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DEPARTAMENTO DE FÍSICA TEÓRICA
FACULTAD DE CIENCIAS
UNIVERSIDAD DE ZARAGOZA

A geometrical version of Noether's theorem in supermechanics

J. F. Cariñena, H. Figueroa

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JOSÉ F. CARIÑENA

Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain

HÉCTOR FIGUEROA

Escuela de Matemática, Universidad de Costa Rica, San José, Costa Rica

and

Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain

In this paper, the notion of supervector field along a morphism of supermanifolds is introduced and is used to obtain a version of Noether's theorem admitting a converse, that is generalizable to higher order supermechanics.

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Introduction

In the last few years, and as one more step in the process of geometrization of physics, it has been shown that the concept of vector field along a map—as well as forms along a map, or even sections along a map—plays an important role in the understanding of classical mechanics and field theory [6,12,7]; for instance, it seems to be the natural geometric object to deal with non-point transformations in classical mechanics. Actually, it was found to be the geometric concept corresponding to (non-point) infinitesimal transformations that are symmetries of the Hamiltonian dynamical system defined by a regular Lagrangian L without being symmetries of L . Moreover, it has also been shown to be very useful to introduce objects like the time evolution operator K relating constraint functions arising in the Lagrangian and Hamiltonian formalism for a singular Lagrangian and other geometric concepts such as the Liouville 1-form, the vertical endomorphism S and the total time derivative operator $\mathbf{T}$, (see e.g. [7]). Furthermore, the use of these vector fields along maps as generalized symmetries of a Lagrangian system furnished the appropriate framework for a geometric approach to Noether's Second Theorem [8].

On the other hand, the study of classical systems including fermionic degrees of freedom, started by Casalbuoni [9,10] several years ago, is receiving, during the last years, a lot of attention from the geometric point of view (see e.g [14] and references therein.) Thus, we feel it is interesting to generalize these concepts of vector fields and forms along maps to graded geometry and apply them to the study of Lagrangian supermechanics, in particular to the converse of Noether's theorem.

Once the above mentioned objects are established, we generalize the tools necessary to discuss higher order Lagrangian supermechanics, even though, for simplicity, we only work here in first order Lagrangian supermechanics.

Our strategy will be to combine Ibort and Marín-Solano [14] approach to Lagrangian supermechanics and Cariñena *et al* [6,7,19] approach to classical Lagrangian mechanics. Thus, in section 1, we review the concepts of supermanifold, morphism and tangent

supermanifold and introduce supervector fields and differential forms along a morphism. Supervector fields along a morphism are no longer given by maps making a diagramme commutative, in the same way as supervector fields are not sections of superbundles; in both cases the corresponding objects are superderivations. In section 2, we study the prolongations of supervector fields along the tangent projection; here our approach is inspired by [6,7,19], nevertheless, to adapt it to our context, it was necessary to consider such prolongations using an algebraic point of view; for instance, we define them via their relation with the total time derivative operator and its prolongations. This was not necessary in the non-graded case, where in fact, more tractable definitions are available. Naturally, this forces a careful study of the total time derivative operator and its prolongations. Finally, in section 3, the analogue of the concept of second order differential equation and the concept of semibasic form are considered in order to prepare a generalized version of Noether's theorem in supermechanics.

1. Generalities on supermanifolds.

1.1. Definition of supermanifolds.

There are several different approaches to arrive at the notion of a supermanifold, all of which try to emphasize a particular way to describe the notion of a manifold. From a theoretical point of view, it seems best to define a supermanifold as a ringed space $(M, \mathcal{A})$ consisting of a topological manifold M and a sheaf of supercommutative $\mathbb{R}$ -superalgebras $\mathcal{A}$, satisfying certain conditions which lead to different notions of supermanifolds. The classification of the definitions was initiated by Rothstein [20] and is discussed in full detail by Bartocci *et al* [1,2].

In this paper we shall be dealing with the so called graded-manifolds, in other words, we shall adopt the definition of supermanifold given in the pioneer works of Kostant [16] and Berezin-Leites [4,17], thus we consider the following definitions:

Definition 1.1.

- i– A superdomain of dimension (m, n) is a ringed space $(U, \mathcal{A}_{m,n}(U))$, where U is an open subset of $\mathbb{R}^m$ and $\mathcal{A}_{m,n}$ is the commutative superalgebra $\mathcal{A}_{m,n} := C^\infty(U) \otimes \wedge(\mathbb{R}^n)$.
- ii– A supermanifold $\mathcal{M}$ of dimension (m, n) is a ringed space $(M, \mathcal{A}_{\mathcal{M}})$ where M is a Hausdorff space with a countable base and $\mathcal{A}_{\mathcal{M}}$ is a sheaf of commutative superalgebras on M such that, for every point $q \in M$ there exists a neighbourhood $\mathcal{U}$ so that the ringed space $(\mathcal{U}, \mathcal{A}_{\mathcal{M}}(\mathcal{U}))$ is isomorphic to a superdomain of dimension (m, n) .

Thus, essentially a supermanifold is obtained by glueing together superdomains; actually already Leites, in [17] see (also [14]) pointed out an alternative way of constructing supermanifolds out of superdomains in the same way one builds up a manifold using local charts. In particular, if $\{(\mathcal{U}_\alpha, \mathcal{A}_\alpha)\}$ is a family of superdomains of dimension (m, n) such that for each pair of indexes (α, β) for which $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$, one has a morphism of superalgebras $\Phi_{\alpha,\beta}: \mathcal{A}_{\alpha,\beta} \rightarrow \mathcal{A}_{\beta,\alpha}$, where $\mathcal{A}_{\alpha,\beta} := \mathcal{A}_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} \subset \mathcal{A}_\alpha$ and $\mathcal{A}_{\beta,\alpha} := \mathcal{A}_{\mathcal{U}_\beta \cap \mathcal{U}_\alpha} \subset \mathcal{A}_\beta$, such that

$$(1.1) \quad \Phi_{\alpha,\alpha} = \text{id} \quad \text{and} \quad \Phi_{\alpha,\beta} \circ \Phi_{\beta,\gamma} \circ \Phi_{\gamma,\alpha} = \text{id},$$

then the space $M = \bigcup_{\alpha} \mathcal{U}_{\alpha}$, together with the sheaf of commutative superalgebras built out of the transition functions $\Phi_{\alpha,\beta}$, is a supermanifold. We refer the reader to [14] for details.

If q^i, q'^i are local coordinates on $\mathcal{U}$ and $\mathcal{U}'$ respectively, and $\theta_{\alpha}, \theta'_{\alpha}$ are the generators of the corresponding Grassmann algebras, then the transitions functions are uniquely determined by equations of the form

$$(1.2a) \quad q'^i = \Phi_{\mathcal{U}',\mathcal{U}}(q'^i) = \phi_0^i(q) + \phi_{\alpha\beta}^i(q) \theta^{\alpha} \theta^{\beta} + \dots$$

$$(1.2b) \quad \theta'^{\alpha} = \Phi_{\mathcal{U}',\mathcal{U}}(\theta'^{\alpha}) = \psi_{\beta}^{\alpha}(q) \theta^{\beta} + \psi_{\beta\gamma\delta}^{\alpha}(q) \theta^{\beta} \theta^{\gamma} \theta^{\delta} + \dots$$

As pointed out in [14], the importance of the transition functions is that they allow us to describe the structure bundle associated to the supermanifold $\mathcal{M}$ via Batchelor' structure theorem [3], that is, the vector bundle $E \rightarrow M$ such that $(M, \mathcal{A}) \cong (M, \Gamma \wedge(E))$, where $\Gamma \wedge(E)$ denotes the sheaf of sections of the Grassmannian of E . Indeed, the transition function $\Psi_{\mathcal{U}',\mathcal{U}}: \mathcal{U}' \cap \mathcal{U} \rightarrow \text{GL}(n, \mathbb{R})$ of E are given by

$$(1.3) \quad \Psi_{\mathcal{U}',\mathcal{U}}(q) = (\psi_{\beta}^{\alpha}(q)).$$

1.2. The tangent supermanifold.

Unlike the non-graded case, the notion of a tangent supermanifold present some difficulties since there are several different possibilities in the literature, and the relations among them are not so clear. Kostant, in [16], associated a tangent bundle to a supermanifold that, in general, is not a supermanifold; Sánchez-Valenzuela defined in [21] a different notion of supertangent bundle which is a supervector bundle. Nevertheless, Ibrort and Marín-Solano [14] noticed that, as far as supermechanics is concerned, the notion of the velocity phase superspace, which corresponds to the space of positions and velocities in ordinary Lagrangian mechanics, is better described by what they called the tangent supermanifold, which they introduced using the transition functions used in the previous subsection, in the same way as the tangent bundle is introduced in ordinary geometry.

Thus, let $\mathcal{M} = (M, \mathcal{A})$ be a supermanifold that has been constructed out a family of superdomains $\{(\mathcal{U}_{\alpha}, \mathcal{A}_{\alpha})\}$; consider the superdomains

$$(1.4) \quad (T\mathcal{U}_{\alpha}, T\mathcal{A}_{\alpha}) := (\mathcal{U}_{\alpha} \times \mathbb{R}^m, C^{\infty}(\mathcal{U}_{\alpha} \times \mathbb{R}^m) \otimes \bigwedge(\mathbb{R}^{2n})),$$

and let $(q^i, v^i, \theta^{\alpha}, \zeta^{\alpha})$ be local supercoordinates on these superdomains; we consider the transition functions

$$(1.5a) \quad q'^i = \Phi_{\mathcal{U}',\mathcal{U}}^i(q'^i) = \phi_0^i(q) + \phi_{\alpha\beta}^i(q) \theta^{\alpha} \theta^{\beta} + \dots$$

$$(1.5b) \quad v'^i = \Phi_{\mathcal{U}',\mathcal{U}}^i(v'^i) = \sum_{j=0}^m \frac{\partial}{\partial q^j} \Phi_{\mathcal{U}',\mathcal{U}}^i(q'^i) v^j + \sum_{\beta=0}^n \frac{\partial}{\partial \theta^{\beta}} \Phi_{\mathcal{U}',\mathcal{U}}^i(q'^i) \zeta^{\beta} \\ = \left(\frac{\partial \phi_0^i}{\partial q^j} + \frac{\partial \phi_{\alpha\beta}^i}{\partial q^j} \theta^{\alpha} \theta^{\beta} + \dots \right) v^j + (-2\phi_{\alpha\beta}^i(q) \theta^{\alpha} + \dots) \zeta^{\beta}$$

$$(1.5c) \quad \theta'^{\alpha} = \Phi_{\mathcal{U}',\mathcal{U}}^{\alpha}(\theta'^{\alpha}) = \psi_{\beta}^{\alpha}(q) \theta^{\beta} + \psi_{\beta\gamma\delta}^{\alpha}(q) \theta^{\beta} \theta^{\gamma} \theta^{\delta} + \dots$$

$$(1.5d) \quad \zeta'^{\alpha} = \Phi_{\mathcal{U}',\mathcal{U}}^{\alpha}(\zeta'^{\alpha}) = \sum_{j=0}^m \frac{\partial}{\partial q^j} \Phi_{\mathcal{U}',\mathcal{U}}^{\alpha}(\theta'^{\alpha}) v^j + \sum_{\beta=0}^n \frac{\partial}{\partial \theta^{\beta}} \Phi_{\mathcal{U}',\mathcal{U}}^{\alpha}(\theta'^{\alpha}) \zeta^{\beta} \\ = \left(\frac{\partial \psi_{\beta}^{\alpha}}{\partial q^j} \theta^{\beta} + \dots \right) v^j + (\psi_{\beta}^{\alpha}(q) + 3\psi_{\beta\gamma\delta}^{\alpha}(q) \theta^{\gamma} \theta^{\delta} + \dots) \zeta^{\beta}.$$

It is easy to check that these transition functions satisfy the cocycle condition (1.1) and therefore, by the procedure described above, they give rise to a supermanifold that will be denoted by $T\mathcal{M}$ and is called the tangent supermanifold of $\mathcal{M}$. Moreover, it is clear that

$$(1.6) \quad T\mathcal{M} = (TM, T\mathcal{A}),$$

where $T\mathcal{A}$ is the sheaf given by the new transition functions, and that $\dim(TM) = (2m, 2n)$. Furthermore, the sheaf $T\mathcal{A}$ can be identified with the sheaf of sections of the Grassmannian of a vector bundle $E^1 \rightarrow TM$ whose transition functions $\Psi_{\alpha',\alpha}^1: (\mathcal{U}' \times \mathbb{R}^m) \cap (\mathcal{U} \times \mathbb{R}^m) \rightarrow \text{GL}(2n, \mathbb{R})$ are given by

$$(1.7) \quad \Psi_{\alpha',\alpha}^1(q, v) = \begin{pmatrix} \psi_{\beta}^{\alpha} & 0 \\ \frac{\partial \psi_{\beta}^{\alpha}}{\partial q^i} v^i & \psi_{\beta}^{\alpha} \end{pmatrix}.$$

On the other hand, if $\tau_{*}: TE \rightarrow TM$ denotes the projection map whose coordinate presentation is

$$(1.8) \quad \tau_{*}(q^i, \theta^{\alpha}, v^i, \zeta^{\alpha}) = (q^i, v^i),$$

it is easy to check that the transition functions of this vector bundle are also given by (1.7), hence E^1 and TE are isomorphic vector bundles and the sheaf $T\mathcal{A}$ can be identified with the sheaf of sections of the Grassmannian of the vector bundle $\tau_{*}: TE \rightarrow TM$.

As we shall be using the tangent supermanifold $T(T\mathcal{M})$, we notice that

$$(1.9) \quad T(T\mathcal{M}) = (T(TM), T(T\mathcal{A})),$$

where the sheaf $T(T\mathcal{A})$ is isomorphic to the sheaf of sections of the Grassmannian of the vector bundle $T(TE) \rightarrow T(TM)$ whose transition functions $\Psi_{\alpha',\alpha}^{1,1}: (\mathcal{U}' \times \mathbb{R}^{3m}) \cap (\mathcal{U} \times \mathbb{R}^{3m}) \rightarrow \text{GL}(4n, \mathbb{R})$ are given by

$$(1.10) \quad \Psi_{\alpha',\alpha}^{1,1}(q, v, u, a) = \begin{pmatrix} \psi_{\beta}^{\alpha} & 0 & 0 & 0 \\ \frac{\partial \psi_{\beta}^{\alpha}}{\partial q^i} v^i & \psi_{\beta}^{\alpha} & 0 & 0 \\ \frac{\partial \psi_{\beta}^{\alpha}}{\partial q^i} u^i & 0 & \psi_{\beta}^{\alpha} & 0 \\ \frac{\partial^2 \psi_{\beta}^{\alpha}}{\partial q^i \partial q^j} v^i u^j + \frac{\partial \psi_{\beta}^{\alpha}}{\partial q^i} a^i & \frac{\partial \psi_{\beta}^{\alpha}}{\partial q^i} u^i & \frac{\partial \psi_{\beta}^{\alpha}}{\partial q^i} v^i & \psi_{\beta}^{\alpha} \end{pmatrix}.$$

Higher order tangent supermanifolds are defined using similar ideas; for simplicity, we shall describe here only the tangent supermanifold of order 2. Thus, we consider the family of superdomains

$$(1.11) \quad (T^2\mathcal{U}_\alpha, T^2\mathcal{A}_\alpha) := (\mathcal{U}_\alpha \times \mathbb{R}^{2m}, C^\infty(\mathcal{U}_\alpha \times \mathbb{R}^{2m}) \otimes \bigwedge(\mathbb{R}^{3n})),$$

and the transition functions

$$(1.12a) \quad q'^i = \Phi_{\mathcal{U}', \mathcal{U}}^2(q'^i) = \phi_0^i(q) + \phi_{\alpha\beta}^i(q) \theta^\alpha \theta^\beta + \dots$$

$$(1.12b) \quad v'^i = \Phi_{\mathcal{U}', \mathcal{U}}^2(v'^i) = \left(\frac{\partial \phi_0^i}{\partial q^j} + \frac{\partial \phi_{\alpha\beta}^i}{\partial q^j} \theta^\alpha \theta^\beta + \dots \right) v^j + (-2\phi_{\alpha\beta}^i(q) \theta^\alpha + \dots) \zeta^\beta$$

$$(1.12c) \quad \theta'^\alpha = \Phi_{\mathcal{U}', \mathcal{U}}^2(\theta'^\alpha) = \psi_\beta^\alpha(q) + \psi_{\beta\gamma\delta}^\alpha(q) \theta^\beta \theta^\gamma \theta^\delta + \dots$$

$$(1.12d) \quad \zeta'^\alpha = \Phi_{\mathcal{U}', \mathcal{U}}^2(\zeta'^\alpha) = \left(\frac{\partial \psi_\beta^\alpha}{\partial q^j} \theta^\beta + \dots \right) v^j + (\psi_\beta^\alpha + 3\psi_{\beta\gamma\delta}^\alpha \theta^\gamma \theta^\delta + \dots) \zeta^\beta$$

$$(1.12e) \quad a'^i = \Phi_{\mathcal{U}', \mathcal{U}}^2(a'^i) = \left(\frac{\partial^2 \phi_0^i}{\partial q^j \partial q^k} + \frac{\partial^2 \phi_{\alpha\beta}^i}{\partial q^j \partial q^k} \theta^\alpha \theta^\beta + \dots \right) v^j v^k \\ + \left(-2 \frac{\partial \phi_{\alpha\beta}^i}{\partial q^k} \theta^\alpha + \dots \right) \zeta^\beta v^k \\ + \left(\frac{\partial \phi_0^i}{\partial q^j} + \frac{\partial \phi_{\alpha\beta}^i}{\partial q^j} \theta^\alpha \theta^\beta + \dots \right) a^j \\ + \left(-2 \frac{\partial \phi_{\alpha\beta}^i}{\partial q^j} \theta^\alpha + \dots \right) v^j \zeta^\beta + (-2\phi_{\alpha\beta}^i + \dots) \zeta^\beta \zeta^\alpha \\ + (-2\phi_{\alpha\beta}^i \theta^\alpha + \dots) \xi^\beta \\ (1.12f) \quad \xi'^\alpha = \Phi_{\mathcal{U}', \mathcal{U}}^2(\xi'^\alpha) = \left(\frac{\partial^2 \psi_\beta^\alpha}{\partial q^j \partial q^i} \theta^\beta + \dots \right) v^j v^i \\ + \left(\frac{\partial \psi_\beta^\alpha}{\partial q^j} + 3 \frac{\partial \psi_{\beta\gamma\delta}^\alpha}{\partial q^j} \theta^\gamma \theta^\delta + \dots \right) v^j \zeta^\beta \\ + \left(\frac{\partial \psi_\beta^\alpha}{\partial q^i} \theta^\beta + \dots \right) a^i \\ + \left(\frac{\partial \psi_\beta^\alpha}{\partial q^i} + 3 \frac{\partial \psi_{\beta\gamma\delta}^\alpha}{\partial q^i} \theta^\gamma \theta^\delta + \dots \right) v^i \zeta^\beta \\ + (6\psi_{\beta\gamma\delta}^\alpha \theta^\delta + \dots) \zeta^\beta \zeta^\gamma + (\psi_\beta^\alpha + 3\psi_{\beta\gamma\delta}^\alpha \theta^\gamma \theta^\delta + \dots) \xi^\beta.$$

Again it is easy to verify that they satisfy (1.1), so we get a supermanifold that will be denoted by $T^2\mathcal{M}$; moreover,

$$(1.13) \quad T^2\mathcal{M} = (T^2M, T^2\mathcal{A}),$$

where the sheaf $T^2\mathcal{A}$ is isomorphic to the sheaf of sections of the Grassmannian of a vector bundle E^2 whose transition functions $\Psi_{\alpha', \alpha}^2: (\mathcal{U}' \times \mathbb{R}^{2m}) \cap (\mathcal{U} \times \mathbb{R}^{2m}) \rightarrow \text{GL}(3n, \mathbb{R})$ are given

by

$$(1.14) \quad \Psi_{\alpha', \alpha}^2(q, v, a) = \begin{pmatrix} \psi_\beta^\alpha & 0 & 0 \\ \frac{\partial \psi_\beta^\alpha}{\partial q^i} v^i & \psi_\beta^\alpha & 0 \\ \frac{\partial^2 \psi_\beta^\alpha}{\partial q^i \partial q^j} v^i v^j + \frac{\partial \psi_\beta^\alpha}{\partial q^i} a^i & 2 \frac{\partial \psi_\beta^\alpha}{\partial q^i} v^i & \psi_\beta^\alpha \end{pmatrix}.$$

Now, using the natural coordinates in the ordinary second order tangent bundles T^2E and T^2M [11,19] it is simple to check that the vector bundle $T^2\tau: T^2E \rightarrow T^2M$, where the projection map $T^2\tau$, in these coordinates, is presented as

$$(1.15) \quad T^2\tau(q^i, \theta^\alpha, v^i, \zeta^\alpha, a^i, \xi^\alpha) = (q^i, v^i, a^i),$$

has transition functions also given by (1.12); hence E^2 is isomorphic to T^2E and we can assume that

$$(1.16) \quad T^2\mathcal{M} \cong (T^2M, \Gamma \bigwedge(T^2E)),$$

when $\mathcal{M} \cong (M, \Gamma \bigwedge(E))$.

To avoid confusion, from now on, we shall use the following convention: local supercoordinates of the supermanifold $T^k\mathcal{M}$ will have the subscript k and those of $T(T^k\mathcal{M})$ will have $1, k$ as a subscript. (We consider this convention necessary since the supercoordinates of the different supermanifolds are superfunctions of different superalgebras, in principle not comparable, eventhough, once we choose specific supercoordinates our notation is such that it tries to keep a relation between these specific choices.)

1.3. Morphism of supermanifolds.

A morphism from the supermanifold $\mathcal{N} = (N, \mathcal{B})$ to the supermanifold $\mathcal{M} = (M, \mathcal{A})$ is defined as a morphism between local ringed spaces; thus, it consists of a pair (ϕ, φ) where $\phi: N \rightarrow M$ is a continuous map and $\varphi: \mathcal{A} \rightarrow \phi_*\mathcal{B}$ is a morphism of sheaves –so for every open set $\mathcal{U}$ of M one has a morphism of superalgebras $\varphi_{\mathcal{U}}: \mathcal{A}(\mathcal{U}) \rightarrow \mathcal{B}(\phi^{-1}(\mathcal{U}))$ – such that the corresponding morphism between the stalks are local (i.e. the maximal ideal of $\mathcal{A}_q$ gets mapped into the maximal ideal of $\mathcal{B}_p$ if $\phi(p) = q$).

Nevertheless, the easiest way to described a morphism of supermanifolds is to tell the image of the elements of a local supercoordinate system for a family of superdomains covering the underlying topological manifold, since, as Leites [17] pointed out, morphisms of superdomains are completely determined by their actions on the elements of a local supercoordinate system.

It is important to emphasize that eventhough the structural sheaves of supermanifolds are locally identified with sheaves of sections of the exterior algebra of some vector bundles, morphisms on supermanifolds are not necessarily morphisms of vector bundles [17].

1.4. Supervector fields along a morphism.

Vector fields along a map, and in general the concept of sections along a map, turn out to be important tools in the theory of ordinary (higher order) Lagrangian mechanics [6,7,19]. We now discuss the corresponding concept in supergeometry.

Supervector fields on a supermanifold $\mathcal{M} = (M, \mathcal{A}_{\mathcal{M}})$ are defined as the elements of the locally free sheaf of $\mathcal{A}_{\mathcal{M}}$ -modules $\mathfrak{X}(\mathcal{A}_{\mathcal{M}}) := (\text{Der } \mathcal{A}_{\mathcal{M}})_0 \oplus (\text{Der } \mathcal{A}_{\mathcal{M}})_1$ where $(\text{Der } \mathcal{A}_{\mathcal{M}})_\gamma$ is the set of sheaf morphisms $X: \mathcal{A}_{\mathcal{M}} \rightarrow \mathcal{A}_{\mathcal{M}}$ satisfying the graded Leibnitz identity

$$(1.17) \quad X(fg) = X(f)g + (-1)^{|f||X|} fX(g),$$

whenever $f \in \mathcal{A}_{\mathcal{M}}$ is homogeneous.

Thus, in analogy with the non-graded case we consider the following definition:

Definition 1.2. A homogeneous supervector field along a morphism $(\phi, \varphi): (N, \mathcal{B}_{\mathcal{N}}) \rightarrow (M, \mathcal{A}_{\mathcal{M}})$ is a morphism of sheaves over M , $X: \mathcal{A} \rightarrow \phi_* \mathcal{B}$ such that for each open subset $\mathcal{U}$ of M

$$(1.18) \quad X(fg) = X(f)\varphi_{\mathcal{U}}(g) + (-1)^{|X||f|}\varphi_{\mathcal{U}}(f)X(g),$$

whenever $f \in \mathcal{A}(\mathcal{U})$ is homogeneous. The sheaf of supervector fields along $\Phi := (\phi, \varphi)$ will be denoted by $\mathfrak{X}(\Phi)$.

Note that supervector fields on a supermanifold $\mathcal{M}$ are supervector fields along the identity morphism $\text{id} = (\text{id}_M, \text{id}_{\mathcal{A}})$, in other words, $\mathfrak{X}(\mathcal{A}_{\mathcal{M}}) = \mathfrak{X}(\text{id})$. Moreover the argument used to prove that supervector fields are local operators, see for instance [1, p. 60], can easily be modified to prove that supervector fields along a morphism are also local operators; hence, if $\mathcal{W} \subset \mathcal{U}$ are open sets one can introduce a restriction morphism $\mathfrak{X}(\Phi)(\mathcal{U}) \rightarrow \mathfrak{X}(\Phi)(\mathcal{W})$ and therefore the sheaf $\mathcal{U} \mapsto \text{Der}_{\phi_* \mathcal{B}}(\mathcal{A}(\mathcal{U}), \mathcal{B}(\phi^{-1}(\mathcal{U})))$ coincides with the sheaf $\mathfrak{X}(\Phi)$, which is then a sheaf of graded $\phi_* \mathcal{B}$ -modules.

When the supermanifolds $\mathcal{N}$ and $\mathcal{M}$ are just the usual manifolds $(N, C^\infty(N))$ and $(M, C^\infty(M))$ then, in this context, a differentiable map $\phi: N \rightarrow M$, is regarded as the pair (ϕ, ϕ^*) [21, 1], where the sheaf morphism $\phi^*: C^\infty(M) \rightarrow C^\infty(N)$ is given by

$$(1.19) \quad \phi_{\mathcal{U}}^*(h) := h \circ \phi,$$

hence a supervector field along such morphism would be a map $X: C^\infty(M) \rightarrow C^\infty(N)$ such that

$$(1.20) \quad X(fg) = X(f)\phi^*(g) + \phi^*(f)X(g),$$

in other words, X would be a ϕ^* -derivation and therefore a vector field along the differentiable map ϕ according to the standard notion in non-graded geometry [7, 19].

1.5. Superforms along a morphism.

The sheaf $\Omega^1(\mathcal{A})$ of 1-superforms is defined as the dual sheaf of $\mathfrak{X}(\mathcal{A})$, so 1-superforms are sheaf morphisms $\omega: \mathfrak{X}(\mathcal{A}) \rightarrow \mathcal{A}$ and the grading in $\Omega^1(\mathcal{A})$ is given by

$$(1.21) \quad (\Omega^1(\mathcal{A}))_\gamma = \{ \omega : \omega(\mathfrak{X}(\mathcal{A})_\delta) \subset \mathcal{A}_{\gamma+\delta} \}.$$

In general, k -superforms are defined as

$$(1.22) \quad \Omega^k(\mathcal{A}) := \bigwedge^k (\Omega^1(\mathcal{A})),$$

where the wedge product is to be understood in the sense of graded algebras. Furthermore, one can develop a calculus on superforms similar to the one in ordinary geometry; in particular, one can define exterior derivative, interior products, Lie derivatives. Some caution is mandatory though; for instance, the exterior derivative is defined as an operator acting on the left, the presence of odd coordinates prevents the $\Omega^k(\mathcal{A})$'s from being zero at some stage [5], and so on, we refer the reader to [16, 13, 1] for details.

Superforms along a morphism $\Phi := (\phi, \varphi)$ are defined in the same way replacing the sheaf $\Omega^1(\mathcal{A})$ by the sheaf $\Omega^1(\Phi) := \text{Hom}(\mathfrak{X}(\Phi), \phi_* \mathcal{B})$.

Definition 1.3. The sheaf of superforms along a morphism $(\phi, \varphi): (N, \mathcal{B}_{\mathcal{N}}) \rightarrow (M, \mathcal{A}_{\mathcal{M}})$ are the sheaves

$$(1.23) \quad \Omega^k(\Phi) := \bigwedge^k (\Omega^1(\Phi)).$$

Again, it is easy to verify that superforms are superforms along the identity morphism of a supermanifold and that when the supermanifolds are ordinary manifolds this definition agrees with the one given in ordinary geometry.

2. Prolongations of supervector fields along a morphism.

2.1. A local basis on $\text{Der}(\tau_{\mathcal{M}})$.

Most of the time, we shall be dealing with supervector fields along a particular morphism, the projection of the tangent supermanifold of a given supermanifold on the supermanifold: let $\mathcal{M} = (M, \mathcal{A}_{\mathcal{M}})$ be a supermanifold of dimension $\dim \mathcal{M} = (m, n)$ and let $T\mathcal{M} = (TM, T\mathcal{A})$ be the tangent supermanifold, and consider the projection map $\tau_{\mathcal{M}}: T\mathcal{M} \rightarrow \mathcal{M}$ defined by $\tau_{\mathcal{M}} = (\tau_M, \tau_{\mathcal{A}})$, where $\tau_M: TM \rightarrow M$ is the usual projection and $\tau_{\mathcal{A}}: \mathcal{A} \rightarrow \tau_{M*} T\mathcal{A}$ is the morphism of sheaves given by

$$(2.1) \quad \tau_{\mathcal{A}}(\mathcal{U}): \mathcal{A}(\mathcal{U}) \cong C^\infty(\mathcal{U}) \otimes \bigwedge(\mathbb{R}^n) \longrightarrow T\mathcal{A}(\tau_{\mathcal{M}}^{-1}(\mathcal{U})) \cong C^\infty(\mathcal{U} \times \mathbb{R}^n) \otimes \bigwedge(\mathbb{R}^{2n})$$

$$f_0 + f_\alpha \theta_0^\alpha + f_{\alpha\beta} \theta_0^\alpha \theta_0^\beta + \dots \longmapsto F_0 + F_\alpha \theta_1^\alpha + F_{\alpha\beta} \theta_1^\alpha \theta_1^\beta + \dots$$

where $F_0 = f_0 \circ \tau_M$, $F_\alpha = f_\alpha \circ \tau_M$ and so on. (In general an element of $C^\infty(\mathcal{U} \times \mathbb{R}^n) \otimes \bigwedge(\mathbb{R}^{2n})$ is of the form:

$$(2.2) \quad F = F_0 + F_\alpha \theta_1^\alpha + F_{\alpha\beta} \theta_1^\alpha \theta_1^\beta + F_{\alpha\beta\gamma} \theta_1^\alpha \theta_1^\beta \theta_1^\gamma + \dots$$

Note that τ_M is surjective and $\tau_{\mathcal{A}}(\mathcal{U})$ is injective, hence, $\tau_{\mathcal{M}}$ is an epimorphism.

To simplify the notation we shall use, from now on, the following convention: for every open subset $\mathcal{U}$ of M for which $\mathcal{A}(\mathcal{U}) \cong C^\infty(\mathcal{U}) \otimes \bigwedge(\mathbb{R}^n)$, we write

$$(2.3a) \quad T\mathcal{A}(\mathcal{U}) = T\mathcal{A}(\tau_{\mathcal{M}}^{-1}(\mathcal{U}))$$

$$(2.3b) \quad T(T\mathcal{A})(\mathcal{U}) = T(T\mathcal{A})((\tau_M \circ \tau_{TM})^{-1}(\mathcal{U}))$$

$$(2.3c) \quad T^2\mathcal{A}(\mathcal{U}) = T^2\mathcal{A}((\tau_M \circ \tau_{TM} \circ i_{1,1})^{-1}(\mathcal{U}))$$

$$(2.3d) \quad T(T^2\mathcal{A})(\mathcal{U}) = T(T^2\mathcal{A})((\tau_M \circ \tau_{TM} \circ i_{1,1} \circ \tau_{T^2M})^{-1}(\mathcal{U}))$$

$$(2.3e) \quad T^3\mathcal{A}(\mathcal{U}) = T^3\mathcal{A}((\tau_M \circ \tau_{TM} \circ i_{1,1} \circ \tau_{3,2})^{-1}(\mathcal{U}))$$

On the other hand, let $\partial_{q_0^i}$ and $\partial_{\theta_0^\alpha}$: $\mathcal{A}(\mathcal{U}) \cong C^\infty(\mathcal{U}) \otimes \wedge(\mathbb{R}^n) \rightarrow T\mathcal{A}(\mathcal{U}) \cong C^\infty(\mathcal{U} \times \mathbb{R}^n) \otimes \wedge(\mathbb{R}^{2n})$ be the $\tau_{\mathcal{A}}$ -derivations defined by

$$(2.4) \quad \partial_{q_0^i}(f_{\alpha_1 \dots \alpha_k} \theta_0^{\alpha_1} \dots \theta_0^{\alpha_k}) = \frac{\partial f_{\alpha_1 \dots \alpha_k} \circ \tau_M}{\partial q_1^i} \theta_1^{\alpha_1} \dots \theta_1^{\alpha_k} = \frac{\partial f_{\alpha_1 \dots \alpha_k} \circ \tau_M \circ \psi^{-1}}{\partial q_1^i} \circ \psi \theta_1^{\alpha_1} \dots \theta_1^{\alpha_k},$$

and

$$(2.5) \quad \partial_{\theta_0^\alpha}(f_{\alpha_1 \dots \alpha_k} \theta_0^{\alpha_1} \dots \theta_0^{\alpha_k}) = \begin{cases} (-1)^{j-1} F_{\alpha_1 \dots \alpha_k} \theta_1^{\alpha_1} \dots \widehat{\theta_1^{\alpha_j}} \dots \theta_1^{\alpha_k} & \text{if } \alpha = \alpha_j \\ 0 & \text{if } \alpha \neq \alpha_i \quad i = 1, \dots, k, \end{cases}$$

where $F_{\alpha_1 \dots \alpha_k} = f_{\alpha_1 \dots \alpha_k} \circ \tau_M$ and $(\tau_M^{-1}(\mathcal{U}), \psi)$ is the chart of TM associated to the one of M .

It is easy to check that if f is homogeneous then

$$(2.6) \quad \begin{aligned} \partial_{q_0^i}(fg) &= \partial_{q_0^i}(f)\tau_{\mathcal{A}}(g) + \tau_{\mathcal{A}}(f)\partial_{q_0^i}(g) \\ \partial_{\theta_0^\alpha}(fg) &= \partial_{\theta_0^\alpha}(f)\tau_{\mathcal{A}}(g) + (-1)^{|f|}\tau_{\mathcal{A}}(f)\partial_{\theta_0^\alpha}(g) \end{aligned}$$

and that $\partial_{q_0^i}$ is even whereas $\partial_{\theta_0^\alpha}$ is odd.

We remind the reader that $\text{Der}(\tau_{\mathcal{M}})$ is a sheaf of graded modules, in other words, $\text{Der}(\tau_{\mathcal{M}})(\mathcal{U})$ is a $T\mathcal{A}(\mathcal{U})$ -module, the product being defined by

$$(2.7) \quad (gX)(f) = gX(f).$$

Since

$$(2.8) \quad \partial_{q_0^i}(q_0^j) = \delta_i^j, \quad \partial_{q_0^i}(\theta_0^\beta) = 0, \quad \partial_{\theta_0^\alpha}(q_0^j) = 0, \quad \text{and} \quad \partial_{\theta_0^\alpha}(\theta_0^\beta) = \delta_\alpha^\beta,$$

it follows that $\partial_{q_0^i}$ and $\partial_{\theta_0^\alpha}$ are linearly independent; moreover, one has

Proposition 2.1. *Let $X \in \text{Der}(\tau_{\mathcal{M}})$; if $X^i = X(q_0^i)$ and $X^\alpha = X(\theta_0^\alpha)$, then*

$$(2.9) \quad X = \sum_{i=1}^m X^i \partial_{q_0^i} + \sum_{\alpha=1}^n X^\alpha \partial_{\theta_0^\alpha}.$$

In other words, the $\partial_{q_0^i}$'s and the $\partial_{\theta_0^\alpha}$'s form a local basis of $\text{Der}(\tau_{\mathcal{M}})(\mathcal{U})$.

Proof: This is proved along the same lines as the similar proposition for the usual super-vector fields. We refer the reader to [17, Lemma 2.2.3]. $\square$

2.2. The total time derivative operator.

In non-graded geometry, the total time derivative operator $d_{\mathbf{T}}$ can be defined as the operator [7, 11, 19]

$$(2.10) \quad d_{\mathbf{T}}f = \widehat{df},$$

where, for each function $f \in C^\infty(\mathcal{U})$, $\widehat{df}$ denotes the function $\widehat{df}: TM \rightarrow \mathbb{R}$ given by

$$(2.11) \quad \widehat{df}(v_q) = df_q(v_q);$$

moreover, from the Leibniz's rule one has

$$(2.12) \quad d(\widehat{fg}) = g \circ \tau_M \widehat{df} + f \circ \tau_M \widehat{dg}.$$

Thus, if now f is a superfunction, say $f = f_0 + f_\alpha \theta_0^\alpha + f_{\alpha\beta} \theta_0^\alpha \theta_0^\beta + \dots \in C^\infty(\mathcal{U}) \otimes \wedge(\mathbb{R}^n)$, then we define the superfunction $\widehat{df} \in C^\infty(\mathcal{U} \times \mathbb{R}^n) \otimes \wedge(\mathbb{R}^{2n})$ by

$$(2.13) \quad \begin{aligned} \widehat{df} &= \widehat{df}_0 + \widehat{df}_\alpha \tau_{\mathcal{A}}(\theta_0^\alpha) + \tau_{\mathcal{A}}(f_\alpha) \widehat{d\theta_0^\alpha} + \widehat{df}_{\alpha\beta} \tau_{\mathcal{A}}(\theta_0^\alpha \theta_0^\beta) + \tau_{\mathcal{A}}(f_{\alpha\beta}) d(\theta_0^\alpha \theta_0^\beta) \dots \\ &= \widehat{df}_0 + \widehat{df}_\alpha \theta_1^\alpha + F_\alpha \widehat{d\theta_0^\alpha} + \widehat{df}_{\alpha\beta} \theta_1^\alpha \theta_1^\beta + F_{\alpha\beta} (\widehat{d\theta_0^\alpha} \theta_1^\beta + \theta_1^\alpha \widehat{d\theta_0^\beta}) \dots \\ &= \widehat{df}_0 + \widehat{df}_\alpha \theta_1^\alpha + F_\alpha \zeta_1^\alpha + \widehat{df}_{\alpha\beta} \theta_1^\alpha \theta_1^\beta + F_{\alpha\beta} (\zeta_1^\alpha \theta_1^\beta + \theta_1^\alpha \zeta_1^\beta) + \dots, \end{aligned}$$

where, as before, $F_\alpha = f_\alpha \circ \tau_M$ and so on.

Furthermore, in analogy with the non-graded case, we define the operator $\mathbf{T}: \mathcal{A}(\mathcal{U}) \rightarrow T\mathcal{A}(\mathcal{U})$ by

$$(2.14) \quad \mathbf{T}f = \widehat{df};$$

since in the non-graded case, the derivation associated to this operator corresponds to what physicists call the total derivative with respect to time, we call $\mathbf{T}$ the total time derivative operator. Moreover, it follows immediately from the definition that $\mathbf{T}$ is an element of degree 0 of $\text{Der}(\tau_{\mathcal{M}})$. We also note that $\mathbf{T}$ is well defined, for if $\mathcal{V}$ is another open subset of M and we denote the supercoordinates here with primes then

$$(2.15) \quad \widehat{df}' = \widehat{df}'.$$

Indeed,

$$(2.16) \quad \widehat{df}' = \widehat{df}'_0 + \widehat{df}'_\alpha \theta_1'^\alpha + \tau_{\mathcal{A}} f'_\alpha \zeta_1'^\alpha + \widehat{df}'_{\alpha\beta} \theta_1'^\alpha \theta_1'^\beta + \tau_{\mathcal{A}} f'_{\alpha\beta} (\zeta_1'^\alpha \theta_1'^\beta + \theta_1'^\alpha \zeta_1'^\beta) \dots,$$

and

$$(2.17) \quad \widehat{df}' = \widehat{df}'_0 + \widehat{df}'_\alpha \theta_1'^\alpha + (\tau_{\mathcal{A}} f'_\alpha)' \zeta_1'^\alpha + \widehat{df}'_{\alpha\beta} \theta_1'^\alpha \theta_1'^\beta + (\tau_{\mathcal{A}} f'_{\alpha\beta})' (\zeta_1'^\alpha \theta_1'^\beta + \theta_1'^\alpha \zeta_1'^\beta) \dots;$$

the equality follows since

$$(2.18) \quad (\tau_{\mathcal{A}} f'_\alpha)' = (f'_\alpha \circ \tau_M)' = f'_\alpha \circ \tau_M = \tau_{\mathcal{A}} f'_\alpha;$$

and

$$(2.19) \quad \widehat{df}'_\alpha = \widehat{df}'_\alpha,$$

since $\widehat{df}$ is well-defined on M .

On the other hand, using the ideas of previous section, we can write

$$(2.20) \quad \mathbf{T} = \sum_{i=1}^m v_i^i \partial_{q_0^i} + \sum_{\alpha=1}^n \zeta_1^\alpha \partial_{\theta_0^\alpha},$$

since

$$(2.21) \quad \mathbf{T}q_0^i = \widehat{dq_0^i} = v_1^i \quad \text{and} \quad \mathbf{T}\theta_0^\alpha = \widehat{d\theta_0^\alpha} = \zeta_1^\alpha.$$

2.9. Prolongations of $\mathbf{T}$.

In the non-graded case the best description of the prolongations of a vector field along, say, the projection τ_M is done in terms of a generalization, to higher orders, of the vector field $\mathbf{T}$. In this subsection we shall define the corresponding generalizations in our context.

First we consider the morphism $\mathcal{I}_{1,1}: (T^2M, T^2\mathcal{A}) \rightarrow (T(TM), T(T\mathcal{A}))$ of supermanifolds given by the pair $(i_{1,1}, I_{1,1})$ where, as before, $i_{1,1}: T^2M \rightarrow T(TM)$ is the map whose coordinate expression is

$$(2.22) \quad (q_2^i, v_2^i, a_2^i) \mapsto (q_{1,1}^i, v_{1,1}^i, a_{1,1}^i)$$

and $I_{1,1}: T(T\mathcal{A})(\mathcal{U}) \rightarrow T^2\mathcal{A}(\mathcal{U})$ is the homomorphism of superalgebras described by the assignments

$$(2.23a) \quad q_{1,1}^i \mapsto q_2^i \quad u_{1,1}^i \mapsto v_2^i$$

$$(2.23b) \quad v_{1,1}^i \mapsto v_2^i \quad a_{1,1}^i \mapsto a_2^i$$

$$(2.23c) \quad \theta_{1,1}^\alpha \mapsto \theta_2^\alpha \quad \eta_{1,1}^\alpha \mapsto \zeta_2^\alpha$$

$$(2.23d) \quad \zeta_{1,1}^\alpha \mapsto \zeta_2^\alpha \quad \xi_{1,1}^\alpha \mapsto \xi_2^\alpha.$$

(for instance, $F(q_{1,1}, v_{1,1}; u_{1,1}, a_{1,1}) \mapsto f(q_2, v_2, a_2) := F(q_{1,1}, v_{1,1}; v_{1,1}, a_{1,1})$). That $I_{1,1}$ is well defined follows immediately from the expressions of the transition functions of both supermanifolds; moreover, it is clear that $i_{1,1}$ is injective whereas $I_{1,1}$ is surjective for each $\mathcal{U}$, hence $\mathcal{I}_{1,1}$ is a monomorphism. Now we combine it with the morphism $\tau_{\mathcal{M}}$ to get the morphism $\mathcal{T}_{2,1} = \tau_{T\mathcal{M}} \circ \mathcal{I}_{1,1}: (T^2M, T^2\mathcal{A}) \rightarrow (T(TM), T(T\mathcal{A}))$.

To define $\mathbf{T}^{(1)}$ we mimic the construction of the non-graded case. Applying the ideas of subsection 2.1. to the tangent supermanifold $T\mathcal{M} = (TM, T\mathcal{A})$ we obtain a canonical supervector field along the morphism $\tau_{T\mathcal{M}}$ which we shall denote by $\mathcal{C}$ instead of $\mathbf{T}$, next we define the supervector field along the morphism $\mathcal{T}_{2,1}$, $\mathbf{T}^{(1)}: T\mathcal{A}(\mathcal{U}) \rightarrow T^2\mathcal{A}(\mathcal{U})$ by

$$(2.24) \quad \mathbf{T}^{(1)} = I_{1,1} \circ \mathcal{C}.$$

Motivated by subsection 2.2 we write

$$(2.25) \quad \mathcal{C} = \sum_{i=1}^m u_{1,1}^i \partial_{q_1^i} + \sum_{i=1}^m a_{1,1}^i \partial_{v_1^i} + \sum_{\alpha=1}^n \eta_{1,1}^\alpha \partial_{\theta_1^\alpha} + \sum_{\alpha=1}^n \xi_{1,1}^\alpha \partial_{\zeta_1^\alpha},$$

where $\partial_{q_1^i}$, $\partial_{v_1^i}$, $\partial_{\theta_1^\alpha}$ and $\partial_{\zeta_1^\alpha}: T\mathcal{A}(\mathcal{U}) \rightarrow T(T\mathcal{A}(\mathcal{U}))$ are defined in a similar way as (2.4) and (2.5). Hence if $F \in T\mathcal{A}(\mathcal{U}) \cong C^\infty(\mathcal{U} \times \mathbb{R}^n) \otimes \wedge(\mathbb{R}^{2n})$, say,

$$(2.26) \quad F = F_0(q_1, v_1) + F_\alpha(q_1, v_1) \theta_1^\alpha + {}_\alpha F(q_1, v_1) \zeta_1^\alpha + \cdots,$$

then

$$(2.27) \quad \begin{aligned} \mathbf{T}^{(1)}F(q_2, v_2, a_2, \theta_2, \zeta_2, \xi_2) &= \sum_{i=1}^m v_2^i \left(\partial_{q_1^i} F_0 \right) \circ i_{1,1}(q_2, v_2, a_2) \\ &+ \sum_{i=1}^m a_2^i \left(\partial_{q_1^i} F_0 \right) \circ i_{1,1}(q_2, v_2, a_2) \\ &+ \sum_{\alpha=1}^n \sum_{i=1}^m v_2^i \left(\partial_{q_1^i} F_\alpha \right) \circ i_{1,1}(q_2, v_2, a_2) \theta_2^\alpha \\ &+ \sum_{\alpha=1}^n \sum_{i=1}^m a_2^i \left(\partial_{v_1^i} F_\alpha \right) \circ i_{1,1}(q_2, v_2, a_2) \theta_2^\alpha \\ &+ \sum_{\alpha=1}^n F_\alpha \left(\tau_{TM} \circ i_{1,1}(q_2, v_2, a_2) \right) \zeta_2^\alpha \\ &+ \sum_{\alpha=1}^n \sum_{i=1}^m v_2^i \left(\partial_{q_1^i} {}_\alpha F \right) \circ i_{1,1}(q_2, v_2, a_2) \zeta_2^\alpha \\ &+ \sum_{\alpha=1}^n \sum_{i=1}^m a_2^i \left(\partial_{v_1^i} {}_\alpha F \right) \circ i_{1,1}(q_2, v_2, a_2) \zeta_2^\alpha \\ &+ \sum_{\alpha=1}^n {}_\alpha F \left(\tau_{TM} \circ i_{1,1}(q_2, v_2, a_2) \right) \xi_2^\alpha + \cdots; \end{aligned}$$

since the superfunctions at which $\partial_{q_1^i} F$, $\partial_{v_1^i} F$, $\partial_{\theta_1^\alpha} F$ and $\partial_{\zeta_1^\alpha} F$ do not depend on the variables u_2, a_2, η_2, ξ_2 , we can, somewhat loosely, write

$$(2.28) \quad \mathbf{T}^{(1)} = \sum_{i=1}^m v_2^i \partial_{q_1^i} + \sum_{i=1}^m a_2^i \partial_{v_1^i} + \sum_{\alpha=1}^n \zeta_2^\alpha \partial_{\theta_1^\alpha} + \sum_{\alpha=1}^n \xi_2^\alpha \partial_{\zeta_1^\alpha},$$

with the understanding that $\partial_{q_1^i}$ should be replaced by $\partial_{q_1^i} \circ I_{1,1}$ and so on.

The supervector field $\mathbf{T}^{(2)}$ is defined exactly in the same way: first we consider the morphism $\mathcal{I}_{2,1}: (T^3M, T^3\mathcal{A}) \rightarrow (T(T^2M), T(T^2\mathcal{A}))$ given by the pair $(i_{2,1}, I_{2,1})$ where $i_{2,1}: T^3M \rightarrow T(T^2M)$ is the function whose coordinate presentation is

$$(2.29) \quad (q_3^i, v_3^i, a_3^i, b_3^i) \mapsto (q_{1,2}^i, v_{1,2}^i, a_{1,2}^i, v_{1,2}^i, a_{1,2}^i, b_{1,2}^i)$$

and $I_{2,1}: T(T^2\mathcal{A})(\mathcal{U}) \rightarrow T^3\mathcal{A}(\mathcal{U})$ is the morphism of superalgebras described by the assignments

$$(2.30a) \quad q_{1,2}^i \mapsto q_3^i \quad r_{1,2}^i \mapsto v_3^i$$

$$\begin{aligned}
(2.30b) \quad & v_{1,2}^i \mapsto v_3^i & w_{1,2}^i & \mapsto a_3^i \\
(2.30c) \quad & a_{1,2}^i \mapsto a_3^i & b_{1,2}^i & \mapsto b_3^i \\
(2.30d) \quad & \theta_{1,2}^\alpha \mapsto \theta_3^\alpha & \rho_{1,2}^\alpha & \mapsto \zeta_3^\alpha \\
(2.30e) \quad & \zeta_{1,2}^\alpha \mapsto \zeta_3^\alpha & \lambda_{1,2}^\alpha & \mapsto \xi_3^\alpha \\
(2.30f) \quad & \xi_{1,2}^\alpha \mapsto \xi_3^\alpha & \sigma_{1,2}^\alpha & \mapsto \sigma_3^\alpha.
\end{aligned}$$

Next we consider the canonical supervector field along $\tau_{T^2\mathcal{M}}$, which, in analogy with $\mathbf{T}$ and $\mathcal{C}$, is the element of $\mathfrak{X}(\tau_{T^2\mathcal{M}})$ given by

$$(2.31) \quad \mathcal{E} := \sum_{i=1}^m r_{1,2}^i \partial_{q_2^i} + \sum_{i=1}^m w_{1,2}^i \partial_{v_2^i} + \sum_{i=1}^m b_{1,2}^i \partial_{a_2^i} + \sum_{\alpha=1}^n \rho_{1,2}^\alpha \partial_{\theta_2^\alpha} + \sum_{\alpha=1}^n \lambda_{1,2}^\alpha \partial_{\zeta_2^\alpha} + \sum_{\alpha=1}^n \sigma_{1,2}^\alpha \partial_{\xi_2^\alpha},$$

where $(q_{1,2}^i, v_{1,2}^i, a_{1,2}^i, r_{1,2}^i, w_{1,2}^i, b_{1,2}^i, \theta_{1,2}^\alpha, \zeta_{1,2}^\alpha, \xi_{1,2}^\alpha, \rho_{1,2}^\alpha, \lambda_{1,2}^\alpha, \sigma_{1,2}^\alpha)$ are the coordinates in $T(T^2\mathcal{M})$.

Finally, we define $\mathbf{T}^{(2)}: T^2\mathcal{A}(\mathcal{U}) \rightarrow T^3\mathcal{A}(\mathcal{U})$ by

$$(2.32) \quad \mathbf{T}^{(2)} := I_{2,1} \circ \mathcal{E},$$

which clearly is a supervector field along the morphism $\mathcal{T}_{3,2} := \tau_{T^2\mathcal{M}} \circ I_{2,1} := (\tau_{3,2}, T_{3,2})$, where $T_{3,2}: T^2\mathcal{A}(\mathcal{U}) \rightarrow T^3\mathcal{A}(\mathcal{U})$ is the morphism described by

$$\begin{aligned}
(2.33a) \quad & q_2^i \mapsto q_3^i \\
(2.33b) \quad & v_2^i \mapsto v_3^i \\
(2.33c) \quad & a_2^i \mapsto a_3^i \\
(2.33d) \quad & \theta_2^\alpha \mapsto \theta_3^\alpha \\
(2.33e) \quad & \zeta_2^\alpha \mapsto \zeta_3^\alpha \\
(2.33f) \quad & \xi_2^\alpha \mapsto \xi_3^\alpha.
\end{aligned}$$

Again, we write

$$(2.34) \quad \mathbf{T}^{(2)} := \sum_{i=1}^m v_3^i \partial_{q_2^i} + \sum_{i=1}^m a_3^i \partial_{v_2^i} + \sum_{i=1}^m b_3^i \partial_{a_2^i} + \sum_{\alpha=1}^n \zeta_3^\alpha \partial_{\theta_2^\alpha} + \sum_{\alpha=1}^n \xi_3^\alpha \partial_{\zeta_2^\alpha} + \sum_{\alpha=1}^n \sigma_3^\alpha \partial_{\xi_2^\alpha},$$

with the understanding that $\partial_{q_2^i}$ is replaced by $\partial_{q_2^i} \circ I_{2,1}$, etcetera.

2.4. Prolongations of supervector fields along $\tau_{\mathcal{M}}$.

The first prolongation of a supervector field X along $\tau_{\mathcal{M}}$ would be a supervector field along the morphism $\mathcal{T}_{2,1} := \tau_{T\mathcal{M}} \circ I_{1,1}$, in other words, along the pair $(\tau_{2,1}, T_{2,1})$, where $\tau_{2,1}$ is the standard projection and $T_{2,1} = I_{2,1} \circ \tau_{T\mathcal{M}}$. Nevertheless, motivated by the ideas of the previous subsection, we shall introduce such first prolongations using supervector

fields along the projection $\tau_{T\mathcal{M}}$. Thus, we denote by $\tilde{X}$ the supervector field along $\tau_{T\mathcal{M}}$ that has the following expression in local supercoordinates:

$$(2.35) \quad \tilde{X} = \sum_{i=1}^m (F^i \partial_{q^i} + H^i \partial_{v^i}) + \sum_{\alpha=1}^n (G^\alpha \partial_{\theta^\alpha} + K^\alpha \partial_{\zeta^\alpha}),$$

where $F^i, H^i, G^\alpha, K^\alpha$ are the superfunctions on $T(T\mathcal{A})(\mathcal{U})$ given by

$$(2.36) \quad \begin{aligned} \tilde{X}(q^i) &= F^i = \tau_{T\mathcal{A}}(X_{q^i}), & \tilde{X}(\theta^\alpha) &= G^\alpha = \tau_{T\mathcal{A}}(X_{\theta^\alpha}), \\ \tilde{X}(v^i) &= H^i = \mathcal{C}(X_{q^i}), & \tilde{X}(\zeta^\alpha) &= K^\alpha = \mathcal{C}(X_{\theta^\alpha}), \end{aligned}$$

and $X_{q^i}, X_{\theta^\alpha}$ are the local supercoordinates of X :

$$(2.37) \quad X = \sum_{i=1}^m X_{q^i} \partial_{q^i} + \sum_{\alpha=1}^n X_{\theta^\alpha} \partial_{\theta^\alpha}.$$

Definition 2.1. The first prolongation of a supervector field X along $\tau_{\mathcal{M}}$ is the supervector field along $\mathcal{T}_{2,1}$ given by

$$(2.38) \quad X^{(1)} = I_{1,1} \circ \tilde{X}.$$

Thus if we keep in mind our notation and write

$$(2.39) \quad X^{(1)} = \sum_{i=1}^m X_{q^i}^{(1)} \partial_{q_1^i} + \sum_{i=1}^m X_{v_1^i}^{(1)} \partial_{v_1^i} + \sum_{\alpha=1}^n X_{\theta_1^\alpha}^{(1)} \partial_{\theta_1^\alpha} + \sum_{\alpha=1}^n X_{\zeta_1^\alpha}^{(1)} \partial_{\zeta_1^\alpha},$$

then

$$(2.40) \quad \begin{aligned} X_{q^i}^{(1)} &= T_{2,1}(X_{q^i}) & X_{\theta^\alpha}^{(1)} &= T_{2,1}(X_{\theta^\alpha}) \\ X_{v^i}^{(1)} &= \mathbf{T}^{(1)}(X_{q^i}) & X_{\zeta^\alpha}^{(1)} &= \mathbf{T}^{(1)}(X_{\theta^\alpha}). \end{aligned}$$

This motivates the following definition

Definition 2.2. The second prolongation of a supervector field X along $\tau_{\mathcal{M}}$ is the supervector field along $\mathcal{T}_{3,2}$ given by

$$(2.41) \quad \begin{aligned} X^{(2)} &= \sum_{i=1}^m X_{q^i}^{(2)} \partial_{q_2^i} + \sum_{i=1}^m X_{v_2^i}^{(2)} \partial_{v_2^i} + \sum_{i=1}^m X_{a_2^i}^{(2)} \partial_{a_2^i} \\ &+ \sum_{\alpha=1}^n X_{\theta^\alpha}^{(2)} \partial_{\theta_2^\alpha} + \sum_{\alpha=1}^n X_{\zeta^\alpha}^{(2)} \partial_{\zeta_2^\alpha} + \sum_{\alpha=1}^n X_{\xi^\alpha}^{(2)} \partial_{\xi_2^\alpha}, \end{aligned}$$

where

$$(2.42) \quad \begin{aligned} X_{q^i}^{(2)} &= T_{3,2}(X_{q^i}^{(1)}) & X_{\theta^\alpha}^{(2)} &= T_{3,2}(X_{\theta^\alpha}^{(1)}) \\ X_{v^i}^{(2)} &= T_{3,2}(X_{v_1^i}^{(1)}) & X_{\zeta^\alpha}^{(2)} &= T_{3,2}(X_{\zeta^\alpha}^{(1)}) \\ X_{a^i}^{(2)} &= \mathbf{T}^{(2)}(X_{v_1^i}^{(1)}) & X_{\xi^\alpha}^{(2)} &= \mathbf{T}^{(2)}(X_{\xi^\alpha}^{(1)}). \end{aligned}$$

The main properties of these supervector fields are given in the following propositions.

Proposition 2.2. *If X is a supervector field along $\tau_{\mathcal{A}}$, then the following projectability conditions hold:*

- a- $X^{(1)} \circ \tau_{\mathcal{A}} = T_{2,1} \circ X$.
- b- $X^{(2)} \circ T_{2,1} = T_{3,2} \circ X^{(1)}$.
- c- $X^{(2)} \circ T_{2,0} = T_{3,1} \circ X$.

Proof: We start with a-. It is enough to consider superfunctions on $\mathcal{A}(\mathcal{U})$ of the form $f \theta_0^{\alpha_1} \dots \theta_0^{\alpha_k}$. Since $\tau_{\mathcal{A}}(f \theta_0^{\alpha_1} \dots \theta_0^{\alpha_k}) = f \circ \tau_{\mathcal{A}} \theta_1^{\alpha_1} \dots \theta_1^{\alpha_k}$, then

$$(2.43) \quad \tilde{X}(\tau_{\mathcal{A}}(f \theta_0^{\alpha_1} \dots \theta_0^{\alpha_k})) = F^i \frac{\partial f \circ \tau_M \circ \tau_{TM}}{\partial q_2^i} \theta_{1,1}^{\alpha_1} \dots \theta_{1,1}^{\alpha_k} + G^{\alpha_i} (-1)^{\alpha_i-1} f \circ \tau_M \circ \tau_{TM} \theta_{1,1}^{\alpha_1} \dots \widehat{\theta_{1,1}^{\alpha_i}} \dots \theta_{1,1}^{\alpha_k};$$

it follows from (2.36) that

$$(2.44) \quad X^{(1)} \circ \tau_{\mathcal{A}}(f \theta_0^{\alpha_1} \dots \theta_0^{\alpha_k}) = T_{2,1}(X_{q^i}) \frac{\partial f \circ \tau_M}{\partial q_1^i} \circ \tau_{2,1} \theta_2^{\alpha_1} \dots \theta_2^{\alpha_k} + T_{2,1}(X_{\theta^{\alpha_i}}) (-1)^{\alpha_i-1} f \circ \tau_M \circ \tau_{2,1} \theta_2^{\alpha_1} \dots \widehat{\theta_2^{\alpha_i}} \dots \theta_2^{\alpha_k}.$$

On the other hand, since

$$(2.45) \quad X(f \theta_0^{\alpha_1} \dots \theta_0^{\alpha_k}) = X_{q^i} \frac{\partial f \circ \tau_M}{\partial q_1^i} \theta_1^{\alpha_1} \dots \theta_1^{\alpha_k} + X_{\theta^{\alpha_i}} (-1)^{\alpha_i-1} f \circ \tau_M \theta_1^{\alpha_1} \dots \widehat{\theta_1^{\alpha_i}} \dots \theta_1^{\alpha_k},$$

it is clear that the $T_{2,1} \circ X(f \theta_0^{\alpha_1} \dots \theta_0^{\alpha_k})$ agrees with the right hand side of (2.44). Relation b- is proved using the same arguments and c- follows from b-. Indeed,

$$(2.46) \quad X^{(2)} \circ T_{2,0} = X^{(2)} \circ T_{2,1} \circ \tau_{\mathcal{A}} = T_{3,2} \circ X^{(1)} \tau_{\mathcal{A}} = T_{3,2} \circ T_{2,1} \circ X = T_{3,1} \circ X. \quad \square$$

Proposition 2.3. *Let X be a supervector field along the projection $\tau_{\mathcal{A}}$ then*

- a- $X^{(1)} \circ \mathbf{T} = \mathbf{T}^{(1)} \circ X$.
- b- $X^{(2)} \circ \mathbf{T}^{(1)} = \mathbf{T}^{(2)} \circ X^{(1)}$.

Proof: Taking into consideration (2.36) and the equations

$$(2.47) \quad \frac{\partial}{\partial q_2^j} \left(\frac{\partial f \circ \tau_M}{\partial q_1^i} \circ \tau_{TM} \right) = \frac{\partial}{\partial q_2^j} \left(\frac{\partial f \circ \tau_M}{\partial q_1^i} \circ \tau_{TM} \right),$$

$$(2.48) \quad \frac{\partial f \circ \tau_M}{\partial q_1^i} \circ \tau_{TM} = \frac{\partial f \circ \tau_M \circ \tau_{TM}}{\partial q_2^i},$$

it is easy to check that

$$(2.49) \quad \begin{aligned} \mathbf{T}^{(1)} \circ X(f \theta_0^{\alpha_1} \dots \theta_0^{\alpha_k}) &= X^{(1)} \circ \mathbf{T}(f \theta_0^{\alpha_1} \dots \theta_0^{\alpha_k}) \\ &= F^j \left\{ v_2^i \frac{\partial}{\partial q_2^j} \left(\frac{\partial f \circ \tau_M}{\partial q_1^i} \circ \tau_{2,1} \right) \theta_2^{\alpha_1} \dots \theta_2^{\alpha_k} \right. \\ &\quad + (-1)^{\alpha_i-1} \zeta_2^{\alpha_i} \frac{\partial f \circ \tau_M}{\partial q_1^i} \circ \tau_{2,1} \theta_2^{\alpha_1} \dots \widehat{\theta_2^{\alpha_i}} \dots \theta_2^{\alpha_k} \} \\ &\quad + H^i \frac{\partial f \circ \tau_M}{\partial q_1^i} \circ \tau_{2,1} \theta_2^{\alpha_1} \dots \theta_2^{\alpha_k} \\ &\quad + G^{\alpha_j} \left\{ (-1)^{\alpha_j-1} v^i \frac{\partial f \circ \tau_M}{\partial q_1^i} \circ \tau_{2,1} \theta_2^{\alpha_1} \dots \widehat{\theta_2^{\alpha_j}} \dots \theta_2^{\alpha_k} \right. \\ &\quad + \zeta_2^{\alpha_i} \left[\sum_{j < i} (-1)^{\alpha_i+\alpha_j-2} f \circ \tau_M \circ \tau_{2,1} \theta_2^{\alpha_1} \dots \widehat{\theta_2^{\alpha_i}} \dots \widehat{\theta_2^{\alpha_j}} \dots \theta_2^{\alpha_k} \right. \\ &\quad + \sum_{j > i} (-1)^{\alpha_i+\alpha_j-1} f \circ \tau_M \circ \tau_{2,1} \theta_2^{\alpha_1} \dots \widehat{\theta_2^{\alpha_i}} \dots \widehat{\theta_2^{\alpha_j}} \dots \theta_2^{\alpha_k} \} \} \\ &\quad + K^{\alpha_i} (-1)^{\alpha_i-1} f \circ \tau_M \circ \tau_{2,1} \theta_2^{\alpha_1} \dots \widehat{\theta_2^{\alpha_i}} \dots \theta_2^{\alpha_k}. \end{aligned}$$

As far as b- is concerned, we notice first that

$$(2.50) \quad \mathbf{T}^{(2)}(X_{q^i}^{(1)}) = T_{3,2}(X_{v^i}^{(1)}).$$

In fact, since $X_{v^i}^{(1)} = X^{(1)} \circ \mathbf{T}(q_0^i) = \mathbf{T}^{(1)} \circ X(q_0^i)$ then

$$(2.51) \quad T_{3,2}(X_{v^i}^{(1)}) = T_{3,2} \circ \mathbf{T}^{(1)} \circ X(q_0^i);$$

on the other hand, by Proposition 2.2.a, $X_{q^i}^{(1)} = X^{(1)} \circ \tau_{\mathcal{A}}(q_0^i) = T_{2,1} \circ X(q_0^i)$, hence

$$(2.52) \quad \mathbf{T}^{(2)}(X_{q^i}^{(1)}) = \mathbf{T}^{(2)} \circ T_{2,1}(q_0^i),$$

and (2.50) follows from Proposition 2.2.b. Similarly one can prove that

$$(2.53) \quad \mathbf{T}^{(2)}(X_{\theta^{\alpha}}^{(1)}) = T_{3,2}(X_{\zeta^{\alpha}}^{(1)}).$$

Using (2.50) and (2.53) relation b- follows using the same kind of computations as before. $\square$

We point out that the definitions of $X^{(1)}$ and $X^{(2)}$ were chosen so that they are precisely the supervector fields that satisfy the equations in Propositions 2.2. and 2.3.

2.5. Φ^* -derivations.

Let $\Phi = (\phi, \varphi): (N, \mathcal{B}) \rightarrow (M, \mathcal{A})$ be a morphism of graded manifolds and let $\mathcal{U}$ be an open subset of M ; the morphism of supercommutative superalgebras

$$(2.54) \quad \varphi_{\mathcal{U}}: \mathcal{A}(\mathcal{U}) \rightarrow \mathcal{B}(\phi^{-1}(\mathcal{U})),$$

can be extended, in a unique way, to a morphism of bigraded supercommutative algebras

$$(2.55) \quad \varphi^*: \Omega(\mathcal{U}, \mathcal{A}) \rightarrow \Omega(\phi^{-1}(\mathcal{U}), \mathcal{B}),$$

that commutes with exterior differentiation [16, theorem 4.4]. Moreover such morphisms are compatible with the restriction maps and agree with $\varphi_{\mathcal{U}}$ on 0-forms.

Definition 2.3. A Φ^* -derivation of bidegree $(r, j) \in \mathbb{Z} \oplus \mathbb{Z}_2$ is a map

$$(2.56) \quad D: \Omega(\mathcal{U}, \mathcal{A}) \rightarrow \Omega(\phi^{-1}(\mathcal{U}), \mathcal{B}),$$

for each open subset $\mathcal{U}$ of M such that:

- a- $D(\Omega^k(\mathcal{U}, \mathcal{A})_i) \subset \Omega^{k+r}(\phi^{-1}(\mathcal{U}), \mathcal{B})_{i+j} \quad \forall (k, i) \in \mathbb{Z} \otimes \mathbb{Z}_2$.
- b- $D(\omega + \nu) = D(\omega) + D(\nu)$.
- c- $D(\omega \wedge \nu) = D(\omega) \wedge \varphi^* \nu + (-1)^{kr+ij} \varphi^* \omega \wedge D(\nu)$, if $|\omega| = (k, i)$.
- d- $D(\lambda \mathbf{1}) = 0$, for each $\lambda \in \mathbb{R}$ ($\mathbf{1}$ denotes the identity of the superalgebra $\mathcal{A}(\mathcal{U})$).

Notice that Φ^* -derivations are local operators:

Proposition 2.4. Let D be a Φ^* -derivation of bidegree (r, j) and let $\mathcal{U}$ be an open subset of M . If ω is a k -form on $(M, \mathcal{A})$ such that $\omega|_{\mathcal{U}} = 0$ then $D(\omega)|_{\phi^{-1}(\mathcal{U})} = 0$.

Proof: Let x be a point of $\phi^{-1}(\mathcal{U})$ and let V be a neighborhood of $\phi(x)$ such that $\overline{V} \subseteq \mathcal{U}$. We use a partition of unity ([16, lemma 2.4, p. 241]) to write $1 = g + h$, where the superfunctions h and g on M are such that $\text{Supp } g \subseteq \mathcal{U}$ and $\text{Supp } h \subseteq M - \overline{V}$.

Since $\omega|_{\mathcal{U}} = 0$ then $g\omega = 0$ and

$$(2.57) \quad 0 = D(g\omega) = D(g)\varphi^* \omega + (-1)^{kr+ij} \varphi^* g D(\omega).$$

The compatibility of φ^* with the restriction maps implies that $\varphi^* \omega|_{\phi^{-1}(V)} = 0$; hence

$$(2.58) \quad 0 = \varphi^*(g)|_{\phi^{-1}(V)} D(\omega)|_{\phi^{-1}(V)} = \varphi^*(g|_V) D(\omega)|_{\phi^{-1}(V)},$$

and since $\varphi^*(g|_V) \neq 0$, we conclude that $D(\omega)|_{\phi^{-1}(V)} = 0$ and the proposition follows from the arbitrariness of the point x . $\square$

The main consequence of the previous proposition is that the Φ^* -derivations are determined by their actions on 0-forms and 1-forms. In fact, writting ω in local coordinates the conditions b- and c- of the definition allow us to write $D(\omega)$ in terms of images under D of 0-forms and 1-forms. Moreover, since $\Omega^1(M, \mathcal{A})$ is locally generated by exact forms, the action of a Φ^* -derivation D on a general 1-form is determined by the action of D on 0-forms and exact 1-forms.

In particular, to each homogeneous supervector field $X \in \mathfrak{X}(\Phi)$ we associate the Φ^* -derivation of bidegree $(-1, |X|)$ defined by

$$(2.59) \quad i_X f = 0 \quad \text{and} \quad i_X df = X(f),$$

for each superfunction f on M . For arbitrary $X \in \mathfrak{X}(\Phi)$ we shall write $i_X = i_{X_0} + i_{X_1}$ where X_0 and X_1 are the homogeneous components of X .

We also associate to $X \in \mathfrak{X}(\Phi)$ the Φ^* -derivation of bidegree $(0, |X|)$ given by

$$(2.60) \quad d_X f = X(f) \quad \text{and} \quad d_X d^{\mathcal{M}} f = d^{\mathcal{N}} X(f).$$

It follows that

$$(2.61) \quad d_X = i_X d^{\mathcal{M}} + d^{\mathcal{N}} i_X.$$

Indeed; both are Φ^* -derivations of the same bidegree that agree on 0-forms and exact 1-forms, as it is easy to check from the defining relations. It is also easy to check, using (2.61), that d_X commutes with exterior derivatives:

$$(2.62) \quad d_X \circ d^{\mathcal{M}} = d^{\mathcal{N}} \circ d_X.$$

Now we extend the projectability condition and the relations of Proposition 2.2. to the Φ^* -derivation d_X and prove similar conditions for the Φ^* -derivation i_X .

Proposition 2.5. Let X be a supervector field along $\tau_{\mathcal{M}}$. Then for $k = 1, 2$ the following projectability conditions hold: (we use the notation $X^{(0)} = X$ and $T_{1,0} = \tau_{\mathcal{A}}$)

- a- $d_{X^{(k)}} \circ T_{k,k-1}^* = T_{k+1,k}^* \circ d_{X^{(k-1)}}$.
- b- $i_{X^{(k)}} \circ T_{k,k-1}^* = T_{k+1,k}^* \circ i_{X^{(k-1)}}$.

Proof: Since both $T_{k,k-1}^*$ and $T_{k+1,k}^*$ are morphisms of superalgebras, then $d_{X^{(k)}} \circ T_{k,k-1}^*$ and $T_{k+1,k}^* \circ d_{X^{(k-1)}}$ are both $T_{k+1,k-1}^*$ -derivations of bidegree $(0, |X|)$. By Proposition 2.2. they agree on 0-forms (in general, $\varphi = \varphi^*$ on 0-forms); moreover since all the operators in relation a- commute with the exterior derivatives, Proposition 2.2. also implies that the two derivations agree on exact 1-forms and therefore they must be equal, so a- is proved.

Similarly, since

$$(2.63) \quad \begin{aligned} T_{k+1,k}^* \circ i_{X^{(k-1)}}(df) &= T_{k+1,k}^* d_{X^{(k-1)}} f = d_{X^{(k)}} \circ T_{k,k-1}^* f \\ &= i_{X^{(k)}} d T_{k,k-1}^* f \\ &= i_{X^{(k)}} \circ T_{k,k-1}^*(df), \end{aligned}$$

b-) follows using the same argument. $\square$

Proposition 2.6. Let X be a supervector field along $\tau_{\mathcal{M}}$. Then, for $k = 1, 2$,

- a- $d_{X^{(k)}} \circ d_{T^{(k-1)}} = d_{T^{(k)}} \circ d_{X^{(k-1)}}$.
- b- $i_{X^{(k)}} \circ d_{T^{(k-1)}} = d_{T^{(k)}} \circ i_{X^{(k-1)}}$.

Proof: Since the d 's commute with exterior derivatives, Proposition 2.3. implies that

$$(2.64) \quad d_{X^{(k)}} \circ d_{T^{(k-1)}}(df) = d_{T^{(k)}} \circ d_{X^{(k-1)}}(df).$$

On the other hand, assume that relation a-) holds on k -forms and let ω be a $k+1$ -form; locally we can write ω as $\omega = \nu \wedge df$, where ν is a k -form.

Now, $d_{T^{(k)}}$ and $d_{T^{(k-1)}}$ are derivations of bidegree $(0,0)$, whereas $d_{X^{(k-1)}}$ and $d_{X^{(k)}}$ are of bidegree $(0, |X|)$, hence using the induction hypothesis and the Proposition 2.5a. we have

$$\begin{aligned}
(2.65) \quad d_{X^{(k)}} \circ d_{T^{(k-1)}}(\omega) &= d_{X^{(k)}}(d_{T^{(k-1)}}\nu \wedge T_{k,k-1}^* df + T_{k,k-1}^* \nu \wedge d_{T^{(k-1)}} df) \\
&= d_{X^{(k)}} \circ d_{T^{(k-1)}}\nu \wedge T_{k+1,k}^* \circ T_{k,k-1}^* df \\
&\quad + (-1)^{|X||\nu|} T_{k+1,k}^* \circ d_{T^{(k-1)}}\nu \wedge d_{X^{(k)}} \circ T_{k,k-1}^* df \\
&\quad + d_{X^{(k)}} \circ T_{k,k-1}^* \nu \wedge T_{k+1,k}^* \circ d_{T^{(k-1)}} df \\
&\quad + (-1)^{|X||\nu|} T_{k+1,k}^* \circ T_{k,k-1}^* \nu \wedge d_{X^{(k)}} \circ d_{T^{(k-1)}} df \\
&= d_{T^{(k)}} \circ d_{X^{(k-1)}}\nu \wedge T_{k+1,k}^* \circ T_{k,k-1}^* df \\
&\quad + (-1)^{|X||\nu|} d_{T^{(k)}} \circ T_{k,k-1}^* \nu \wedge T_{k+1,k}^* \circ d_{X^{(k-1)}} df \\
&\quad + T_{k+1,k}^* \circ d_{X^{(k-1)}}\nu \wedge d_{T^{(k)}} \circ T_{k,k-1}^* df \\
&\quad + (-1)^{|X||\nu|} T_{k+1,k}^* \circ T_{k,k-1}^* \nu \wedge d_{T^{(k)}} \circ d_{X^{(k-1)}} df \\
&= d_{T^{(k)}}(d_{X^{(k-1)}}\nu \wedge T_{k+1,k}^* df) \\
&\quad + (-1)^{|X||\nu|} d_{T^{(k)}}(T_{k,k-1}^* \nu \wedge d_{X^{(k-1)}} df) \\
&= d_{T^{(k)}} \circ d_{X^{(k-1)}}(\nu \wedge df);
\end{aligned}$$

thus, a-) follows by induction on the order of the forms.

As far as b-) is concerned, we notice that

$$\begin{aligned}
(2.66) \quad d_{T^{(k)}} \circ i_{X^{(k-1)}}(df) &= d_{T^{(k)}} \circ d_{X^{(k-1)}} f = d_{X^{(k)}} \circ d_{T^{(k-1)}} f \\
&= i_{X^{(k)}} d d_{T^{(k-1)}} f \\
&= i_{X^{(k)}} d_{T^{(k-1)}}(df);
\end{aligned}$$

then again, b-) follows by induction on the order of the forms. $\square$

Lemma 2.1. Let $\Phi = (\phi, \varphi): (N, \mathcal{B}) \rightarrow (M, \mathcal{A})$ and $\Pi = (\pi, \varpi): (P, \mathcal{C}) \rightarrow (N, \mathcal{B})$ be morphisms of supermanifolds. If Y is a supervector field along Φ and Z is a supervector field along Π then

$$a- \quad i_{Z \circ \varphi} = i_Z \circ \varphi^*.$$

$$b- \quad i_{\varpi \circ Y} = \varpi^* \circ i_Y.$$

Proof: It is easy to check that both $i_{Z \circ \varphi}$ and $i_Z \circ \varphi^*$ are $(\Phi \circ \Pi)^*$ -derivations of bidegree $(-1, |Z|)$; moreover they agree on exact 1-forms; in fact, since φ^* commutes with exterior differentiation then

$$(2.67) \quad i_{Z \circ \varphi} df = Z \circ \varphi(f) = Z(\varphi(f)) = i_Z d(\varphi(f)) = i_Z \circ \varphi^*(df),$$

for each $f \in \mathcal{A}$. Hence the $(\Phi \circ \Pi)^*$ -derivations are equal.

Relation b-) is proved in the same manner. $\square$

3. Noether's theorem in graded geometry.

3.1. Super SODE's.

Our next goal is to study super SODE's which is nothing but a natural generalization of a Second Order Differential Equation in the graded context that we have been considering.

Definition 3.1. A super SODE is a supervector field Γ on $T\mathcal{M}$ such that

$$(3.1) \quad \Gamma \circ \tau_{\mathcal{A}} = \mathbf{T}$$

Since $\mathbf{T}(q_0^i) = v_1^i$ and $\mathbf{T}(\theta_0^\alpha) = \zeta_1^\alpha$, it follows from (3.1) that

$$(3.2) \quad \Gamma(q_1^i) = v_1^i \quad \text{and} \quad \Gamma(\theta_1^\alpha) = \zeta_1^\alpha,$$

hence the local expression of Γ is

$$(3.3) \quad \Gamma = v_1^i \partial_{q_1^i} + \zeta_1^\alpha \partial_{\theta_1^\alpha} + F^i \partial_{v_1^i} + G^\alpha \partial_{\zeta_1^\alpha},$$

where F^i and G^α are arbitrary superfunctions; moreover, since $\mathbf{T}$ is even, Γ must also be even and therefore F^i and G^α are, respectively, even and odd superfunctions.

In analogy with the non-graded case we consider the set $\mathcal{D}$ of all morphism of superalgebras $\delta: T^2\mathcal{A}(\mathcal{U}) \rightarrow T\mathcal{A}(\mathcal{U})$ satisfying

$$(3.4) \quad \delta \circ T_{2,1} = \text{id}_{T\mathcal{A}(\mathcal{U})}.$$

Proposition 3.1. If $\Gamma \in \mathcal{D}$, then $\delta \circ \mathbf{T}^{(1)}$ is a super SODE.

Proof: It is easy to check that $\delta \circ \mathbf{T}^{(1)}$ is an even superderivation. On the other hand, condition (3.1) follows immediately from the projectability condition. $\square$

The converse is also true:

Proposition 3.2. If Γ is super SODE, then there exists $\gamma \in \mathcal{D}$ such that $\Gamma = \gamma \circ \mathbf{T}^{(1)}$.

Proof: If Γ is written in local supercoordinates as $\Gamma = v_1^i \partial_{q_1^i} + \zeta_1^\alpha \partial_{\theta_1^\alpha} + F^i \partial_{v_1^i} + G^\alpha \partial_{\zeta_1^\alpha}$, then we consider the morphism of superalgebras $\gamma: T^2\mathcal{A}(\mathcal{U}) \rightarrow T\mathcal{A}(\mathcal{U})$ described by the assignments

$$(3.5a) \quad q_2^i \mapsto q_1^i, \quad \theta_2^\alpha \mapsto \theta_1^\alpha,$$

$$(3.5b) \quad v_2^i \mapsto v_1^i, \quad \zeta_2^\alpha \mapsto \zeta_1^\alpha,$$

$$(3.5c) \quad a_2^i \mapsto F_1^i, \quad \xi_2^\alpha \mapsto G_1^\alpha.$$

First we notice that $\gamma \in \mathcal{D}$; for if $H = H_0 + H_\alpha \theta_1^\alpha + {}_\alpha H \zeta_1^\alpha + \dots \in T\mathcal{A}(\mathcal{U})$, then

$$(3.6) \quad \gamma \circ T_{2,1}(H) = \gamma(H_0 \circ \tau_{2,1} + H_\alpha \circ \tau_{2,1} \theta_2^\alpha + {}_\alpha H \circ \tau_{2,1} \zeta_2^\alpha + \dots) = H,$$

since $H_0 \circ \tau_{2,1} + H_\alpha \circ \tau_{2,1} \theta_2^\alpha + {}_\alpha H \circ \tau_{2,1} \zeta_2^\alpha + \dots$ does not depend neither on a_2^i nor on ξ_2^α .

On the other hand,

$$(3.7) \quad \Gamma(H) = v_1^i \partial_{q_1^i} H_0 + F^i \partial_{v_1^i} H_0 + v_1^i \partial_{q_1^i} H_\alpha \theta_1^\alpha + H_\alpha \zeta_1^\alpha + F^i \partial_{v_1^i} H_\alpha \theta_1^\alpha + v_1^i \partial_{q_1^i} H \zeta_1^\alpha + F^i \partial_{v_1^i} H \theta_1^\alpha + G^\alpha H + \dots,$$

and

$$(3.8) \quad \mathbf{T}^{(1)}(H) = v_2^i \partial_{q_1^i} H_0 + a_2^i \partial_{v_1^i} H_0 + v_2^i \partial_{q_1^i} H_\alpha \theta_2^\alpha + a_2^i \partial_{v_1^i} H_\alpha \theta_2^\alpha + H_\alpha \circ \tau_{2,1} \zeta_2^\alpha + v_2^i \partial_{q_1^i} H \zeta_2^\alpha + a_2^i \partial_{v_1^i} H \zeta_2^\alpha + \xi_2^\alpha H \circ \tau_{2,1} + \dots;$$

since γ is a morphism it is clear that

$$(3.9) \quad \Gamma(H) = \gamma \circ \mathbf{T}^{(1)}(H).$$

It remains to prove that the definition of γ does not depend on the choice of coordinates. Since F^i and G^α are coordinates of a supervector field, under a change of supercoordinates they transform as

$$(3.10) \quad F'^i = \Gamma(v_1'^i) = \Gamma\left(\frac{\partial \phi^i}{\partial q_1^j} v_1^j\right) = v_1^k \frac{\partial}{\partial q_1^k} \left(\frac{\partial \phi^i}{\partial q_1^j} v_1^j\right) + F^k \frac{\partial}{\partial v_1^k} \left(\frac{\partial \phi^i}{\partial q_1^j} v_1^j\right) = v_1^k v_1^j \frac{\partial^2 \phi^i}{\partial q_1^k \partial q_1^j} + F^j \frac{\partial \phi^i}{\partial q_1^j},$$

and

$$(3.11) \quad G'^\alpha = \Gamma(\zeta_1'^\alpha) = \Gamma\left(\frac{\partial \psi_\beta^\alpha}{\partial q_1^j} v_1^j \theta_1^\beta + \psi_\beta^\alpha \zeta_1^\beta\right) = \frac{\partial^2 \psi_\beta^\alpha}{\partial q_1^k \partial q_1^j} v_1^k v_1^j \theta_1^\beta + 2 \frac{\partial \psi_\beta^\alpha}{\partial q_1^j} \zeta_1^\beta v_1^j + F^i \frac{\partial \psi_\beta^\alpha}{\partial q_1^i} \theta_1^\beta + G^\beta \psi_\beta^\alpha.$$

On the other hand, one has

$$(3.12) \quad \alpha_2'^i = \frac{\partial^2 \phi^i}{\partial q_2^k \partial q_2^j} v_2^k v_2^j + \frac{\partial \phi_2^i}{\partial q_2^j} \alpha_2^j, \quad \xi_2'^\alpha = \frac{\partial^2 \psi_\beta^\alpha}{\partial q_2^k \partial q_2^j} v_2^k v_2^j \theta_2^\beta + 2 \frac{\partial \psi_\beta^\alpha}{\partial q_2^j} \zeta_2^\beta v_2^j + \frac{\partial \psi_\beta^\alpha}{\partial q_2^j} a_2^j \theta_2^\beta + \psi_\beta^\alpha \xi_2^\beta;$$

it is now an easy computation to check that γ is well-defined. $\square$

Thus, we have obtained two equivalent definitions of a super SODE; there is yet another equivalent definition in terms of the vertical superendomorphism [14,15], that we shall not consider here.

Following [18], we say that a supervector field Y on $T\mathcal{M}$ is Newtonoid with respect to a given super SODE Γ if

$$(3.13) \quad S([Y, \Gamma]) = 0,$$

where S denotes the vertical superendomorphism [14]. From the theory around the version of Noether's theorem given by Marino and Mukunda [18], generalized by Ibrat and Marín [14] in graded geometry, we know that the candidates for symmetries of a Hamiltonian dynamical system determined by a super-Lagrangian L are the elements of the set of all the Newtonoid supervector fields, denoted by $\mathfrak{X}_\Gamma$; the following result is therefore interesting.

Theorem 3.1. *There exists a one to one correspondence between $\mathfrak{X}(\tau_{\mathcal{M}})$ and $\mathfrak{X}_\Gamma$.*

Proof: We consider the map $J_\Gamma: \mathfrak{X}(\tau_{\mathcal{M}}) \rightarrow \mathfrak{X}_\Gamma$ defined by

$$(3.14) \quad J_\Gamma(X) = \gamma \circ X^{(1)},$$

where γ is the morphism corresponding to Γ by the previous Proposition. Using Propositions 2.2 and 2.3, and the fact that Γ is a super SODE, we prove that $J_\Gamma(X)$ is really in $\mathfrak{X}_\Gamma$.

$$(3.15) \quad \begin{aligned} [\Gamma, J_\Gamma(X)] \circ \tau_{\mathcal{A}} &= \Gamma \circ \gamma \circ X^{(1)} \circ \tau_{\mathcal{A}} - \gamma \circ X^{(1)} \circ \Gamma \circ \tau_{\mathcal{A}} \\ &= \Gamma \circ \gamma \circ T_{2,1} \circ X - \gamma \circ X^{(1)} \circ \mathbf{T} \\ &= \Gamma \circ X - \gamma \circ \mathbf{T}^{(1)} \circ X \\ &= (\Gamma - \gamma \circ \mathbf{T}^{(1)}) \circ X = 0. \end{aligned}$$

Now let $Y \in \mathfrak{X}_\Gamma$ and set $X = Y \circ \tau_{\mathcal{A}}$; clearly $X \in \mathfrak{X}(\tau_{\mathcal{M}})$; moreover,

$$(3.16) \quad \gamma \circ X^{(1)} \circ \tau_{\mathcal{A}} = \gamma \circ T_{2,1} \circ X = X = Y \circ \tau_{\mathcal{A}},$$

hence,

$$(3.17) \quad (Y - \gamma \circ X^{(1)}) \tau_{\mathcal{A}} = 0;$$

in other words, $Y - \gamma \circ X^{(1)}$ is a vertical field and since both, $\gamma \circ X^{(1)}$ and Y are in $\mathfrak{X}_\Gamma$ they must be equal and the map $Y \mapsto X$ is the inverse of J_Γ . $\square$

3.2. Φ -semibasic forms.

Definition 3.2. Let $\Phi = (\phi, \varphi): (N, \mathcal{B}) \rightarrow (M, \mathcal{A})$ be a morphism of graded manifolds.

- a- A supervector field Y on $(N, \mathcal{B})$ is called vertical with respect to Φ if $Y \circ \varphi \equiv 0$.
- b- A k -form on $(N, \mathcal{B})$ is called Φ -semibasic if $i_Y \omega = 0$ for every vertical supervector field Y on $(N, \mathcal{B})$.

When Φ is a submersion one can choose supercoordinates [17] $(q^i, v^j, \theta^\alpha, \zeta^\beta)$ on $\mathcal{N}$ and $(\bar{q}^i, \bar{\theta}^\alpha)$ on $\mathcal{M}$, where $i = 1, \dots, m$, $j = 1, \dots, r - m$, $\alpha = 1, \dots, n$ and $\beta = 1, \dots, s - n$ if $\dim \mathcal{M} = (m, n)$ and $\dim \mathcal{N} = (r, s)$, such that

$$(3.18) \quad \varphi(\bar{q}^i) = q^i \quad \text{and} \quad \varphi(\bar{\theta}^\alpha) = \theta^\alpha.$$

In these supercoordinates, a vertical supervector field Y would be of the form

$$(3.19) \quad Y = H^j \frac{\partial}{\partial q^j} + K^\beta \frac{\partial}{\partial \theta^\beta};$$

whereas a Φ -semibasic form ω would be written in terms of these supercoordinates as

$$(3.20) \quad \omega = A^i dq^i + C^\alpha d\theta^\alpha,$$

where H^j, K^β, A^i and C^α are superfunctions.

Our interest in Φ -semibasic forms lies in the following theorem:

Theorem 3.2. *If Φ is a submersion then $\bigwedge(\Phi)$ is isomorphic to $\bigwedge_{sb}(\Phi)$, the set of all Φ -semibasic forms.*

Proof: First we consider the map $\iota: \bigwedge^k(\Phi) \rightarrow \bigwedge^k_{sb}(\Phi)$ defined by

$$(3.21) \quad \iota(\omega)(Y_1, \dots, Y_k) := \omega(Y_1 \circ \varphi, \dots, Y_k \circ \varphi).$$

On the other hand, we define $j: \bigwedge^k_{sb}(\Phi) \rightarrow \bigwedge^k(\Phi)$ in the following way: since Φ is a submersion we choose supercoordinates on $\mathcal{M}$ and $\mathcal{N}$ so that (3.18) holds. Now, for each supervector field X on $\mathcal{U}$ along Φ , we can choose a supervector field Y on $\text{Der}(\mathcal{B}(\phi^{-1}(\mathcal{U})))$ such that $X = Y \circ \varphi$ on $\mathcal{A}(\mathcal{U})$ (for instance, if $X = X^i \frac{\partial}{\partial q^i} + X^\alpha \frac{\partial}{\partial \theta^\alpha}$, we may take $Y = \varphi(X^i) \frac{\partial}{\partial q^i} + Y^j \frac{\partial}{\partial w^j} + \varphi(X^\alpha) \frac{\partial}{\partial \theta^\alpha} + Y^\beta \frac{\partial}{\partial \zeta^\beta}$, where Y^j and Y^β are arbitrary superfunctions on $\mathcal{B}(\phi^{-1}(\mathcal{U}))$). Thus, if $\nu \in \bigwedge^k_{sb}(\Phi)$ we define $j(\nu) \in \bigwedge^k(\Phi, \mathcal{U})$ by

$$(3.22) \quad j(\nu)(X_1, \dots, X_k) := \nu(Y_1, \dots, Y_k),$$

where Y_i is a supervector field in $\text{Der}(\mathcal{B}(\phi^{-1}(\mathcal{U})))$ such that $X_i = Y_i \circ \varphi$. Note that since ν is Φ -semibasic, the definition of $j(\nu)$ does not depend on the choice of the Y^i 's. Moreover, it is clear that ι and j are linear maps such that $\iota(j\nu) = \nu$ for every $\nu \in \bigwedge^k_{sb}(\Phi)$, whereas $j(\iota\omega) = \omega$ for every $\omega \in \bigwedge^k(\Phi, \mathcal{U})$, hence ι is an isomorphism. $\square$

For convenience, from now on, we shall use the notation $j\nu = \tilde{\nu}$.

Lemma 3.1. *Let ω be a $\tau_{k,0}$ -semibasic 1-form on $T^k\mathcal{M}$ ($k = 1, 2$) and let X be a supervector field along $\tau_{1,0}$, then*

$$(3.23) \quad i_{X^{(*)}}\omega = T_{k+1,k}\tilde{\omega}(T_{k,1} \circ X).$$

Proof: First of all, it is enough to consider the case when ω is an exact 1-form, say $\omega = dF$ for some $F \in T^k\mathcal{A}$. Actually, since ω is $\tau_{k,0}$ -semibasic, $F = T_{k,0}f$ for some $f \in \mathcal{A}$. Then

$$(3.24) \quad i_{X^{(*)}}dF = i_{X^{(*)}} \circ T_{k,0}df = T_{k+1,1} \circ i_X df = T_{k+1,1}X(f).$$

On the other hand, if Y is a supervector field on $T^k\mathcal{M}$ such that $T_{k,1} \circ X = Y \circ T_{k,0}$, then

$$(3.25) \quad (dF)^\vee(T_{k,1} \circ X) = d(T_{k,0}f)(Y) = Y \circ T_{k,0}(f) = T_{k,1} \circ X(f),$$

hence

$$(3.26) \quad T_{k+1,k}(dF)^\vee(T_{k,1} \circ X) = T_{k+1,1} \circ X(f) = i_{X^{(*)}}dF. \quad \square$$

3.3. Noether's theorem.

The situation in Lagrangian supermechanics is quite similar to that of ordinary Lagrangian mechanics [14]. In particular, for a regular superlagrangian $L \in T\mathcal{A}$, the Euler-Lagrange equation:

$$(3.27) \quad i_{\Gamma_L}\Omega_L = dE_L$$

has a unique solution Γ_L ; moreover the Euler-Lagrange supervector field Γ_L is a super SODE [14].

As in ordinary mechanics the superenergy is defined in supermechanics as $E_L := \Delta L - L$; thus, using the Cartan identity we rewrite (3.27) as

$$(3.28) \quad dL = d\Delta L + i_{\Gamma}d\theta_L = \mathcal{L}_{\Gamma}\theta_L,$$

where $\mathcal{L}_{\Gamma}\theta_L$ denotes the Lie derivative with respect to Γ_L [16] of the Cartan 1-form θ_L . On the other hand, since Γ is a super SODE there exists a morphism of superalgebras γ such that $\mathcal{L}_{\Gamma} = \gamma^* \circ d_{T^{(1)}}$ and $\gamma^* \circ T_{2,1}^* = \text{id}$, hence the Euler-Lagrange equation can be rewritten as

$$(3.29) \quad 0 = \gamma^* \circ d_{T^{(1)}}\theta_L - \gamma^* \circ T_{2,1}^*dL = \gamma^*(d_{T^{(1)}}\theta_L - T_{2,1}^*dL);$$

the graded 1-form on $T^2\mathcal{M}$ inside the parenthesis is called the Euler-Lagrange 1-form and will be denoted by

$$(3.30) \quad \delta L := d_{T^{(1)}}\theta_L - T_{2,1}^*dL.$$

Our interest in the Euler-Lagrange superform lies in the following proposition:

Proposition 3.3. *Let L be a regular superlagrangian; a superfunction $G \in T\mathcal{A}$ is a first integral of the motion if, and only if, there exists $X \in \mathfrak{X}(\tau_{\mathcal{M}})$ such that*

$$(3.31) \quad \mathbf{T}^{(1)}G = -(\delta L)^\vee(T_{2,1} \circ X).$$

Proof: Since Γ_L is a super SODE, a superfunction G is a constant of the motion if and only if,

$$(3.32) \quad 0 = \gamma \circ \mathbf{T}^{(1)}G = \Gamma_L G.$$

Now, if G satisfies (3.31) then by definition of $(\delta L)^\vee$ and (3.29)

$$(3.33) \quad \gamma \circ \mathbf{T}^{(1)}G = -\gamma(\delta L)^\vee(T_{2,1} \circ X) = -\gamma^* \circ \delta L(X^{(1)}) = 0,$$

since $X^{(1)} \circ T_{1,0} = T_{2,1} \circ X$.

Conversely, let $G \in T\mathcal{A}$ be a constant of the motion. Since L is regular there exists a supervector field Y on $T\mathcal{M}$ such that

$$(3.34) \quad dG + i_Y\omega_L = 0.$$

Now let $\tilde{Y} \in \mathfrak{X}(T^2\mathcal{M})$ be a supervector field that projects onto Y (i.e. $T_{2,1} \circ \tilde{Y} = \tilde{Y} \circ T_{2,1}$); then by Lemma 2.1.

$$(3.35) \quad i_{\tilde{Y}} \circ T_{2,1} = i_{\tilde{Y} \circ T_{2,1}} = i_{T_{2,1} \circ Y} = T_{2,1}^* \circ i_Y;$$

a simple computation, using this equation, shows that $i_{T(1)} \circ i_Y - i_Y \circ i_{T(1)}$ is a $\tau_{2,1}^*$ -derivation of bidegree $(-2, |X|)$, hence

$$(3.36) \quad i_{T(1)} \circ i_Y - i_Y \circ i_{T(1)} = 0.$$

On the other hand, (3.34) together with (3.36) gives us

$$(3.37) \quad \begin{aligned} 0 &= i_{T(1)} dG + i_{T(1)} \circ i_Y \omega_L = T^{(1)}G - i_Y \circ i_{T(1)} \omega_L \\ &= T^{(1)}G - i_Y(i_{T(1)} \omega_L - T_{2,1}^* dE_L) \\ &\quad - i_Y \circ T_{2,1}^* dE_L \\ &= T^{(1)}G - i_Y \delta L - T_{2,1} \circ Y(E_L), \end{aligned}$$

but since G is a constant of the motion, $Y(E_L) = \Gamma(G) = 0$, hence

$$(3.38) \quad 0 = T^{(1)}G - i_Y \delta L = T^{(1)}G - (\delta L)^\vee(\bar{Y});$$

thus, since δL is $\tau_{2,0}$ -semibasic, taking $X := Y \circ T_{1,0}$ we conclude that

$$(3.39) \quad 0 = T^{(1)}G - (\delta L)^\vee(T_{2,1} \circ X). \quad \square$$

The technical part of the proof of Noether's theorem is concentrated in the following lemma:

Lemma 3.2. *Let X be a supervector field along $\tau_{1,0}$, then*

$$(3.40) \quad X^{(1)}L = T^{(1)}(\bar{\theta}_L(X)) - (\delta L)^\vee(T_{2,1} \circ X).$$

Proof: Since δL is $\tau_{2,0}$ -semibasic we use Lemma 3.1. together with Propositions 2.5 and 2.6 in order to get

$$(3.41) \quad \begin{aligned} T_{3,2}((\delta L)^\vee(T_{2,1} \circ X)) &= i_{X^{(2)}} \delta L = i_{X^{(2)}}(d_{T(1)} \theta_L - T_{2,1}^* dL) \\ &= d_{T(1)}^{(2)} \circ i_{X^{(1)}}(\theta_L) - T_{3,2} \circ i_{X^{(1)}}(dL), \end{aligned}$$

and using the fact that θ_L is $\tau_{1,0}$ -semibasic, again Lemma 3.1. gives us

$$(3.42) \quad \begin{aligned} T_{3,2}((\delta L)^\vee(T_{2,1} \circ X)) &= d_{X^{(2)}} \circ T_{2,1}(\bar{\theta}_L(X)) - T_{3,2} \circ i_{X^{(1)}}(dL) \\ &= T_{3,2}(T^{(1)}(\bar{\theta}_L(X)) - X^{(1)}(L)), \end{aligned}$$

and since $T_{3,2}$ is injective the lemma is proved. $\square$

Definition 3.3. Let $L \in T\mathcal{A}$ be a super-Lagrangian superfunction; a supervector field X along $\tau_{1,0}$ is called a symmetry of L if there exists a superfunction $F \in T\mathcal{A}$ such that

$$(3.43) \quad X^{(1)}L = T^{(1)}F.$$

Now we are ready to prove the following version of Noether's theorem which generalizes the one given by Cariñena et al [6,7].

Theorem 3.3. *Let X be a symmetry of a regular super-Lagrangian L ; then the superfunction $G := F - \bar{\theta}_L(X)$ is a constant of the motion. Conversely, if G is a first integral of the motion given by L , there exists a symmetry X of L such that $G = F - \bar{\theta}_L(X)$, where F is the superfunction associated to X via (3.43).*

Proof: If X is a symmetry of L and F is a superfunction satisfying (3.43), then by the previous lemma

$$(3.44) \quad 0 = X^{(1)}L - T^{(1)}F = -(\delta L)^\vee(T_{2,1} \circ X) + T^{(1)}(\bar{\theta}_L(X) - F) = -(\delta L)^\vee(T_{2,1} \circ X) - T^{(1)}G,$$

so by Proposition 3.3. G is a constant of the motion.

On the other hand, if G is a constant of the motion, then

$$(3.45) \quad T^{(1)}G = (\delta L)^\vee(T_{2,1} \circ X),$$

for some $X \in \mathfrak{X}(\tau_{1,0})$ and again, by Lemma 3.2.,

$$(3.46) \quad X^{(1)}L = -T^{(1)}(-G + \bar{\theta}_L(X)),$$

hence the superfunction $F := G - \bar{\theta}_L(X)$ satisfies (3.43) and the theorem is proved. $\square$

Remark 3.1. We have therefore established a one-to-one correspondence between the generalized symmetries according to Definition 3.3. and first integrals of the dynamics defined by the super-Lagrangian L . On the other hand, as in the non-graded case, the advantage of our approach versus the one followed by Ibort and Marín-Solano in [14], is that it admits a simple generalization to higher order Lagrangian supermechanics, whereas their's can not, see [6,7] for details. Finally, we want to point out that the tools developed here can be used to study the case when the super-Lagrangian is singular.

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