdot D)^2 - \frac{i g_A}{2m} \{S \cdot D, v \cdot u\} - \frac{i}{4m} [S^\mu, S^\nu][(1 + \mathcal{K}_v) f^+_{\mu\nu} + \frac{\mathcal{K}_S - \mathcal{K}_V}{2} \text{Tr} f^+_{\mu\nu}]\right\} H$$

where $f^+_{\mu\nu} = \epsilon(u^\dagger Q u + u Q u^\dagger) F_{\mu\nu}$, $Q = (1 + \tau_3)/2$ and $F_{\mu\nu}$ is the canonical photon field strength tensor. Note that $\mathcal{K}_S, \mathcal{K}_V$, the isoscalar and isovector anomalous magnetic moments of the nucleon in the chiral limit, are combinations of the low-energy constants $c_6$ and $c_7$ discussed in [7]. We note here that the other low-energy constants of order $q^2$, which are called $c_{1,2,3,4}$, do not contribute at all to our final results for the chiral expansion of the S- and P-waves (to the order we are working). There are two terms from $\mathcal{L}^{(3)}_{\pi N}$ which enter [10],

$$\mathcal{L}^{(3)}_{\pi N} = \hat{H}\left\{b_{14} v^\lambda \epsilon^{3 \mu \nu \rho} \text{Tr} (f^+_{\mu\nu} u_\rho) + b_{22} (i v^\lambda \epsilon^{3 \mu \nu \rho} S_\rho f^V_{\mu\nu} v \cdot D + \text{h.c.})\right\} H + \ldots$$

with $f^V_{\mu\nu} = f_{\mu\nu} - (1/2) \text{Tr} f_{\mu\nu}$. Although the coefficients $b_{14}$ and $b_{22}$ are infinite [10], in the case at hand only the finite combination $b_P \sim 4 b_{14} + \mathcal{O}(q^2) b_{22}$ contributes and the low-energy constant $b_P$ enters the P-wave $P_3$ in the following way:

$$P_3^\omega(\omega) = \epsilon b_P \omega \left|\frac{q}{\l}\right|.$$  

Note that for the other multipoles $E_{0^+}$ and $P_{1,2}$ one can not construct any gauge and Lorentz invariant chirally symmetric counter term at $\mathcal{O}(q^3)$ respecting the discrete symmetries P, C, and T. Finally, the minimal set of terms from $\mathcal{L}^{(4)}_{\pi N}$ which give a contribution to $E_{0^+}(\omega)$ (for the proton) is of the form

$$\mathcal{L}^{(4)}_{\pi N} = \hat{H}\left\{-4\pi F a_2 S^{\mu} v^\nu f^+_{\mu\nu} (v \cdot D v \cdot u) - 2\pi i F a_1 S^{\mu} v^\nu f^+_{\mu\nu} \chi-\right\} H + \ldots$$

and show up in $E_{0^+}$ in the following way,

$$E_{0^+}^\omega(\omega) = \epsilon a_1(\lambda) \omega M_\pi^2 + \epsilon a_2(\lambda) \omega^3.$$  

These counter terms are necessary to absorb the divergences generated by the loops at order $q^4$. Here, $\lambda$ is the scale introduced via the dimensional regularization. Ideally, this scale dependence is cancelled from the one in the corresponding loop contribution. If one, however, estimates the contact terms via resonance saturation, one is left with
a small but spurious scale dependence as detailed in ref. [13]. We also note that the low-energy constants $b_P$, $a_1$ and $a_2$ are a priori different for the proton and the neutron. The dots in eqs. (3.6,3.8) stand for the respective chiral vertices accompanied by the low-energy constants. We omit the superscript $'$(p,n)$'$ here since from the context it is obvious which reaction we are considering. There are furthermore many terms in $\mathcal{L}^{(3,4)}_{\pi N}$ which come from the $1/m$-expansion of the relativistic $\pi N$ Dirac Lagrangian. These generate the $1/m$-expanded Born contributions to the S- and P-waves given in the appendix. Finally, we note that we will work in the Coulomb gauge $\epsilon_0 = 0$ since in that gauge one has not direct lowest-order photon-nucleon coupling (which is proportional to $\epsilon \cdot v$) and thus many diagrams vanish.

**IV. QCD ANALYSIS**

In this section, we briefly outline the $\mathcal{O}(q^3)$ calculations for the P-waves and the $\mathcal{O}(q^4)$ for $E_{0+}$. The explicit expressions are relegated to the appendix. Next, we estimate the low-energy constants which enter the expressions for $E_{0+}$ and $P_3$. We then turn to the discussion of the low-energy theorem (LET) for $E_{0+}$ and exhibit novel LETs for the P-waves. This constitutes one of the major results of this investigation. We end this section with a short discussion of the imaginary part of $E_{0+}$ and its relation to the Fermi-Watson theorem.

**IV.1. CHIRAL EXPANSION TO ORDER $\mathcal{O}(q^3)$**

We wish to calculate the T-matrix elements to order $q^3$. Since the photon polarization vector counts as $\mathcal{O}(q)$, one gets the S- and P-wave multipoles with an accuracy of $\mathcal{O}(q^2)$. In the Coulomb gauge, we have tree contributions from $\mathcal{L}^{(2,3)}_{\pi N}$ to $E_{0+}$ and $P_i$ ($i = 1, 2, 3$) of the type

$$E_{0+} \sim \frac{\omega}{m^2} f_2(\frac{\omega}{M_\pi}), \quad \frac{\omega^2}{m^2} f_3(\frac{\omega}{M_\pi})$$

$$P_i \sim \sqrt{\frac{q_i^2}{m^2}}, \quad \sqrt{\frac{q_i}{m^3}} g_i(\frac{\omega}{M_\pi})$$

(4.1)

where $f_2$, $f_3$ and $g_i$ are dimensionless functions of their arguments. In the one-loop diagrams, we include the pion mass difference to account for the most important isospin-breaking effect. $E_{0+}(\omega)$ is given by the triangle and rescattering diagrams (see fig.1a,b) and the other two diagrams shown in that figure contribute to $P_{1,2}(\omega)$. To this order, there is no loop contribution to $P_3(\omega)$. The pion mass difference in the loops leads to the cusp in $E_{0+}$ of the square-root type

$$E_{0+}(\omega) = \text{coefficient} \cdot \sqrt{1 - \frac{\omega^2}{\omega_c^2} + \ldots}$$

(4.2)
where the ellipsis stands for polynomial pieces (in $\omega$). We will come back to the coefficient multiplying the square root when we discuss the final state theorem. At this order, there is no mass and coupling constant renormalization due to the vanishing of the tree couplings. Finally, there are also finite renormalization terms at this order. The anomalous magnetic moment of the proton (neutron) enters the expressions for $E_{0+}$ and $P_{1,2}$. In addition, there is the finite coefficient $b_P$ defined in eq. (3.7) contributing to $P_3$. The estimate of its numerical value will be given in section 4.3.

**IV.2. CHIRAL EXPANSION TO ORDER $\mathcal{O}(q^4)$**

For the electric dipole amplitude, we consider one more order. This is motivated by the fact that the relativistic calculation of ref.[3] points towards the importance of higher orders. On more general grounds, we remind the reader of the well-known fact that in S-wave observables it is often mandatory to go beyond $\mathcal{O}(q^3)$. However, we should also stress already at this point that a calculation up to and including $\mathcal{O}(q^4)$ might not be sufficiently accurate. Furthermore, we do not differentiate here between the neutral and charged pion masses and use only $M_{\pi^+}$ to get the proper cusp effect at this order. In the polynomial pieces (of the loops), this inflicts a theoretical error proportional to $(\omega - M_{\pi^+})/M_{\pi^+} \sim 4\%$ which is smaller than the uncertainty from the determination of the low-energy constants. The dominant unitarity (cusp) effect is already accounted for at order $q^3$, compare eq. (4.2). At order $q^4$, there are tree diagrams which lead to terms of the order $\omega^3/m^4$ and there are many loop graphs. The latter can be categorized as follows: (i) loop graphs with one vertex from $\mathcal{L}_{\pi^0 N}^{(1)}$ and one vertex from $\mathcal{L}_{\pi^0 N}^{(2)}$, (ii) with vertices from $\mathcal{L}_{\pi^0 N}^{(1)}$ but a nucleon propagator from $\mathcal{L}_{\pi^0 N}^{(2)}$, and (iii) the $1/m$ corrections to the loops calculated at order $q^3$. Of course, many of the loop diagrams account for mass and coupling constant renormalization,

\[
(\vec{m}, \vec{g}_{\pi^0 N}, \vec{\kappa}_{\pi^0 N}, F, M) \rightarrow (m, g_{\pi^0 N}, \kappa_{\pi^0 N}, F, M) . \tag{4.3}
\]

Finally, there are the novel counterterms which contribute to $E_{0+}$ as given in eq.(3.9). Therefore, the expressions for $E_{0+}$ and $P_{1,2,3}$ calculated to order $q^3$ and $q^4$, respectively, take the generic form

\[
E_{0+}(\omega) = E_{0+}^{\text{Born}}(\omega) + E_{0+}^{q^3\text{-loop}}(\omega) + E_{0+}^{q^4\text{-loop}}(\omega) + E_{0+}^{\text{ct}}(\omega)
\]

\[
P_i(\omega) = P_i^{\text{Born}}(\omega) + P_i^{q^3\text{-loop}}(\omega) , \quad i = 1, 2
\]

\[
P_3(\omega) = P_3^{\text{Born}}(\omega) + P_3^{\text{ct}}(\omega) \tag{4.4}
\]

where

where 'Born' subsumes the nucleon–pole and anomalous magnetic moment contributions.\textsuperscript{2)} The explicit expressions for the various terms appearing in eq.\textsuperscript{4,44} can be found in the appendix. Here, we just remark that the Born and counterterm contributions are real and that the loop contributions are complex for $\omega > \omega_c$. We also point out again that the leading terms for $E_0^\pm$ and $P_{1,2}$ appear at the same chiral power, namely $\mathcal{O}(q^2)$. Indeed, the P-waves are proportional to $|\vec{q}|$, but not to $|\vec{q}|^3$ as usually assumed. However, since $|\vec{k}|$ only varies by four per cent from $\omega_{\text{thr}}$ to $\omega_c$, this makes no visible effect. Note furthermore that $P_3$ is essentially given by the contact term $P_3^c$ since the Born contribution to this multipole is very small. It is also instructive to notice that the magnetic part of the nucleon Born terms is dominated by the $M_1^+$ and $M_2^-$ multipoles with $2M_1^+ + M_1^- \simeq 0$ whereas static $\Delta$ exchange leads approximatively to $P_1 = -P_2 \simeq P_3/4$, i.e. $2M_1^+ + M_1^-$ is much larger than $M_1^+ - M_1^-$. This is, of course, a particular feature of the threshold region, further up in energy the $M_1^+$ multipole quickly becomes dominant.

\textbf{IV.3. ESTIMATION OF THE LOW-ENERGY CONSTANTS}

The most difficult task is to pin down the values of the low–energy constants (LECs) $a_1$, $a_2$ and $b_p$. We concentrate on the reaction $\gamma p \rightarrow \pi^0 p$ and mention the necessary modifications for the case $\gamma n \rightarrow \pi^0 n$ in the end of this section. We will follow two approaches here. In the first one, we will use the available total and differential cross section data to fix these coefficients. However, as we will see, this leads to unnaturally large values of the constants $a_1$ and $a_2$ because they subsume the effects of higher loops not yet calculated. Therefore and secondly, we will estimate the LECs making use of the resonance saturation principle\textsuperscript{11}. This can be formulated as follows. Consider meson resonances ($M=V, A, S, P$) and baryonic excitations ($N^* = \Delta(1232), N^*(1440), \ldots$) chirally coupled to the Goldstone bosons (collected in $U$) and the matter fields ($N$). Integrating out the meson and nucleon excitations,

\begin{equation}
\int [dM][dN^*] \exp i \int dx \hat{\mathcal{L}}_{\text{eff}}[U, M, N, N^*] = \exp i \int dx \mathcal{L}_{\text{eff}}[U, N] \tag{4.5}
\end{equation}

one is left with a string of higher dimensional operators contributing to $\mathcal{L}_{\text{eff}}[U, N]$ in a manifestly chirally invariant manner and with coefficients given entirely in terms of resonance masses and coupling constants of these resonance fields to the Goldstone bosons. A specific example for the baryon sector is discussed in ref.\textsuperscript{12}. For the case at

\textsuperscript{2)} This decomposition facilitates the comparison with the existing literature but is not a consequence of the chiral expansion.
hand, we have mesonic and baryonic contributions. As already discussed in [3], there is
t-channel vector meson exchange, here the \( \rho(770) \) and the \( \omega(782) \). These contribute to\( a_{1,2} \) as follows (\( V = \rho^0 + \omega \)),

\[
a_2^V = \frac{5}{48 \pi^3 m F_\pi^3} = 4.45 \text{ GeV}^{-1}, \quad a_1^V = -\frac{2}{5} a_2^V = -1.78 \text{ GeV}^{-1} \tag{4.6}
\]

where we have used \( g_{\rho N} = g/2 \), \( g_{\omega N} = 3g/2 \), \( M_\rho = M_\omega = \sqrt{2} g F_\pi \), \( g = 5.8545 \), \( \kappa_\rho = 6 \),\( \kappa_\omega = 0 \) and \( G_{\pi \rho \gamma} = g/(10 \pi^2 F_\pi) = (1/3) G_{\pi \omega \gamma} \) together with \( m = 938.28 \text{ MeV} \) and \( F_\pi = 93 \text{ MeV} \). Notice that the values of these coupling constants are somewhat simplified and given in part by the gauged Wess–Zumino action. However, as it will turn out, the vector meson contribution to the P-wave \( P_3 \) is rather small and in the S-wave, there are uncertainties due to higher order effects so that the accuracy given by these values is sufficient. We could as well take the vector dominance value of \( \kappa_\omega = -0.12 \) (and so on) without any noticeable change in the numerical results to be discussed later. The sign of the vector meson contribution is fixed from the sign of the triangle anomaly. The P-wave contribution is found as

\[
b_p^V = \frac{5}{64 \pi^3 F_\pi^3} = 3.13 \text{ GeV}^{-3} . \tag{4.7}
\]

The result given in eq.(4.7) is not affected by the tensor coupling in agreement with the considerations presented in ref.[3].

From the baryon sector, the by far largest contribution comes from the \( \Delta(1232) \) resonance. It is mandatory to consider the \( \Delta \) as a fully relativistic spin–3/2 field before integrating it out. Therefore, the contribution to the various low-energy constants will depend on the two \( \Delta N \gamma \) coupling constants \( g_1 \) and \( g_2 \), the \( \pi N \Delta \) coupling \( g_{\pi N \Delta} = (3/\sqrt{2}) g_{\pi N} \) and the off-shell parameters \( X, Y \) and \( Z \) (see also the discussion in ref.[13]). This procedure leads to

\[
a_1^\Delta = \frac{C g_1}{6} \left[ -\frac{2m_\Delta^2 + 2m_\Delta m + m^2}{m_\Delta - m} + 2m(2Y - Z - 2YZ) - 2m_\Delta(Y + Z + 4YZ) \right] + \frac{C g_2}{8} \left[ m(2X + 1) + m_\Delta \right] \tag{4.8a}
\]

\[
a_2^\Delta = \frac{C g_1}{12} \left[ 10m_\Delta^2 - 7m_\Delta m - 8m^2 \right] + \frac{C g_2}{8} \left[ m(2X + 1)(1 - 6Z) + 2m_\Delta(4XZ + X + Z) \right] + \frac{C g_2}{16} \left[ 2m_\Delta(4XZ + X + Z) \right] \tag{4.8b}
\]
with \( C = g_{\pi N}/(6\sqrt{2}\pi m^3 m^2) = 0.40 \text{ GeV}^{-5} \) and \( g_{\pi N} = 13.4 \). Throughout, we use the Goldberger–Treiman relation to fix \( g_A \), i.e. \( g_A = g_{\pi N}F_\pi/m \). In what follows, we will adopt two strategies. First, we keep \( g_1 = g_2 = 5 \) fixed and vary \( X, Y, Z \) in the ranges given in ref.[14] and then also allow to vary \( g_1 \) and \( g_2 \) within the ranges given in [14]. As it will turn out, the results are fairly insensitive to the variation of \( g_1 \) and \( g_2 \).

Inspection of eqs.(4.8a,b) reveals a rather large uncertainty indicted from the relatively poor knowledge of the off–shell parameters. We will come back to this when we discuss the numerical results. Furthermore, the \( \Delta \) contributes prominently to \( b_P \),

\[
b_P^\Delta = \frac{Cg_1m}{2} \left[ \frac{2m^2 + m\Delta m - m^2}{m\Delta - m} + 2m(Y + Z + 2YZ) + 2m\Delta(Y + Z + 4YZ) \right]
\]  

(4.9)

Notice that to the order we are working, \( b_P^2 \) receives no contribution proportional to \( g_2 \). To get an idea about the size of \( b_P^\Delta \), we use the static isobar model (which involves no off–shell parameters) and find

\[
b_P^{\Delta,\text{static}} = \frac{g_1g_{\pi N}(m\Delta - m)}{6\sqrt{2}\pi m^2((m\Delta - m)^2 - M^2_\pi)} = 12.3 \text{ GeV}^{-3}
\]  

(4.10)

which is considerably larger than the vector meson contribution, eq.(4.8). For the neutron, the only difference is the sign of the \( \rho^0 \)-meson contribution. We get

\[
a_i^{V,n} = \frac{1}{8} a_i^{V,p} \quad (i = 1, 2); \quad b_P^{V,n} = \frac{4}{5} b_P^{V,p}.
\]  

(4.11)

In this case, we have no data to fit these LECs and must use the resonance exchange estimates.

**IV.4. LOW–ENERGY THEOREMS (LETs) FOR THE S– AND P–WAVES**

We consider here the expansion in \( \mu = M/\mu \) of \( E_{0+}, P_1 \) and \( P_2 \) at threshold. Since \( P_3 \) is completely dominated by the contact term proportional to \( b_P \), a similar expansion for this combination of the P–wave multipoles does not make sense. First, however, let us briefly state what is meant by a LET following ref.[15]. We consider as a LET of order \( q^n \) a general prediction of CHPT to \( \mathcal{O}(q^n) \). General prediction means a strict consequence of the Standard Model depending on some low–energy constants like \( F_\pi, m, g_A, \kappa_p, \ldots \), but without any model assumption for these parameters. This gives a precise prescription for obtaining higher–order corrections to the leading order LETs which can e.g. be obtained from current algebra.
First, we study the electric dipole amplitude. For that, we work in the isospin limit \( m_u = m_d \) and to first order in the electromagnetic coupling constant. The \( \mu \)-expansion takes the form

\[
E_{0+, \text{thr}}^i = -\frac{e}{{\pi} m} \mu \{ C_1^i + \mu C_2^i + \mu^2 C_3^i + \mathcal{O}(\mu^3) \}, \quad i = p, n \tag{4.12}
\]

with

\[
C_1^p = 1, \quad C_1^n = 0
\]

\[
C_2^p = -\frac{1}{2}(3 + \kappa_p + \frac{m^2}{8 F_\pi^2}), \quad C_2^n = -\frac{1}{2}(\kappa_n - \frac{m^2}{8 F_\pi^2})
\]

\[
C_3^p = \frac{3}{4} \left( \frac{5}{2} + \kappa_p \right) - \frac{m^2}{16 \pi^2 F_\pi^2} \left[ \left( 10 + \frac{8}{3} g_A^2 \right) \ln \frac{M_\pi}{\Lambda} - g_A^2 \left( \frac{\pi^2}{4} - \frac{5 \pi}{3} + \frac{11}{9} \right) - 4(1 + \frac{5 \pi^2}{16}) \right]
\]

\[
- \frac{8 \pi m}{g_\pi} (a_1^p(\lambda) + a_2^p(\lambda))
\]

\[
C_3^n = \frac{3}{4} \kappa_n - \frac{m^2}{16 \pi^2 F_\pi^2} \left[ \left( 2 + 4 g_A^2 \right) \ln \frac{M_\pi}{\Lambda} - g_A^2 \left( \frac{\pi^2}{4} - \frac{5 \pi}{3} + \frac{11}{9} \right) + \left( \frac{4}{9} - \frac{5 \pi^2}{4} \right) \right]
\]

\[
- \frac{8 \pi m}{g_\pi} (a_1^n(\lambda) + a_2^n(\lambda)) \tag{4.12a}
\]

At present, the LETs given by (4.12) do not have too much predictive power since the only way to determine the LECs \( a_1(\lambda) \) and \( a_2(\lambda) \) are the threshold data from Mainz \( [1] \) (for the proton). Using the best fit values (see section 5), one finds \( a_1^p(m) + a_2^p(m) = 2.7 \) GeV\(^{-4} \) leading to\(^3\)

\[
E_{0+, \text{thr}}^p = -3.45 \left( 1 - 1.26 + 0.55 + \ldots \right) \cdot 10^{-3}/M_{\pi^+} = -1.0 \cdot 10^{-3}/M_{\pi^+} \tag{4.13}
\]

where the ellipsis stands for terms of order \( \mu^4 \) and higher. Setting \( a_1^p(m) + a_2^p(m) = 0 \), the 0.55 would read 0.64 and the corresponding value for \( E_{0+, \text{thr}}^p = -1.33 \cdot 10^{-3}/M_{\pi^+} \).

For the neutron, we find (using the same values for the LECs) \( E_{0+, \text{thr}}^n = 3.64(1 - 0.29 + \ldots) \cdot 10^{-3}/M_{\pi^+} = 2.59 \cdot 10^{-3}/M_{\pi^+} \) which shows a better convergence since the term of order \( \mu \) is absent and thus the contribution from the triangle diagram appears already at lowest order. We note that the electric dipole amplitude for \( \pi^0 \) production off neutrons is sizeable and of opposite sign to the one for production off protons. The lesson to be learned is that the \( \mu \) expansion of \( E_{0+} \) converges very slowly (as already

\(^3\) Notice that the individual values of \( a_1^p(m) \) and \( a_2^p(m) \) are much larger than their sum so that this determination is afflicted with a substantial uncertainty. Even the sum \( a_1 + a_2 \) is strongly affected if one chooses either the charged or the neutral pion mass in eq.(3.9).
anticipated in ref.[3]) and thus one has at least to go one order higher before one can make a reasonably accurate theoretical prediction. This clearly shows that the electric dipole amplitude is not a good testing ground for the chiral dynamics of QCD. Such a behaviour is, however, not too surprising. We remind the reader of similar large higher order S–wave effects in the scalar form factor of the nucleon [16] or the scalar form factor of the pion [17] (just to name two such cases). To further tighten the prediction on $E_{0^+}$, it is conceivable that one has to perform a dispersive analysis supplemented with CHPT constraints to get a handle on the higher orders. It is again important to stress that the LETs for the electric dipole amplitude have been derived in the exact isospin limit. It is not meaningful to compare the number (4.13) with the data, since isospin breaking and other higher order effects are substantial.

We now turn to the P–waves. From the explicit formulae given in the appendix we derive the LETs for the slope of $P_1$ and $P_2$ at threshold (for the proton)

$$\frac{1}{|q|} P_{1, \text{thr}} = \frac{e g \pi N}{8 \pi m^2} \left( 1 + \kappa_p + \mu \left[ -1 - \frac{\kappa_p}{2} + \frac{g^2 N (10 - 3\pi)}{48 \pi} \right] + \mathcal{O}(\mu^2) \right)$$

$$\frac{1}{|q|} P_{2, \text{thr}} = \frac{e g \pi N}{8 \pi m^2} \left( -1 - \kappa_p + \frac{\mu}{2} \left[ 3 + \kappa_p - \frac{g^2 N}{12 \pi} \right] + \mathcal{O}(\mu^2) \right)$$

for (4.14a)

with $\kappa_p = 1.793$ and similarly for the reaction $\gamma n \to \pi^0 n$,

$$\frac{1}{|q|} P_{1, \text{thr}} = \frac{e g \pi N}{8 \pi m^2} \left( -\kappa_n + \frac{\mu}{2} \left[ \kappa_n + \frac{g^2 N (10 - 3\pi)}{24 \pi} \right] + \mathcal{O}(\mu^2) \right)$$

$$\frac{1}{|q|} P_{2, \text{thr}} = \frac{e g \pi N}{8 \pi m^2} \left( \kappa_n - \frac{\mu}{2} \left[ \kappa_n + \frac{g^2 N}{12 \pi} \right] + \mathcal{O}(\mu^2) \right)$$

for (4.14b)

with $\kappa_n = -1.913$ the anomalous magnetic moment of the neutron. These are examples of quickly converging $\mu$ expansions.\(^4\)

$$\frac{1}{|q|} P_{1, \text{thr}}^p = 0.512 (1 - 0.062) \text{ GeV}^{-2} = 0.480 \text{ GeV}^{-2}$$

$$\frac{1}{|q|} P_{2, \text{thr}}^p = -0.512 (1 - 0.0008) \text{ GeV}^{-2} = -0.512 \text{ GeV}^{-2}$$

$$\frac{1}{|q|} P_{1, \text{thr}}^n = 0.351 (1 - 0.020) \text{ GeV}^{-2} = 0.344 \text{ GeV}^{-2}$$

$$\frac{1}{|q|} P_{2, \text{thr}}^n = -0.351 (1 + 0.107) \text{ GeV}^{-2} = -0.389 \text{ GeV}^{-2}$$

\(^4\) To be precise, we mean that the leading term is much bigger than the first correction in contrast to what happens in the electric dipole amplitude.
From these expressions, a few interesting observations can be made. First, we note that the multipole $E_{1+}$ is not exactly zero since in that case one would have $P_1 = -P_2$. Commonly, this multipole is set to zero when one analyzes the threshold data. We will come back to this small multipole when we discuss the polarization observables. Second, these P-wave LETs help to constrain the existing fits to the threshold region [9]. They favor the solution which leads to a strong energy-dependence of the electric dipole amplitude. If we pull out by hand a factor $|\vec{k}|^5$, these LETs translate into

$$P_{1, \text{thr}}^p = 10.3 |\vec{k}| |\vec{q}| 10^{-3} / M_{\pi^+}^2$$

and

$$P_{2, \text{thr}}^p = -10.9 |\vec{k}| |\vec{q}| 10^{-3} / M_{\pi^+}^3$$

(4.16)

to be compared e.g. with the value of $P_1 = (8.8 \pm 0.6) |\vec{k}| |\vec{q}| 10^{-3} / M_{\pi^+}^2$ given by Drechsel and Tiator [18]. Third, we also would like to stress that the corresponding $P_1$ and $P_2$ of the relativistic calculation [3] agree quite nicely with the LET (remember that in the relativistic formulation some higher order terms are included). For example, at $E_\gamma = 151$ MeV, the LET predicts $P_1 = 2.47$ and $P_2 = -2.48$ while the P-wave multipoles of ref. [3] lead to $P_1 = 2.43$ and $P_2 = -2.60$ (all in units of $10^{-3} / M_{\pi^+}$) (for $\gamma p \rightarrow \pi^0 p$). These novel P-wave LETs should be tested more accurately and they can also serve to constrain the empirical analysis. It is amusing to note that this is a good testing ground for chiral dynamics in contrast to common folklore.

**IV.5. RELATION TO THE FERMI–WATSON THEOREM**

Here, we wish to elaborate briefly on the imaginary part of the electric dipole amplitude. By virtue of the Fermi–Watson theorem, it is related to the $\pi N$ scattering phases via (see e.g. [20])

$$\text{Im} E_{0+}^{\pi^0 p} = \text{Re} E_{0+}^{(0)} \tan(\delta_1) + \frac{1}{3} \text{Re} E_{0+}^{(1/2)} \tan(\delta_1) + \frac{2}{3} \text{Re} E_{0+}^{(3/2)} \tan(\delta_3)$$

(4.17)

with $\delta_{1,3}$ the $\pi N$ S–wave phases for total isospin $1/2$ and $3/2$, respectively. Close to threshold, we can approximate the $\tan(\delta_{2f})$ by $a_{2f} |\vec{q}|$ (in the respective channels) and also drop the term proportional to $a^+ \text{Re} E_{0+}^{\pi^0 p}$ which is a factor of 200 smaller than $\sqrt{2} a^- \text{Re} E_{0+}^{\pi^0 p}$, i.e.

$$\text{Im} E_{0+}^{\pi^0 p} \simeq \sqrt{2} a^- M_{\pi} \text{Re} E_{0+}^{\pi^0 p} \sqrt{\frac{\omega^2}{M_{\pi}^2} - 1}.$$ 

(4.18)

\[5\] We stress again that this is not the correct threshold behaviour of the P-wave multipoles.
Therefore, the strength of the imaginary part of $E_{0+}$ in the threshold region is governed by the product $\sqrt{2} a^{-} M_{\pi} \text{Re} E_{0+}^{-} = 3.7 \cdot 10^{-3}/M_{\pi+}$. This is in fact the coefficient which we did not write explicitly in eq.(4.2). The order $q^4$ calculation leads to

$$\frac{eg_{\pi N} M_{\pi}^2}{32\pi^2 m F_{\pi}^2} \left(1 - \frac{5\mu}{2}\right) = 2.7 \cdot 10^{-3}/M_{\pi+},$$

(4.19)

which shows that the strength of $\text{Im} \ E_{0+}$ is underestimated by approximately 30%. As already noted a couple of times, this indicates that for an accurate description of the electric dipole amplitude even in the threshold region one has to go beyond order $q^4$. Of course, we should also stress that the imaginary part is in any case less accurately calculated. Here, the first contributions to $\text{Re} \ E_{0+}$ are of order $q^2$ whereas the corresponding imaginary part starts at $O(q^3)$. Finally, we wish to point out why the phase related to the direct $\pi^0 p$ scattering process only appears at two-loop accuracy making use of the Fermi–Watson theorem. Above the $\pi^0 p$ but below the $\pi^+ n$ threshold, we have $\text{Im} \ E_{0+}^{\pi^0 p} = \text{Re} \ E_{0+}^{\pi^0 p} \cdot \tan(\delta(\pi^0 p))$. In the T-matrix, $a^+$ starts at order $q^2$ and $\text{Re} \ E_{0+}^{\pi^0 p}$ is $O(q)$. Furthermore, there is an additional factor $q$ from the relation $\tan(\delta(\pi^0 p) = a^+ |q|$ so that the imaginary part starts out at order $q^4$, i.e. is $O(q^5)$ in the full amplitude which is a two–loop effect.

V. RESULTS AND DISCUSSION

In this section, we will discuss first a two–parameter model which describes most of the physics in the threshold region and draw some general conclusions from it. We then turn to the detailed numerical investigation of the reaction $\gamma p \rightarrow \pi^0 p$ making use of the Fermi–Watson theorem. Above the $\pi^0 p$ but below the $\pi^+ n$ threshold, we have $\text{Im} \ E_{0+}^{\pi^0 p} = \text{Re} \ E_{0+}^{\pi^0 p} \cdot \tan(\delta(\pi^0 p))$. In the T-matrix, $a^+$ starts at order $q^2$ and $\text{Re} \ E_{0+}^{\pi^0 p}$ is $O(q)$. Furthermore, there is an additional factor $q$ from the relation $\tan(\delta(\pi^0 p) = a^+ |q|$ so that the imaginary part starts out at order $q^4$, i.e. is $O(q^5)$ in the full amplitude which is a two–loop effect.

V.1. A REALISTIC TWO–PARAMETER MODEL FOR $E_{0+}(\omega)$

Tree diagrams and resonance exchanges lead to an almost constant $\text{Re} \ E_{0+}$ in the threshold region, 144.7 MeV $\leq E_{\gamma} < 160$ MeV. The unitarity corrections due to the opening of the $\pi^+ n$ channel at 6.8 MeV above threshold have a square–root behaviour as discussed in section 4.1. This claim is further substantiated by the fact that in the isospin limit one finds essentially no energy dependence in $\text{Re} \ E_{0+}$ [3] but only after inclusion of the pion mass difference (and, to a lesser extent, the proton–neutron mass
difference) a strong energy dependence develops [19]. To a good approximation, we can therefore parametrize the electric dipole amplitude in the threshold region as

\[ E_{0+}(\omega) = -a - b\sqrt{1 - (\omega/\omega_c)^2} \]

(5.1)

which immediately leads to a square-root type behaviour for \( \text{Im} E_{0+} \),

\[ \text{Im} E_{0+}(\omega) = b\sqrt{(\omega/\omega_c)^2 - 1} \Theta(\omega - \omega_c) \]

(5.2)

which means that the coefficient \( b \) is constrained by the Fermi–Watson theorem (cf. section 4.5). We now fit the Mainz data with this form for \( E_{0+} \) and the P-waves as given by the chiral expansion, i.e. we have three parameters, namely \( a \), \( b \) and \( b_P \), to fit 126 data points (total and differential cross sections). We find using standard minimization procedures

\[
\begin{align*}
a &= (0.28 \pm 0.07) \cdot 10^{-3}/M_{\pi^+}, \\
b &= (4.62 \pm 0.49) \cdot 10^{-3}/M_{\pi^+}, \\
b_P &= (15.64 \pm 0.25) \text{ GeV}^{-3}.
\end{align*}
\]

(5.3)

Several remarks on these numbers are in order. First, the value for \( b_P \) is close to the static \( \Delta \) exchange estimate eq.(4.10), i.e. the multipole \( P_3 \) is completely dominated by \( \Delta \) exchange even close to threshold.\(^6\) Second, the fitted value for \( b \) is somewhat larger than what one would get from the Fermi–Watson theorem, \( b_{FW} = 3.7 \) (in canonical units of \( 10^{-3}/M_{\pi^+} \)). The source of this discrepancy is two-fold. First, in the derivation of \( \text{Im} E_{0+} \) from the Fermi–Watson theorem we assumed exact isospin symmetry and made the further approximation that the phase shift is simply the product of the scattering length times the momentum. The result obtained was, however, applied to a situation involving some isospin breaking. Second, if the remeasured threshold data lead to somewhat smaller values of \( E_{0+} \) in the threshold region, this difference of 20\% would diminish. Furthermore, the rather simple but physically motivated form eqs.(5.1,5.2) leads to a good fit with a \( \chi^2 \)/datum of 1.89. At the respective thresholds, this gives

\[
\begin{align*}
\text{Re} E_{0+}(\omega_0) &= -1.52 \cdot 10^{-3}/M_{\pi^+} \\
\text{Re} E_{0+}(\omega_c) &= -0.28 \cdot 10^{-3}/M_{\pi^+}
\end{align*}
\]

(5.4)

which translates into a difference of \( \delta E_{0+} = E_{0+}(\omega_c) - E_{0+}(\omega_0) = 1.24 \cdot 10^{-3}/M_{\pi^+} \) between the two thresholds. Due to the reflection properties of (5.1), this also means that

\(^6\) The sign of the multipole \( P_3 \) is determined from existing multipole analyses at somewhat higher energies, see e.g. refs.[24]. Of course, only in the polarization observables to be discussed later this sign plays a role.
the imaginary part of $E_{0+}$ at $\omega_R = 2\omega_c - \omega_0 = 145.25$ MeV (equivalent to $E^R = 158.27$ MeV) should be

$$\text{Im} E_{0+}(\omega_R) = 1.24 \cdot 10^{-3}/M_{\pi^+} \quad .$$

(5.5)

Stated differently, a strong variation of the real part between the $\pi^0p$ and the $\pi^+n$ thresholds reflects itself in a rapid growth of the imaginary part and vice versa. This stringent constraint rooted in dispersion theory has not yet been discussed in the various examinations of the energy-dependence of the electric dipole amplitude as given by the Mainz data. If one uses a square-root behaviour of the imaginary part in the threshold region, the value of Bergstrom [9], $\text{Im} E_{0+}(180 \text{ MeV}) = 2.0$ translates into $\text{Im} E_{0+}(\omega_R) = 1.0$ and the four values below 182 MeV given by Müllensiefen [20] lead to $\text{Im} E_{0+}(\omega_R) = (1.0 \pm 0.1)$ (all in canonical units). These numbers are consistent with a direct calculation of $\text{Im} E_{0+}(\omega_R) = 3.7 \cdot \sqrt{\omega_R^2/\omega_c^2 - 1} = 1.0$, and they indicate that the variation of $\text{Re} E_{0+}$ between the $\pi^0p$ and the $\pi^+n$ thresholds is indeed less strong as commonly believed. We furthermore stress that the relative smallness of the parameter $a = 0.3$ indicates that there have to be large corrections to the Born result of 2.3 (in canonical units). Only with the inclusion of loop diagrams, here the triangle graph and its crossed partner, it is possible to understand such large corrections to $a$. This can be considered a success of CHPT. A last important point is the following. In the approximation (5.1), $\text{Re} E_{0+}(\omega)$ is strictly constant for $\omega > \omega_c$. At present, the data are not accurate enough to clearly differentiate between a constant or slowly varying energy dependence above $\omega_c$. We have therefore added a linear term of the type $c(1 - \omega/\omega_c)$ to eq.(5.1) and redone the fitting. The values for $a$ and $b$ are somewhat changed leading to $\text{Re} E_{0+}(\omega_0) = -1.57 \cdot 10^{-3}/M_{\pi^+}$ and $\text{Re} E_{0+}(\omega_c) = -0.43 \cdot 10^{-3}/M_{\pi^+}$, not very different form eq.(5.4) with a comparable $\chi^2$/datum of 1.89. However, the one-$\sigma$ uncertainties on $a$ and $b$ are considerably larger for this type of fit. This means that a reshuffling between the linear and the square-root term is possible (using the existing data). We will come back to this when we discuss the fit with the counter terms proportional to $a_1(\lambda)$ and $a_2(\lambda)$. We end this section by stressing again our believe that the pertinent ingredients of the threshold behaviour of the electric dipole amplitude are indeed given by the form eqs.(5.1,5.2).

V.2. CHPT RESULTS FOR $\gamma p \rightarrow \pi^0p$

We now turn to the discussion of the results making full use of the formalism outlined in section 4. First, we consider $a_1(\lambda)$, $a_2(\lambda)$ and $b_P$ as completely unconstrained parameters and use the best fit to the Mainz total and differential cross section data to determine their values. This will be called the “free fit” in what follows. Second,
we vary these coefficients within the bounds given by the resonance exchange picture as discussed in section 4.3. This means in particular that we vary the off-shell parameters $X, Y$ and $Z$ (for fixed $g_1 = g_2 = 5$). In principle, one should also vary the vector meson couplings within some bounds but as discussed before, the uncertainty with respect to the $\Delta$ parameters is by far larger and we thus use a fixed vector meson contribution to the various LECs. We will, however, also present a fit in which $g_1$ and $g_2$ are allowed to vary within their bounds. We already note here that the results are essentially indistinguishable from the fit with $g_1 = g_2 = 5$. This procedure will be coined the “resonance fit”. With $M_{\pi^+} = 140.11$ MeV, the $\pi^+n$ threshold is located at its physical value, $E_\gamma = 151.43$ MeV.

First, we show results for the free fit. We find

$$a_1(m) = (-55.45 \pm 3.34) \text{ GeV}^{-4},$$
$$a_2(m) = (58.15 \pm 3.14) \text{ GeV}^{-4},$$
$$b_P = (15.80 \pm 0.23) \text{ GeV}^{-3}.$$  \hspace{1cm} (5.6)

Notice that the value for $b_P$ is in excellent agreement with the resonance exchange estimate, $b_P^R + b_P^V = (12.3 + 3.1) \text{ GeV}^{-3} = 15.4 \text{ GeV}^{-3}$, using eqs.(4.7,4.10). Such a value is essentially a consequence of the bell-shaped differential cross sections for $E_\gamma > 150$ MeV. The result of the free fit for the differential cross sections is shown in Fig.2 by the solid line (the fit to the data at $E_\gamma = 156.1$ MeV is not exhibited) and similarly in Fig.3 for the total cross section.\footnote{Notice that the few Saclay data were not used in the fitting procedure. Including them would not alter any of our conclusions.} This fit has a $\chi^2$/datum of 1.88. If one multiplies the values of $a_1(m)$ and $a_2(m)$ by $M_{\pi^+}^2$, one notices indeed that their individual contributions to $E_{0^+}$ at threshold are of the order $\pm 5.6 \cdot 10^{-3}/M_{\pi^+}$ which is considerably larger than their sum. The corresponding real part of the electric dipole amplitude is shown in fig.4a, with

$$\text{Re} E_{0^+}(\omega_0) = -1.56 \cdot 10^{-3}/M_{\pi^+} \quad \text{Re} E_{0^+}(\omega_e) = -0.32 \cdot 10^{-3}/M_{\pi^+}$$ \hspace{1cm} (5.7)

not very different from the constant plus square-root fit discussed in the previous section. However, after the $\pi^+n$ threshold, $\text{Re} E_{0^+}(\omega)$ rises in contrast to the two-parameter model. We believe that this is an artefact of the strong energy dependence from the polynomial contact terms proportional to $a_{1,2}$ with their large coefficients. This is reminiscent of the fit we discussed before when we added a linearly growing term to eq.(5.1). We interpret the unnaturally large values of $a_1(m)$ and $a_2(m)$ as a signal of the importance of higher loop effects not accounted for by our order $q^4$ calculation. After all,
from the discussion of the unitarity corrections leading to the cusp effect it is rather obvious that one can not expect a strong energy dependence due to some polynomial terms (in $\omega$). This is exactly what happens in this free fit - the unconstrained parameters try to make up for some higher order effects. Clearly, the situation would be much more satisfactory if one could determine the LECs $a_{1,2}$ from some other reaction. The imaginary part of the electric dipole amplitude (cf. fig.4b) is not affected by such uncertainties. It shows the expected square-root type rise and stays below $0.7 \cdot 10^{-3}/M_{\pi^+}$ for $E_\gamma < 160$ MeV as already elaborated on in section 4.5. In Fig.5, we exhibit the conventional P-wave multipoles $M_{1+}$, $M_{1-}$ and $E_{1+}$. These show the empirically expected pattern $M_{1+} > -M_{1-} \gg E_{1+}$ but still $P_1 \simeq -P_2$ and the $\Delta$ is most visible in $P_3$. We note that the P-waves are improved compared to the relativistic $O(q^3)$ calculation [3]. It is furthermore important to stress that the imaginary parts of these P-waves are tiny (they increase with $|q|^3$) and that consequently the cusp at the $\pi^+n$ threshold is not visible. Isospin breaking effects are also small, typically of the size $((M_{\pi^+} - M_{\pi^0})/M_{\pi^+})^{3/2} \sim 0.6\%$. The energy-dependence of $P_{1,2}/|q|$ in the threshold region $\omega_0 \leq \omega \leq \omega_R$ is very weak, i.e. these reduced P-waves stay constant on the level of 2%. These observations are at the heart of the usefulness of the P-wave LETs, eqs.(5.14).

Before discussing the polarization observables, let us consider the resonance fit. First, we keep $g_1 = g_2 = 5$ fixed.\footnote{Note that the empirical width $\Gamma(\Delta \to N\gamma)$ demands $g_1 \simeq 5$.} We find as best values with a $\chi^2$/datum of 2.02

$$X = 2.24 \pm 1.87, \ Y = 0.13 \pm 0.52, \ Z = 0.28 \pm 0.75 \ .$$ (5.8)

This corresponds to $a_1^I(m) = 1.3 \text{ GeV}^{-4}$, $a_2^I(m) = 2.7 \text{ GeV}^{-4}$ and $b_p = 15.9 \text{ GeV}^{-3}$. As already stated, the magnitude of $a_{1,2}(m)$ is considerably smaller as in the case of the free fit for the reasons discussed. The corresponding differential and total cross sections are shown in Figs.1 and 2 as the dashed lines. They are very similar to the free fit, the sole exception being the first two MeV above the $\pi^0p$ threshold. The resonance fit leads to a smaller $E_{0+,\text{thr}}$ and weaker energy dependence as shown in Fig.4a. Specifically, we have

$$\text{Re} \ E_{0+}(\omega_0) = -1.16 \cdot 10^{-3}/M_{\pi^+} \quad \text{Re} \ E_{0+}(\omega_c) = -0.43 \cdot 10^{-3}/M_{\pi^+}$$ (5.9)

This means that the cusp is less pronounced. We point out, however, that the energy-dependence of $E_{0+}$ for the resonance fit follows closely the generic form constant plus square root, eq.(5.1) (with somewhat different values for $a$ and $b$). The large one-\sigma
uncertainties on $X$, $Y$ and $Z$ signal that the presently available data are not yet accurate enough to pin down the electric dipole amplitude very tightly. In the resonance fit, $\text{Re } E_{0+}(\omega > \omega_c)$ stays flat as does the two-parameter model discussed above. Consequently, the relation between the imaginary part at $\omega_R$ and the difference in the real part between $\omega_0$ and $\omega_c$ is fulfilled (cf fig.4b). The $P$-waves are essentially the same as for the free fit. If we relax the condition that $g_1 = g_2 = 5$, we find a very similar fit with the following values: $X = 0.55 \pm 0.32$, $Y = 0.65 \pm 0.02$, $Z = 0.24 \pm 0.01$, $g_1 = 3.94 \pm 0.06$ and $g_2 = 4.49 \pm 4.24$. Since $g_2$ enters only the $S$-wave, its value is determined within large uncertainties. For example, this 5-parameter fit leads to $E_{0+,\text{thr}} = -1.17$, only marginally different from the 3-parameter fit.

We now turn to the polarization observables defined in eq.(2.7). In Fig.6, we show $\Sigma(\theta)$, $T(\theta)$ and $P(\theta)$ for $E_\gamma = 153.7$ MeV. Since none of these is sensitive to $\text{Re } E_{0+}$, the predictions based on the free fit and on the resonance fits are essentially the same (compare the solid to the dashed lines in Fig.6). From the size of the effect we conclude that the target asymmetry $T(\theta)$ is best suited to pin down the imaginary part of the electric dipole amplitude. The photon asymmetry $\Sigma$ is very sensitive to the ratio of the $P$-wave multipoles $E_{1+}/M_{1-}$ (for fixed $M_{1+}$).\(^9\) However, note that our analysis gives a very small $E_{1+}$ in the threshold region so that the sensitivity discussed in ref.[21] is presumably overestimated.

**V.3. CHPT PREDICTIONS FOR $\gamma n \to \pi^0n$**

We now discuss predictions for neutral pion production off the neutron in the threshold region. To fix the low-energy constants, we use the resonance exchange values discussed in section 4.3 (since no data to fit exist). Also, we set $m = m_n = 939.57$ MeV. The threshold of this reaction is at $E_{\gamma,\text{thr}} = 144.66$ MeV and the $p\pi^-$ channel opens at $E_{\gamma} = 148.46$ MeV, i.e. at $\omega_c = 137.86$ MeV, and we choose the charged pion mass accordingly to account for the proper location of the second threshold.

The resulting total cross section from threshold up to $E_\gamma = 160$ MeV is shown in fig.7 and four corresponding differential cross sections in fig.8. We note that $\sigma_{\text{tot}}$ rises quicker than in the case of the proton, this is due to the dual effects of (i) the larger (in magnitude) electric dipole amplitude, $E_{0+}^\pi$ changes from 2.13 to 2.77 between threshold and $E_\gamma = 160$ MeV (in canonical units), and (ii) the even closer proximity of the first open channel. One also notices the cusp effect. The differential cross sections are strongly peaked in forward direction. This can be traced back to the large and positive value of $\text{Re } E_{0+}$, implying a large and positive angular coefficient $B$ (eqs.(2.4,2.5)).

\(^9\) This was pointed out to us by R. Beck, see also ref.[21].
is very different to the case of the proton. At \( \omega_c \), we find \( E^{\pi^0 n}_{0+} = 2.79 \), i.e. the cusp in the electric dipole is of similar size as in the proton case. The polarization observables shown in fig.9 (for \( E_\gamma = 153.7 \text{ MeV} \)) are strongly enhanced in backward direction which is due to the forward peaked \( d\sigma/d\Omega_{cm} \). It is obvious that these rather distinctive features should be tested experimentally.

VI. SUMMARY, CONCLUSIONS AND OUTLOOK

In this paper, we have used heavy baryon chiral perturbation theory to study the reactions \( \gamma p \to \pi^0 p \) and \( \gamma n \to \pi^0 n \) in the threshold region. This is a continuation and improvement on the calculations making use of relativistic baryon CHPT reported in refs.[3,19]. The pertinent results of this study can be summarized as follows (preliminary results were presented in ref.[22]):

- In the threshold region, one can restrict oneself to the inclusion of S- and P-wave multipoles as defined in eq.(2.1). We have calculated the electric dipole amplitude to order \( q^4 \) and the P-waves \( P_{1,2,3} \) to order \( q^3 \). Besides the Born and loop contributions, we have one finite counter term contribution to \( P_3 \) and two scale-dependent ones to \( E_{0+} \). These counter terms can either be determined by a best fit (the so-called “free fit”) to the threshold data (total and differential cross sections) or estimated from resonance exchange (“resonance fit”). In the latter case, the dominant contributions come from the \( \Delta(1232) \) as well as the vector mesons \( \rho \) and \( \omega \). In both cases we get a good fit to the existing data. However, for the free fit, the two S-wave low-energy constants are unnaturally large and of opposite sign. This signals the importance of higher loop effects not yet accounted for. The P-wave low-energy constant is essentially given by \( \Delta \)-exchange and takes a value expected from the static isobar model.

- We have considered the low-energy theorem for \( E^{\pi^0 p}_{0+} \) and shown that the convergence in \( \mu = M_\pi/m \) is indeed very slow (as conjectured in ref.[3]), compare eq.(4.13). We conclude that this multipole is not a good testing ground for the chiral dynamics of QCD. In case of the neutron, the large contribution of order \( \mu^2 \) from the triangle diagram and its crossed partner appears already at leading order and thus the convergence for \( E^{\pi^0 n}_{0+} \) is much better. From our calculation, one expects that \( |E^{\pi^0 n}_{0+}| > |E^{\pi^0 p}_{0+}| \).

- We have derived novel low-energy theorems for the P-waves \( P_1 \) and \( P_2 \) as given in eqs.(4.14,4.15). These are quickly converging expansions in \( \mu \) and they should be used to constrain the data analysis. In particular, the small difference in the magnitudes of \( P_1 \) and \( P_2 \) indicates a small but non-vanishing \( E_{1+} \) multipole.
- We have presented a simple but realistic two-parameter model for the energy dependence of the electric dipole amplitude in the threshold region, cf. eqs.(5.1,5.2). The parameter $b$ is closely related to the strength of the imaginary part of $E_{0+}$ by the Fermi–Watson theorem. In fact, the energy dependence of the real part should reflect itself by a similar rise in the imaginary part above the $\pi^+ n$ threshold. This points towards the importance of an independent determination of the imaginary part at $E_{\gamma}^{R} = 158.3$ MeV.

- We have discussed the polarization observables $\Sigma(\theta)$, $P(\theta)$ and $T(\theta)$ and shown that the target asymmetry $T$ seems to be best suited to determine $\text{Im} E_{0+}$ for the neutral pion production off protons.

- Finally, we have given predictions for the total and differential cross sections as well as for polarization observables for the reaction $\gamma n \rightarrow \pi^0 n$ (with the low-energy constants determined from resonance exchange). These should be determined experimentally since they serve as a further test of the chiral dynamics of QCD.

Where do we go from here? It is imperative to improve upon the S–wave on the theoretical side by either a two–loop calculation or a dispersive representation constrained by CHPT and on the experimental side by more accurate determinations of the total and differential cross section close to threshold. Only with very accurate data one is able to test the proposed constant plus square–root form (5.1) for the S–wave constrained by the Fermi–Watson theorem and the P–wave LETs. Furthermore, a new look at the corrections to the low–energy theorems for charged pion photoproduction seems to be required from the new data on $\pi^-$ production [23]. We hope to come back to these topics in the future.
APPENDIX: EXPRESSIONS FOR S– AND P–WAVE MULTIPOLES

Here, we will give explicit analytical expressions for the S– and P–wave multipoles \( E_{0+}, P_1, P_2, P_3 \) of neutral pion photoproduction from protons and neutrons. The formulae are given in the isospin limit using only one pion mass \( M_\pi \). In order to account for the branch point and unitarity cusp above the physical threshold, the value of \( M_\pi \) has to be chosen appropriately, \( M_{\pi^+} = \omega_c \), as explained in the text. The Born terms for the proton read:

\[
E_{0+}^{\text{Born}}(\omega) = -\frac{eg\pi N}{24\pi m^2} \left\{ \frac{M_\pi^2}{\omega} + 2\omega \right\} + \frac{eg\pi N}{48\pi m^3} \left\{ 6\omega^2 + 4M_\pi^2 - \frac{M_\pi^4}{\omega^2} + \kappa_p(4\omega^2 - M_\pi^2) \right\} \\
+ \frac{eg\pi N}{960\pi m^4} \left\{ -10\frac{M_\pi^6}{\omega^3} + 13\frac{M_\pi^4}{\omega} - 156\omega M_\pi^2 - 72\omega^3 - 10\kappa_p\omega(4\omega^2 + 5M_\pi^2) \right\} 
\]

\[
P_1^{\text{Born}}(\omega) = \frac{eg\pi N}{8\pi m^2} \left\{ 1 - \frac{6\omega}{5m} + \frac{M_\pi^2}{5m\omega} + \kappa_p \left( 1 - \frac{\omega}{2m} \right) \right\} 
\]

\[
P_2^{\text{Born}}(\omega) = \frac{eg\pi N}{8\pi m^2} \left\{ -1 + \frac{13\omega}{10m} + \frac{M_\pi^2}{5m\omega} + \kappa_p \left( -1 + \frac{\omega}{2m} \right) \right\} 
\]

\[
P_3^{\text{Born}}(\omega) = \frac{eg\pi N}{16\pi m^3} \frac{\omega}{\sin^2 \frac{\omega}{2}} 
\]

The Born terms for the neutron are:

\[
E_{0+}^{\text{Born}}(\omega) = \frac{eg\pi N\kappa_n}{48\pi m^3} \left\{ M_\pi^2 - 4\omega^2 \right\} + \frac{eg\pi N\kappa_n}{96\pi m^4} \omega \left\{ 4\omega^2 + 5M_\pi^2 \right\} 
\]

\[
P_1^{\text{Born}}(\omega) = -P_2^{\text{Born}}(\omega) = \frac{eg\pi N}{8\pi m^2} \kappa_n \left\{ -1 + \frac{\omega}{2m} \right\} 
\]

\[
P_3^{\text{Born}}(\omega) = 0 
\]

The loop contributions for the proton and the neutron differ only by a few numerical coefficients:

\[
E_{0+}^{\text{loop}}(\omega) = \frac{egA}{64\pi^2 F_\pi^2} \left\{ \frac{M_\pi^2}{\pi} \arcsin \frac{\omega}{M_\pi} - \omega \sqrt{M_\pi^2 - \omega^2} \right\} 
\]
The counterterm contributions for the protons and neutrons are:

\[ E^{\text{ct}}_{0+}(\omega) = e a^{p,n}_2(\lambda) \omega^3 + e a^{p,n}_1(\lambda) \omega M^2_\pi \]

\[ P^{\text{ct}}_1(\omega) = P^{\text{ct}}_2(\omega) = 0 \]

\[ P^{\text{ct}}_3(\omega) = e b^{p,n}_2 \omega |q| \]
REFERENCES

   L. Tiator, "Meson Photo- and Electroproduction", lecture given at the II TAPS Workshop, Alicante, 1993;
   A.M. Bernstein, private communication.

**FIGURE CAPTIONS**

Fig.1 Feynman diagrams contributing at $\mathcal{O}(q^3)$. The triangle (a) and rescattering (b) diagrams give $E_{0+}(\omega)$ while (c) and (d) contribute to the P-wave multipoles $P_1$ and $P_2$. Crossed graphs are not shown.

Fig.2 Differential cross sections for $\gamma p \rightarrow \pi^0 p$. The solid lines refer to the free fit and the dashed ones to the resonance fit as explained in the text. The data are from Mainz[1].

Fig.3 Total cross section for $\gamma p \rightarrow \pi^0 p$. For notations, see fig.2. The data are from Mainz (diamonds) and Saclay (squares) [1].

Fig.4 The electric dipole amplitude for $\gamma p \rightarrow \pi^0 p$. (a) The real part. For notations, see fig.2. In addition, the $1\sigma$–bands for the free fit are indicated by the dotted lines. (b) Imaginary part.

Fig.5 The P-wave multipoles $M_{1+}$, $M_{1-}$ and $E_{1+}$ for the free fit.

Fig.6 The polarization observables $\Sigma(\theta)$, $T(\theta)$ and $P(\theta)$ for $E_\gamma = 153.7$ MeV. For notations, see fig.2.

Fig.7 Chiral prediction for $\sigma_{tot}(\gamma n \rightarrow \pi^0 n)$. The low–energy constants are estimated from resonance exchange.

Fig.8 Chiral prediction for the differential cross section for $\gamma n \rightarrow \pi^0 n$ at $E_\gamma = 149.1$ (dashed), 151.4 (dotted), 153.7 (solid) and 156.1 (dash-dotted) MeV.

Fig.9 Chiral prediction for the polarization observables $\Sigma, P, T$ for $\gamma n \rightarrow \pi^0 n$ at $E_\gamma = 153.7$ MeV.