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W.G. Davies

AECL Research
Chalk River Laboratories
Chalk River, Ontario, Canada K0J 1J0

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Physical and Environmental Sciences
Chalk River Laboratories
Chalk River, ON K0J 1J0 Canada

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APPLICATION OF LIE-ALGEBRAIC MAPS TO THE ORBIT DYNAMICS OF CYCLOTRONS

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AECL Research, Chalk River Laboratories, Chalk River Ont. K0J 1J0.

1. Introduction

Although cyclotrons are only quasi-periodic machines under operational conditions, many of the important design and operating parameters are global in nature and can be obtained efficiently from the analysis of Lie-algebraic maps [1-2]. In principle, these maps can be static (not accelerated) or dynamic (accelerated), but most of the important design and operational parameters are usually obtained from the static equations of motion. Amongst the global parameters of interest are the radial and vertical tunes, chromaticities, anharmonicities, location of fixed points and the properties of resonances. Also of interest, from a design point of view, are how the general linear and nonlinear properties depend on the symmetry of the magnetic field. One of the advantages of the Lie-algebraic approach is that it allows a consistent derivation of analytic formulas as well as a consistent analysis of numerical results for many important parameters from a single equation of motion: the Hamiltonian of the system. The real power of the Lie-algebraic approach, though, lies in the insight obtained from the analytic analysis of simplified, but realistic, models.

Analytic formulas for many of the above-mentioned global parameters have been obtained by various authors in the past: notably, Hagedoorn and Verster [3], with the most recent, consistent, derivations being by Gordon [4-5] and Gordon and Jeon [6]. These authors use a combination of Hamiltonian and Lagrangian techniques to obtain analytic formulas for the basic orbit parameters and for the analysis of resonances.

From the Hamiltonian for the system, we obtain a nonlinear transfer-map, $M_{i \rightarrow f}$, mapping the initial phase-space variables at position i to the final phase-space variables at position f . This transfer map contains *all* of the information about the dynamics, up to the selected order, at the final location. If the magnetic fields are expressed in the appropriate analytic form then analytic formulas can be obtained for both global and local variables. Alternatively, if the system is so complex that numerical solutions are necessary, the Lie methods can still be used to obtain numerical values of the global and local parameters [7].

In this paper, we will develop a model Hamiltonian for a cyclotron and use the Lie-algebraic method, to derive analytic formulas for the equilibrium orbit, the tunes, and for the analysis of the $\nu_x = 1$ decentering resonance, the $\nu_x = 1/3$ and $\nu_x = 1/4$ resonances as well as the $\nu_x = 2\nu_z$ coupling resonance. In addition, explicit results will be given for higher order effects such as the tune-shifts caused by iterated sextupole fields.

2. Development of the Hamiltonian

The Hamiltonian, $H(r, P_r, z, P_z, \tau, P_\tau)$, in cylindrical coordinates is [8]

$$H = -e r A_\theta - r \left\{ \left(\frac{p_0 + e\varphi}{c} \right)^2 - m^2 c^2 - \left[P_r - e A_r \right]^2 - \left[P_z - e A_z \right]^2 \right\}^{1/2} \quad (1)$$

where p_0 is the *negative* of the total energy of the beam, r is the radius, P_r is the radial momentum; z and P_z are the vertical position and momentum respectively; τ and p_τ are the time-of-flight and kinetic energy respectively; A is the magnetic vector potential with components A_r , A_z and A_θ ; m the particle rest mass, c the speed of light, and e the charge of the particle. Here we are assuming that $\phi = 0$, in keeping with our decision to treat a static system.

2.1 The Isochronous Field

In this section, we derive the form of the magnetic field that makes the orbits "isochronous".¹ The relativistic momentum, $\mathbf{P}$, of the particle is

$$|\mathbf{P}| = \beta\gamma mc, \quad (2)$$

where $\beta = (|\mathbf{v}|/c)$, and $\gamma = 1/\sqrt{1-\beta^2}$. We define the cyclotron to be isochronous if the time for each equilibrium orbit is the same [3], which implies

$$ct = \frac{cr}{|\mathbf{v}|} = \frac{r}{\beta} = \lambda = \text{constant}. \quad (3)$$

We note that the zeroth-order orbital frequency is $\omega = c/\lambda$.

On substituting (2) into $\mathbf{B}\mathbf{r} = \mathbf{P}/e$, we find that the vertical component of the magnetic field B_z is given by

$$B_z = \frac{\beta\gamma mc}{er}, \quad (4)$$

which is proportional to γ . Substituting $\beta = r/\lambda$ from (3) into the expression for γ , we obtain [3]

$$B_z = \frac{mc}{e\lambda} \left[1 - \left(\frac{r}{\lambda} \right)^2 \right]^{-1/2} \quad (5)$$

on the midplane; we note that this equation is singular for $r = \lambda$, which occurs when $\beta = 1$. It is well known that the magnetic field given by (5) produces divergent orbits in the z -plane. The solution to this problem is to introduce "flutter" of the form $[1 + \sum b_n \cos(n(\theta + \phi_n))]$ into the field, where n is the harmonic number; usually $n \geq 3$ or 4 in cyclotrons. Hence the mid-plane magnetic field has the form

¹ An isochronous cyclotron is not isochronous in the true sense of the word, because off-axis rays are not isochronous.

$$B_z(r, \theta, z=0) = \frac{mc}{e\lambda} \left[1 - \left(\frac{r}{\lambda} \right)^2 \right]^{-1/2} \left[1 + \sum_n b_n \cos(n(\theta + \phi_n)) \right]. \quad (6)$$

Here, both b_n and ϕ_n will be, in general, functions of r . The radial dependence of ϕ_n results from the "spiral-angle" of the iron "hills", which generate the flutter-field; this "spiral-angle" is introduced in order to increase the vertical focusing at larger radii. In order to simplify the discussion, we will assume that b_n and ϕ_n are constants with $\theta + \phi_n \Rightarrow \theta$.

Extrapolation of the magnetic field off the midplane is achieved by expanding B_z in a Taylor series and using Maxwell's equations plus midplane symmetry to determine the coefficients as follows: assume $\mathbf{B}$ may be derived from a magnetic scalar potential $\Phi(r, \theta, z)$ via $\mathbf{B} = -\nabla\Phi$; from $\nabla \cdot \mathbf{B} = 0$, we see that Φ satisfies Laplace's equation, $\nabla^2\Phi = 0$. In cylindrical coordinates, the Laplacian may be split into "transverse" and "z" pieces, $\nabla^2 = [\nabla_\perp^2 + \partial_z^2]$ where $\nabla_\perp^2$ and ∂_z^2 commute. Since $B_z = \partial_z\Phi$, it follows that $\nabla_\perp^2 B_z = 0$ also. Assuming midplane symmetry, we expand B_z in a Taylor series about the midplane,

$$B_z(r, \theta, z) = \sum_m \frac{1}{2^m m!} B_{2m} z^{2m}, \quad (7)$$

where usually $m = 0, 1$. Rewriting Laplace's equation as $\partial_z^2 B_z = -\nabla_\perp^2 B_z$, we obtain

$$B_2 = -\nabla_\perp^2 B_z \big|_{z=0}, \quad B_4 = +\nabla_\perp^4 B_z \big|_{z=0}, \text{ etc.} \quad (8)$$

First, we make a multipole expansion of B_z at the radius r^c , where we define

$$B_0 = \frac{mc}{e\lambda} \left[1 - \left(\frac{r^c}{\lambda} \right)^2 \right]^{-1/2} \quad (9)$$

$$Q_x = \frac{r}{B_0} \frac{\partial B_z}{\partial r} \bigg|_{r=r^c}, \quad S_x = \frac{r^2}{B_0} \frac{\partial^2 B_z}{\partial r^2} \bigg|_{r=r^c}, \quad O_x = \frac{r^3}{B_0} \frac{\partial^3 B_z}{\partial r^3} \bigg|_{r=r^c} \quad (10)$$

where Q_x , S_x , and O_x are the normalized quadrupole, sextupole and octupole strengths, respectively. On the midplane, the magnetic field, B_z , now has the form

$$B_z(r, \theta, 0) = B_0 \left[1 + \frac{Q_x x}{r^c} + \frac{1}{2} \frac{S_x x^2}{r^{c2}} + \frac{1}{6} \frac{O_x x^3}{r^{c3}} + \dots \right] \left(1 + \sum_n b_n \cos(n\theta) \right). \quad (11)$$

where $x = (r - r^c)$.

However, the vector potential, $\mathbf{A}$, appears the Hamiltonian, not $\mathbf{B}$. In calculating $\mathbf{A}$, we choose a gauge in which $A_z = 0$. From $\mathbf{B} = -\nabla\Phi$, we find that

$$\Phi = -\int B_z dz \quad (12)$$

which leads to $B_\theta = -1/r \partial_\theta \Phi$. From $\mathbf{B} = \nabla \mathbf{x} \mathbf{A}$, we find that the A_r component is

$$A_r = \int B_\theta dz. \quad (13)$$

Again from $\mathbf{B} = \nabla \mathbf{x} \mathbf{A}$, we find that the rA_θ component is

$$B_z = \frac{1}{r} \frac{\partial(rA_\theta)}{\partial r} - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \Rightarrow rA_\theta = \int \left\{ rB_z + \frac{\partial A_r}{\partial \theta} \right\} dr. \quad (14)$$

In order to simplify the discussion, we will limit ourselves to sextupole (3rd) order. To this order, we obtain

$$\begin{aligned} rA_\theta = B_0 & \left\{ r^c x + \frac{1}{2} \left((1+Q_x)x^2 - Q_x z^2 \right) \right. \\ & \left. + \frac{1}{2r^c} \left(\left(\frac{2}{3}Q_x + \frac{1}{3}S_x \right) x^3 - (Q_x + S_x)xz^2 \right) \right\} \left(1 + \sum_n b_n \cos(n\theta) \right) \end{aligned} \quad (15)$$

and

$$A_r = -\frac{B_0}{2} \sum_n n b_n \sin(n\theta) z^2. \quad (16)$$

Note that A_r is truncated at second order terms because it enters as the square in the Hamiltonian. Inspection of the vector potential shows that the quadrupole and sextupole strengths are mixed. This is the result of working in cylindrical coordinates.

2.2 Perturbation Expansion of the System.

The nonlinear Hamiltonian (1) does not admit a general solution in terms of analytic functions. However, we can find an exact solution for a particular set of initial conditions; namely, $r = r^c$, $P_r = 0$, $z = 0$, $P_z = 0$, $P_\theta = eB_0 r^c$, which generates a circular orbit. Here the "c" superscript denotes our particular initial conditions. In the traditional perturbative approach [9], we make a Taylor-series expansion of the Hamiltonian [10, 9] about the reference orbit (the circular orbit in our case). The Hamiltonian can be written in the form

$$\mathcal{H} = \mathcal{H}^c + \mathcal{H}^S + \mathcal{H}^F, \quad (17)$$

where $\mathcal{H}^c$ is the exactly solvable part, and $\mathcal{H}^S$ is the residual part of the expanded azimuthally-symmetric Hamiltonian, and $\mathcal{H}^F$ represents the "flutter". A perturbative solution of $\mathcal{H}$ of the form

$$F(q^1, p^1) = f_0(q^1, p^1) + \sum_{n=1}^{\infty} \varepsilon^n f_n(q^1, p^1) \quad (18)$$

is substituted into Hamilton's equations and we attempt to solve for the unknown functions f_n ; $F(q^1, p^1)$ stands for (or generates) the solutions q^f and p^f and ε is a "smallness" parameter. This expansion is a double-perturbation expansion; first because we assume the nonlinearities are small, exemplified by the smallness parameter ε ; secondly, the canonical coordinates, q and p , are assumed small relative to r^c and P^c . Thus we might expect rapid convergence of (18) in the vicinity of the reference trajectory. In fact, (18) will have a radius of convergence equal to the location of the first singularity in the equations of motion, which may lie in the complex "plane" $\mathbb{C}^m$. Furthermore, the radius of convergence depends on the "time-step" of the map [11], the shortest time-step having the largest radius of convergence. The long-term behaviour is more complicated. Because the general nonlinear Hamiltonian (1) has resonances (singularities) which are infinitely dense in phase space, the radius of convergence of (18) approaches zero as θ ("time") approaches infinity, and the solution is asymptotic [9]; that is the solution will diverge for $n > N$ where N is some fixed integer. One might conclude that the situation is hopeless, but the beauty of the asymptotic series is that it can still lead to very accurate results over a limited domain. It is perhaps useful to point out that the perturbative techniques used here do not involve averaging over the "fast" coordinate; the map contains all of the information up to the order of truncation.

Now, we proceed to develop the perturbation series in Lie-algebraic form. First, we scale the Hamiltonian by $r^c P^c = e B_0 r^{c2}$ and then expand about the reference trajectory; scaling the parameters in this way makes all of the canonical variable dimensionless and simplifies the mathematics, making the results more transparent. The result is

$$\mathcal{H} = r\tilde{A}_\theta - (\tilde{x}+1) \left\{ 1 - \frac{2\tilde{p}_\tau}{\beta} + \tilde{p}_\tau^2 - (\tilde{p}_x - \tilde{A}_r)^2 - \tilde{p}_z^2 \right\}^{\frac{1}{2}} - \frac{1}{\beta} (\tilde{p}_\tau + \tilde{p}_0^c), \quad (19)$$

where we have used the fact that $p_r^c = 0$. Here $\beta = (v/c)^c$, $r\tilde{A}_\theta = rA_\theta/r^c P^c$ with rA_θ given by (15), and $\tilde{A}_r = A_r/r^c P^c$ with A_r given by (16). The scaled variables have the form [8]

$$\left. \begin{aligned} \tilde{\tau} &= c\tau/r^c & \tilde{p}_\tau &= p_\tau/P^c \\ \tilde{x} &= x/r^c & \tilde{p}_x &= p_x/P^c \\ \tilde{z} &= z/r^c & \tilde{p}_z &= p_z/P^c \end{aligned} \right\}. \quad (20)$$

For notational simplicity, we will drop the $\sim$ on the variables, but it **must be remembered that the coordinates have been scaled and are relative to the reference trajectory**. In keeping with our decision to work to sextupole order, we expand the square root in (19) in a Taylor series up to 3rd order. We obtain

$$\begin{aligned} (x+1)(\dots)^{\frac{1}{2}} = & x - \frac{1}{2}(p_x^2 + p_z^2) + (1+x)A_r p_r - \frac{1}{2}x(p_x^2 + p_z^2) - \frac{p_\tau}{\beta} \\ & - \frac{x p_\tau}{\beta} + \frac{1}{2} \left(1 - \frac{1}{\beta^2} \right) p_\tau^2 + \frac{1}{2} \left(1 - \frac{1}{\beta^2} \right) x p_\tau^2 - \frac{1}{2\beta} p_r^2 p_\tau - \frac{p_x^2 p_\tau}{\beta}. \end{aligned} \quad (21)$$

By substituting rA_θ , (15), A_r , (16), and (21) into (19) we obtain the expanded Hamiltonian in the normalized coordinates. As indicated above, we will solve the perturbed Hamiltonian in two steps: first we solve the azimuthally symmetric Hamiltonian $\mathcal{H} = \mathcal{H}^c + \mathcal{H}^S$, and then we find the solution of the full Hamiltonian given by (17) and (19).

3. Lie-algebraic analysis

In this section, we will outline the general Lie-algebraic properties of the problem and derive some results from the Lie-algebraic solution, which arise from the symmetries of the system.

Hamilton's equations can be written in Poisson Bracket form [10]

$$\frac{dF}{dt} = [F, \mathcal{H}] \quad (22)$$

where in our case the generalized time $t \Rightarrow \theta$; F is any function of the phase space variables $\zeta = (x, p_x, z, p_z, \tau, p_\tau)$. Eq. (22) can be written as the Lie operator

$$\dot{F} = -[\mathcal{H}, F] = -:\mathcal{H}:F \quad (23)$$

where a Lie operator $:g:$ operating on any function F is defined to be the Poisson bracket of g with F , i.e. $[g, F]$. Hence

$$:g: \stackrel{\text{def}}{=} \sum_{i=1}^n \left[\frac{\partial g}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial g}{\partial p_i} \frac{\partial}{\partial q_i} \right], \quad (24)$$

where q and p are the canonically conjugate generalized positions and momenta respectively. In general $n=3$ for single particle Hamiltonian systems.

If we let $F = \zeta$ in (23), we obtain the equation of motion for the canonical coordinates

$$\dot{\zeta} = -:\mathcal{H}(\theta):\zeta. \quad (25)$$

The Poisson bracket $[F, \mathcal{H}]$ in Hamilton's equations, (23), naturally forms a Lie algebra, because it satisfies the commutation relation [12]

$$A*B = (A, B) = AB - BA \quad (26)$$

and the Jacobi identity

$$(A, (B, C)) + (B, (C, A)) + (C, (A, B)) = 0. \quad (27)$$

Note: This algebra is *neither commutative nor associative*.

The solution of Hamilton's equations provides a *mapping*, $\mathcal{M}_f$, of the initial phase space variables at $t = 0$ to the final phase space variables at $t = t_f$, which can be written as a Lie-map as follows:

$$\mathcal{M} \Rightarrow \mathcal{M}_f, \quad \mathcal{M}_f = e^{:f:}; \quad :f:g = [f, g] \quad (28)$$

where

$$e^{:f:} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{:f:_n}{n!}; \quad :f:_0 = \mathcal{I} = \{\text{identity map}\}. \quad (29)$$

The Lie-map, $\mathcal{M}_f$, can be factored [13] into

$$\mathcal{M}_f = \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3 \mathcal{M}_4 \dots = e^{:f_1:} e^{:f_2:} e^{:f_3:} e^{:f_4:} \dots \quad (30)$$

where $\mathcal{M}_f$ and $\mathcal{M}_n$ are *symplectic maps* and f_n are homogeneous polynomials of degree n .

Now, consider the product of two Lie maps, which for the moment we assume to be the result of concatenation. We write

$$\zeta_2 = \mathcal{M}\zeta = \mathcal{M}_g \mathcal{M}_f \zeta = e^{:g:} e^{:f:} \zeta \quad (31)$$

where we mean that

$$\zeta_2 = \mathcal{M}_g \zeta_1, \text{ with } \zeta_1 = \mathcal{M}_f \zeta. \quad (32)$$

The map $\mathcal{M}$ has the following properties

$$\mathcal{M} = \mathcal{M}(t_{g+f}, \zeta) = \mathcal{M}(t_g, \mathcal{M}(t_f, \zeta)). \quad (33)$$

Because $:g:$ is a (differential) Lie operator, we must evaluate $\zeta_2 = \mathcal{M}_g \zeta_1$ in ζ_1 coordinates. Hence we must know $g(\zeta_1)$ not $g(\zeta)$, but we usually want it in terms of ζ . We can do this as follows:

$$g(\zeta_1) = g(e^{:f(\zeta):} \zeta) = e^{:f(\zeta):} g(\zeta) \quad (34)$$

where we have used Eq. (A.2) from Appendix A. Hence,

$$\zeta_2 = e^{:e^{:f(\zeta):} g(\zeta):} e^{:f(\zeta):} \zeta \quad (35)$$

which by the transformation rule (A.4), leads to

$$\zeta_2 = e^{:f(\zeta):} e^{:g(\zeta):} \zeta . \quad (36)$$

A comparison of (31) with (36) shows that if we write the product in *reverse order*, the map will automatically be in the correct coordinates. Now, because the maps are all with respect to the same coordinates, we can formally evaluate the map (36) with (29) as follows:

$$\zeta_2 = \left\{ 1 + :f: + \frac{1}{2}:f:^2 + \dots \right\} \left\{ 1 + :g: + \frac{1}{2}:g:^2 + \dots \right\} \zeta . \quad (37)$$

Eq. (36) is the standard form for writing concatenated or factored maps.

3.1 Solution of the 'time' independent Hamiltonian

If the Hamiltonian is independent of t , as is H^S (17), then the solution can be written immediately as the *Hamiltonian flow* [12]

$$M(t) = \exp \left\{ - \int_0^t :H: dt \right\} \Rightarrow M_f = e^{-t:H:} . \quad (38)$$

The factorization of (38) is usually non-trivial. In general, the Lie-maps or Lie-transformations do not commute so that the factorization of (or recombination) operators from (to) a single exponent is quite complicated and requires "inverting" the *Campbell-Baker-Hausdorff* (CBH) formula. However, if the functions f and g commute, i.e. $[f, g] = 0$, we obtain

$$e^{:h:} = e^{:f+g:} = e^{:f:} e^{:g:} = e^{:g:} e^{:f:} . \quad (39)$$

Hence the commutation relations of the various parts of the Hamiltonian are crucial to their evaluation. If the Hamiltonian is time dependent, and/or contains a linear term, both of which are true for H^F , then the solution is more difficult and we must solve (25) directly.

3.2 Solution of the 'time' dependent Hamiltonian or when an H_1 Term is Present

Recall that the equations of motion for any *arbitrary function* $f(\zeta)$ of the canonical variables ζ is given by (23). The *map* (28) is such a function. Hence, we obtain

$$\dot{M} f(\zeta^1) = -[H(\zeta), M f(\zeta^1)] = M : -H(\zeta^1) : f(\zeta^1) . \quad (40)$$

Since $f(\zeta)$ is arbitrary, (40) must hold irrespective of f and we obtain

$$\dot{M} = M : -H(\zeta^1) : \quad (41)$$

which is the equation of motion for the map. Although $\mathcal{H}$ is now a function of ζ^i , the initial values of the canonical variable, we must treat $\mathcal{H}(\zeta^i)$ as a function, not as numbers.

Eq. (41) is the equation of motion for the total map, but we want a solution for the factored map

$$\mathcal{M} = \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3 \mathcal{M}_4 \cdots \quad (42)$$

where the $\mathcal{M}_n$'s are associated with the corresponding homogeneous terms in $\mathcal{H} = \sum \mathcal{H}_n$. Because $\mathcal{M}_2$ is closed under Poisson bracket operations and constitutes a subgroup of the infinite group of Lie-maps, we can separate out and solve the equation [14]

$$\dot{\mathcal{M}}_2 = \mathcal{M}_2 : -\mathcal{H}_2 : \quad (43)$$

independent of the rest of the map. Following Dragt [14], define $\mathcal{M}_R$ by

$$\mathcal{M}_R = \mathcal{M}_3 \mathcal{M}_4 \cdots \quad (44)$$

so that (42) becomes

$$\mathcal{M} = \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_R. \quad (45)$$

It is convenient to temporarily reorder (45) using the exchange rule (A.4) so that

$$\mathcal{M} = \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_R = \mathcal{M}_1 \mathcal{M}_R^T \mathcal{M}_2 \quad (46)$$

where

$$\mathcal{M}_R^T = \mathcal{M}_2 \mathcal{M}_R \mathcal{M}_2^{-1}. \quad (47)$$

If the $\mathcal{H}_1 = 0$, which implies $\mathcal{M}_1 = \mathcal{I}$ where $\mathcal{I} = \text{identity map}$, we obtain the solution of $\mathcal{M}_R$ by substituting (46) into (41) as follows

$$\dot{\mathcal{M}}_R^T = \mathcal{M}_R^T \mathcal{M}_2 : -\mathcal{H}_R : \mathcal{M}_2^{-1} = \mathcal{M}_R^T : -\mathcal{H}_R :. \quad (48)$$

Equations (43) and (48) can be solved by the methods of Dragt & Forest [14], which leads to the integral series

$$\mathcal{M}(t) = \mathcal{I} + \int_0^t dt' : -\mathcal{H}(t') : + \int_0^t dt' \left\{ \int_0^{t'} dt'' : -\mathcal{H}(t'') : \right\} : -\mathcal{H}(t') : + \cdots \quad (49)$$

If the $\mathcal{H}_1 \neq 0$, we might expect an iterative solution to converge as ϵ^ℓ where ℓ counts the number of iterations. Setting $\mathcal{M}_1 = \mathcal{I}$, we first solve (43) and (48), which then allows us to write (41) as

$$\dot{\mathcal{M}}_1 \mathcal{M}_2 (\mathcal{M}_2^{-1} \mathcal{M}_R^T \mathcal{M}_2) + \mathcal{M}_1 \mathcal{M}_2 (\mathcal{M}_2^{-1} \mathcal{M}_R^T \mathcal{M}_2) (\mathcal{M}_2^{-1} : -\mathcal{H}_R : \mathcal{M}_2)$$

$$= \dot{M}_1(M_2 M_R) = M_1(M_2 M_R) : -\mathcal{H}_1 : \quad (50)$$

which can be transformed into its final form by multiplying on the left by $(M_2 M_R)^{-1}$. This leads to

$$\dot{M}_1 = M_1 : -\mathcal{H}_1^T : = M_1 : -\mathcal{H}_1(M_2 M_3 \cdots \zeta^1) : \quad (51)$$

where

$$: -\mathcal{H}_1^T : = (M_2 M_R) : -\mathcal{H}_1 : (M_2 M_R)^{-1}. \quad (52)$$

We iterate (51) to build a series solution to the problem. In order to iterate, we must evaluate the map $(M = M_2 M_3 \cdots \zeta^1)$ at each iteration. The evaluated map, M_2 , is the matrix R , which is linear in the canonical variables. This becomes the $\mathcal{H}_1$ term in a new Hamiltonian. The M_3 map leads to a quadratic dependence of the canonical variables which becomes the new $\mathcal{H}_2$ term and so on. Hence, we obtain

$$\mathcal{H}_1(M_2^{(0)} M_3^{(0)} \cdots \zeta^1) = \epsilon \mathcal{K}_1^{(1)}(\zeta^1) + \epsilon \mathcal{K}_2^{(1)}(\zeta^1) + \cdots \quad (53)$$

where we show explicitly the ϵ dependence of $\mathcal{H}^{(1)}$ by writing

$$\mathcal{H}_n^{(1)} = \epsilon \mathcal{K}_n^{(1)}. \quad (54)$$

From (53) we see that in order to solve (51) we must write

$$M_1^{(0)} = M_1^{(1)} M_2^{(1)} M_R^{(1)}. \quad (55)$$

Thus at this iteration the new map becomes

$$M_1^{(0)} M_2^{(0)} M_3^{(0)} \cdots = M_1^{(1)} M_2^{(1)} M_3^{(1)} \cdots M_2^{(0)} M_3^{(0)} \cdots \quad (56)$$

where $M_2^{(0)} M_3^{(0)} \cdots$ are known and $M_1^{(0)}$ is still to be determined. From (55), we obtain a new equation for the map that has the form of (41) with Hamiltonian $\mathcal{H}_1$, given by (53). Explicitly,

$$\frac{d}{dt}(M_1^{(1)} M_2^{(1)} M_R^{(1)}) = M_1^{(1)} M_2^{(1)} M_R^{(1)} : -\epsilon \mathcal{K}_1^{(1)}(\zeta^1) - \epsilon \mathcal{K}_2^{(1)}(\zeta^1) - \cdots : \quad (57)$$

The steps from (44) through (56) are repeated until convergence occurs, or until the "feed-down" of the truncated map terminates at $\mathcal{H}_1$; at each iteration, we drop one order in the Hamiltonian such that the rank² is constant. Finally, we integrate (51) in one step to get the $M_1^{(M)}$ map:

²The rank [13] is the sum, $n+m$, of the order of $\epsilon^n \zeta^m$, but for this to be meaningful, ϵ must be commensurate in magnitude with ζ .

$$\mathcal{M}_1^{(M)} = \int_0^\theta \dot{\mathcal{M}}_1^{(M)} d\theta' = \int_0^\theta :-\mathcal{H}_1^{(M)}(\mathcal{M}_2^{(M-1)} \zeta^i, \theta') : d\theta'. \quad (58)$$

Note that since $\mathcal{H}_1 = \varepsilon \mathcal{K}_1$, then $\mathcal{H}_n^{(1)} \propto \varepsilon$ and $\mathcal{H}_n^{(2)} \propto \varepsilon^2 \dots$ when $n \geq 2$. The existence of an $\mathcal{H}_1$ term in the Hamiltonian causes displacement or misalignment of the central trajectory. Furthermore, this displacement, whether or not it is θ -dependent, causes *feed-down* of all higher order aberrations to lower orders³, greatly complicating the factorization.

In a circular accelerator, it is desirable that the reference orbit be a closed orbit (i.e. a "fixed point" of the one-turn map); this is, in fact, necessary for the transformation to normal form.⁴ If the reference trajectory is not a closed orbit, the map, $\mathcal{M}_f$, will have an $\mathcal{M}_1$ factor. The closed orbit can be found from the map by solving for the fixed point given by the equation

$$\zeta_* = \mathcal{M}_f \zeta_*. \quad (59)$$

This is in general an iterative process. Once the fixed point has been found, the map can be transformed by a canonical change of variables $\bar{\zeta} = \mathcal{A} \zeta$ such that

$$\bar{\zeta}_* = \mathcal{A} \zeta_* = 0 \Rightarrow \bar{\zeta}_* = \mathcal{A} \mathcal{M}_f \mathcal{A}^{-1} \bar{\zeta}_* = \bar{\mathcal{M}}_f \bar{\zeta}_* = 0, \quad (60)$$

i.e. $\bar{\mathcal{M}}_f$ maps the origin onto itself.

The Lie-polynomials provide a much more compact representation of the map than a Taylor-series map and preserves the symplectic nature of the map, $\mathcal{M}$, exactly, which the Taylor-series map, when truncated, does not do. Eq. 30 can be transformed into *normal form* [15,13], which provides a very powerful method for identifying and analysing non-linear behavior and resonances in particle accelerators.

3.3 Normal Form

The normal form is the transformation $N = \mathcal{A} \mathcal{M} \mathcal{A}^{-1}$ such that N is in its "simplest" form; $\mathcal{M}$ must be expanded about a "closed orbit" i.e. no $\mathcal{H}_1$ terms in $\mathcal{H}$, which implies no $\mathcal{M}_1$ factors in $\mathcal{M}$. The symplectic map $\mathcal{A}$ is an n^{th} order canonical

³That an $\mathcal{H}_1$ term in the Hamiltonian causes feed-down is easily seen by formally expanding the map according to (37). We have

$$\mathcal{M} = \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3 \dots$$

$$\begin{aligned} &= \left\{ 1 + :f_1: + \frac{1}{2} :f_1^2: + \dots \right\} \left\{ 1 + :f_2: + \frac{1}{2} :f_2^2: + \dots \right\} \left\{ 1 + :f_3: + \frac{1}{2} :f_3^2: + \dots \right\} \\ &= 1 + \underbrace{:f_1: + :f_1: :f_2: + \frac{1}{2} :f_1^2: :f_3:}_{\text{order 1}} + \dots + \underbrace{:f_2: + :f_1: :f_3:}_{2} + \dots + \underbrace{:f_3: + \frac{1}{2} :f_1: :f_3^2:}_{3} + \dots \end{aligned}$$

where we note that each time $:f_1:$ operates on another Lie operator it lowers its order by 1 [13].

⁴In a cyclotron, we can only have a fixed point for unaccelerated or "static" orbits.

transformation that isolates the tune shifts and resonances to n^{th} order. From the factored map, we can normalize M order by order to obtain

$$N = A_n \cdots A_4 \left\{ A_3 (A_2 M A_2^{-1}) A_3^{-1} \right\} A_4^{-1} \cdots A_n^{-1}. \quad (61)$$

We proceed as follows: From the eigenvectors (they must be distinct) of R^5 , we get a canonical transformation that rescales and block diagonalizes R ;

$$R = A_2 R A_2^{-1} \Rightarrow R = \begin{pmatrix} R_1 & 0 \\ & R_2 \\ 0 & R_3 \end{pmatrix} \quad (62)$$

where

$$R_1 = \begin{pmatrix} \cos(\mu_1) & \sin(\mu_1) \\ -\sin(\mu_1) & \cos(\mu_1) \end{pmatrix}, \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}.$$

Next, we normalize M_3

$$\begin{aligned} N &= A_3 A_2 R e^{i f_3} \cdots A_2^{-1} A_3^{-1} = e^{i G_3} R e^{i g_3} \cdots e^{-i G_3} \\ &= R \exp \left\{ -(J - R^{-1}) G_3 + g_3 \right\} \cdots \end{aligned} \quad (63)$$

A further simplification results if we expand $(J - R^{-1})$, and g_3 in action-angle variables as follows. Hence,

$$(J - R^{-1}) |n, m\rangle = \left[1 - \sum_{n+m=3} e^{i(n-m) \cdot \mu} \right] |n, m\rangle, \quad (64)$$

and

$$g_3 = \sum_{n+m=3} A_{nm} |n, m\rangle, \quad |n, m\rangle = J^{(n+m)/2} e^{-i(n-m) \cdot \varphi}, \quad (65)$$

here J is the action with components (J_x, J_z) , φ the angle with components (φ_x, φ_z) n, m, μ are vectors with components (n_1, n_2) etc. Substituting (64) and (65) into (63) and solving for G_3 , we obtain

⁵The matrix R is the evaluated representation of the Lie-map $e^{i f_2}$. As is common in the literature we will use R to represent both the Lie-map and the matrix itself; the choice being determined from the context.

$$G_3 = \sum_{n+m=3} \frac{A_{nm}}{\{1 - \exp[i(n-m) \cdot \mu]\}} \quad (66)$$

All terms in g_3 can be incorporated into G_3 except when $n-m = 0$, or $(n-m) \cdot \mu = 2\pi\ell$, $\ell = \text{integer}$; i.e. $(n_1-m_1)\mu_x + (n_2-m_2)\mu_z \neq 2\pi\ell$; $\ell = 0, \pm 1, \pm 2 \dots$. The integer ℓ characterizes the resonance. In a similar fashion, we could proceed to find G_4 etc.

If we assume no acceleration and no explicit resonances then

$$\mathcal{N} = \exp: \left\{ (\mu_x + \mu'_x p_t + \mu''_x p_t^2 \dots) J_x + (\mu_z + \dots) J_z + a_x J_x^2 + a_z J_z^2 + \dots \right\}; \quad (67)$$

where, μ are the *phase advances*, μ' etc. are the *chromaticities* and all terms proportional to J^n are non-linear *tune shifts*. That we can incorporate all the other non-linearities into $\mathcal{A}$ and "get rid of them" makes the analysis of the tune-shifts and resonances much easier. Because $[J_1^n, J_j^m] = 0$, that is all powers of the actions commute, the transformation from factored to single-exponent form is straight-forward in (67). However, if there are resonances in the system, the CBH formula must be used to convert to single exponent form. Finally, the *normal form* has the important property that

$$\mathcal{M}^n = \mathcal{A}^{-1} \mathcal{N}^n \mathcal{A} \quad (68)$$

where n is the number of concatenations of the map. Use of (68) results in an enormous saving of work when applicable.

4. Solution of the Hamiltonian for the cyclotron

4.1 Solution of the azimuthally symmetric Hamiltonian H^S

In normalized coordinates (see (20),

$$rA_\theta = x + \frac{1}{2}(1+Q_x)x^2 - \frac{1}{2}Q_x z^2 + \left(\frac{1}{3}Q_x + \frac{1}{6}S_x \right) x^3 - \frac{1}{2}(Q_x + S_x)xz^2 \quad (69)$$

to third order. The expanded Hamiltonian can be partitioned as follows:

$$\mathcal{H} = \mathcal{H}_2^x + \mathcal{H}_2^z + \mathcal{H}_2^\tau + \mathcal{H}_3 + \mathcal{H}_3^\tau. \quad (70)$$

$$\mathcal{H}_2^x = \frac{1}{2}[(1+Q_x)x^2 + p_x^2] + \frac{xp_\tau}{\beta^c} \quad (71)$$

$$\mathcal{H}_2^z = -\frac{1}{2}[Q_x z^2 - p_z^2] \quad (72)$$

$$\mathcal{H}_2^\tau = \frac{1-\beta^2}{2\beta^2} p_\tau^2 \quad (73)$$

$$\mathcal{H}_3 = \left(\frac{1}{3}Q_x + \frac{1}{6}S_x \right) x^3 - \frac{1}{2}(Q_x + S_x)xz^2 + \frac{xp_x^2}{2} + \frac{xp_z^2}{2} - \frac{\beta^2-1}{2\beta^2}xp_\tau^2 + \frac{p_x^2p_\tau}{2\beta} + \frac{p_z^2p_\tau}{2\beta} \quad (74)$$

$$\mathcal{H}_3^\tau = \frac{1-\beta^2}{2\beta^3} p_\tau^3. \quad (75)$$

Because the Hamiltonian (70) is independent of θ , the solution is given by (38), which we will now factor into the form of (30).

4.2 Factorization of the Map

Our task is now to factor

$$\mathcal{M} = e^{-\theta:\mathcal{H}:} \quad (76)$$

into

$$\mathcal{M} = \mathcal{M}_2\mathcal{M}_3 \dots = e^{:f_2^x:} e^{:f_3^z:} \dots \quad (77)$$

The above partitioning has been arranged to optimize the commutation properties of the Hamiltonian. It is easy to show that of the 10 possible commutation relations between the terms in the Hamiltonian (70), only two are non-zero, namely $[\mathcal{H}_2^x, \mathcal{H}_3] \neq 0$, and $[\mathcal{H}_2^z, \mathcal{H}_3] \neq 0$. From these results, we find immediately that

$$f_2^x = -\theta\mathcal{H}_2^x \quad (78)$$

$$f_2^z = -\theta\mathcal{H}_2^z \quad (79)$$

$$f_2^\tau = -\theta\mathcal{H}_2^\tau \quad (80)$$

$$f_3^\tau = -\theta\mathcal{H}_3^\tau. \quad (81)$$

The only nontrivial map is f_3 , which neither commutes with f_2^x or f_2^z . So, at this stage, the map can be written as

$$\mathcal{M} = e^{:f_2^x:} e^{:f_2^z:} e^{:f_3^z:} e^{:f_2^\tau:} e^{:f_3^\tau:} \quad (82)$$

where f_2^x , f_2^z , f_2^τ and f_3^τ are given by (78) through (81), respectively. The first and the last two factors in (82) commute with each other. The fact that most of the components of the Hamiltonian commute results in an enormous simplification of the problem.

In order to evaluate the M_3 factor, we must evaluate $M_2 = M_2^x M_2^z M_2^\tau$ explicitly. Because f_2^x , f_2^z and f_2^τ commute, they can be evaluated independently. Let us evaluate f_2^τ first. We see from section 1 of appendix B [16,13] that $M_2^\tau = R^\tau$ is the familiar *transfer matrix*

$$R^\tau = \begin{bmatrix} 1 & \theta \left(\frac{1-\beta^2}{2\beta^3} \right) \\ 0 & 1 \end{bmatrix}. \quad (83)$$

As expected, (83) is just the matrix for a drift space and hence the motion is divergent in the (τ, p_τ) plane. Next, we evaluate M_2^z . But, before we do this it is useful to put (72) in *standard form*. This is done by transforming $\mathcal{H}_2^z$ as follows:

$$\mathcal{H}_2^z = -\frac{1}{2}[Q_x z^2 - p_z^2] = \frac{\nu_z}{2k_z}(z^2 + k_z^2 p_z^2). \quad (84)$$

Solving (84) for k_z and ν_z leads to

$$i\nu_z = i^2 \sqrt{-Q_x} = -\sqrt{Q_x}, \quad ik_z = i/\nu_z \quad (85)$$

where we keep ν_z and k_z real. Hence,

$$f_2^z = \theta \frac{\nu_z}{2k_z}(z^2 + k_z^2 p_z^2) = -\frac{\mu_z}{2}(\gamma_z x^2 + \beta_z p_x^2) \quad (86)$$

where γ_z and β_z are Twiss parameters, which satisfy the condition $\gamma_z \beta_z = 1$. If we substitute $\mu = i\nu_z \theta$ and $k = ik_z$ from (85), we see from section 3 of appendix B that R^z is given by

$$R^z = \begin{bmatrix} \cosh(\nu_z \theta) & k_z \sinh(\nu_z \theta) \\ \frac{1}{k_z} \sinh(\nu_z \theta) & \cosh(\nu_z \theta) \end{bmatrix}. \quad (87)$$

Transforming $\mathcal{H}_2^x$, (71), to *standard form* leads to

$$k_x = 1/\nu_x, \quad \nu_x = \sqrt{1+Q_x}. \quad (88)$$

Whence,

$$f_2^x = -\theta \left(\frac{\nu_x}{2k_x}(x^2 + k_x^2 p_x^2) + \frac{x p_\tau}{\beta^c} \right) = -\frac{\mu_x}{2}(\gamma_x x^2 + \beta_x p_x^2) - \theta \frac{x p_\tau}{\beta^c}. \quad (89)$$

Again, γ_x and β_x are the Twiss parameters, where $\gamma_x \beta_x = 1$. Now, we evaluate f_2^x , (78). Here, we write

$$-\theta \mathcal{H}_2^x \begin{bmatrix} x \\ p_x \\ \tau \\ p_\tau \end{bmatrix} = \nu_x \theta G \begin{bmatrix} x \\ p_x \\ \tau \\ p_\tau \end{bmatrix} \quad (90)$$

where G is given by

$$G = \begin{bmatrix} 0 & k_x & 0 & 0 \\ -1/k_x & 0 & 0 & -1/\nu_x \beta \\ \frac{1}{\nu_x \beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (91)$$

The eigenvalues of G are $\Lambda = [i, -i, 0, 0]$. Hence, $G^4 = -G^2$ and the expansion of section 3 of Appendix B holds. On substitution of (91) into (B.21), we obtain

$$R^x = \begin{bmatrix} \cos(\nu_x \theta) & k_x \sin(\nu_x \theta) & 0 & -\frac{k_x}{\nu_x \beta} (1 - \cos(\nu_x \theta)) \\ -\frac{1}{k_x} \sin(\nu_x \theta) & \cos(\nu_x \theta) & 0 & -\frac{1}{\nu_x \beta} \sin(\nu_x \theta) \\ \frac{1}{\nu_x \beta} \sin(\nu_x \theta) & \frac{k_x}{\nu_x \beta} (1 - \cos(\nu_x \theta)) & 1 & \frac{k_x}{(\nu_x \beta)^2} \sin(\nu_x \theta) \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (92)$$

This matrix is very similar to that given in ref. [17] for a non-uniform field dipole, excluding the vertical plane.

By transforming $\mathcal{H}$ into standard form, we see that the tune and the optical phase advance are readily found without evaluating the map. Traditionally, the optical phase advance $\mu = \nu \theta$ [18] for the maps $R^T R^x$, or R^z (87) is found from

$$\mu = \nu \theta = \cos^{-1} \left(\frac{1}{2} \text{Tr}(R) \right) \quad (93)$$

where $\text{Tr}(R)$ means the trace of the 2x2 sub-matrix of R . The trace of R is related to the eigenvalues of R [18] by

$$\frac{1}{2}\text{Tr}(R) = \frac{1}{2}(\Lambda_1 + \Lambda_2) = \frac{1}{2}(\Lambda + 1/\Lambda) \quad (94)$$

where we note that because of the symplectic condition the eigenvalues of the evaluated map $M_2 = M_2^x M_2^z M_2^\tau$, always occur in reciprocal pairs i.e. $\Lambda_i^1 = 1/\Lambda_i^1$, where i stands for each of the degrees of freedom. The first-order stability of the system is also apparent without evaluating the map.

It can be shown by direct substitution into the equation $\det[R - \Lambda I] = 0$ that the four eigenvalues of (92) are

$$\begin{aligned} \Lambda_{1,2} &= \cos(\nu_z \theta) \pm i \sin(\nu_z \theta) \\ \Lambda_{3,4} &= 1 \end{aligned} \quad (95)$$

and the eigenvalues of (87) are

$$\Lambda_{1,2} = \cosh(\nu_z \theta) \pm \sinh(\nu_z \theta). \quad (96)$$

From (95), we see that inclusion of the dispersive term in the Hamiltonian does not change the phase advance or the "tune" of the system.

We are now in a position to evaluate the M_3 factor in (82). Because, $\mathcal{H}_2^x$ and $\mathcal{H}_2^z$ do not commute with $\mathcal{H}_3$, we must use an "inversion" of the CBH formula to factor the map. From (A.17) we find that

$$f_3 = -\theta \int_0^1 e^{-\tau(-\theta:\mathcal{H}_2:)} \mathcal{H}_3 \, d\tau \quad (97)$$

but, $\exp[-\theta:\mathcal{H}_2:]$ is just R , (87) and (92), so we need only substitute $-\tau\theta$ for θ in (87) and (92) to obtain the coefficients of x , p_x , z , p_z , and p_τ . Here we have neglected the factor in (A.8) containing $\mathcal{O}(\gamma^2)$: because in this case it does not contribute to f_3 , but only to f_4 and beyond.

Making use of (A.2), which implies (A.18), we find (97) becomes

$$f_3 = -\theta \int_0^1 \mathcal{H}_3 \left(e^{-\tau(-\theta:\mathcal{H}_2:)} \right) d\tau = -\theta \int_0^1 \mathcal{H}_3[R(-\tau\theta)] d\tau. \quad (98)$$

Thus the problem has been reduced to a sum of quadratures of trigonometric functions, which, although tedious to evaluate, are elementary. The result is of the form

$$\begin{aligned} f_3 &= A_{(1)} x^3 + A_{(1)} x^2 p_x + A_{(1)} x p_x^2 + A_{(1)} p_x^3 + A_{(1)} x^2 p_\tau \\ &+ A_{(1)} x p_x p_\tau + A_{(1)} x p_\tau^2 + A_{(1)} p_x^2 p_\tau + A_{(1)} p_x p_\tau^2 + A_{(1)} x z^2 \\ &+ A_{(1)} x p_x p_z + A_{(1)} x z p_x + A_{(1)} x p_z^2 + A_{(1)} z^2 p_x + A_{(1)} p_x p_z^2 \\ &+ A_{(1)} z p_z p_\tau + A_{(1)} z^2 p_\tau + A_{(1)} p_z^2 p_\tau + A_{(1)} p_\tau^3 \end{aligned} \quad (99)$$

where the coefficient $A_{(1)}$ has an index $\{i\}$ which corresponds to its associated monomial; for example the coefficient $A_{(1)}$ in $A_{(1)} x p_x p_\tau$ is the coefficient for $x p_x p_\tau$

etc. As we will see in the discussion of the Normal Form, much can be learned without ever having to know the values of the coefficients.

5. Solution of the Hamiltonian Perturbed by the Fourier Terms.

We now extend our analysis to include the H^F term in the Hamiltonian (17). This leads to the following new terms in the Hamiltonian.

$$\mathcal{H}_1^{x'} = \sum_n b_n \cos(n\theta) x \quad (100)$$

$$\mathcal{H}_2^{x'} = \frac{1}{2} \sum_n b_n (1 + Q_x) \cos(n\theta) x^2 \quad (101)$$

$$\mathcal{H}_2^{z'} = -\frac{1}{2} \sum_n b_n Q_x \cos(n\theta) z^2 \quad (102)$$

$$\begin{aligned} \mathcal{H}_3' = \sum_n \left\{ b_n \left(\frac{1}{3} Q_x + \frac{1}{6} S_x \right) \cos(n\theta) x^3 - \frac{1}{2} b_n (Q_x + S_x) \cos(n\theta) x z^2 \right. \\ \left. - \frac{1}{2} n b_n \sin(n\theta) p_x z^2 \right\} \end{aligned} \quad (103)$$

Hence, we write the Hamiltonian in the form

$$\mathcal{H} = \mathcal{H}_1^{x'} + \mathcal{H}_2^x + \mathcal{H}_2^{x'} + \mathcal{H}_2^z + \mathcal{H}_2^{z'} + \mathcal{H}_2^T + \mathcal{H}_3 + \mathcal{H}_3' + \mathcal{H}_3^T. \quad (104)$$

We see that the primed terms in the new Hamiltonian are θ ("time") dependent and we also have an $\mathcal{H}_1$ term. Hence the solution cannot be obtained as a Hamiltonian "flow" (38) but must be integrated by the procedures given in section 3.2.

To simplify the discussion, we assume that $p_T = 0$. Before we can solve for $\mathcal{M}_1^{(0)}$, (55), we must find the $\mathcal{M}_2'$ and $\mathcal{M}_3'$ maps which arise from the $\mathcal{H}_2^{x'}$, $\mathcal{H}_2^{z'}$ and $\mathcal{H}_3'$ terms in the perturbed Hamiltonian. We will treat these as a perturbations and write

$$\mathcal{M}^{(0)} = \mathcal{M}_1 \mathcal{M}_2^{x'} \mathcal{M}_2^x \mathcal{M}_2^{z'} \mathcal{M}_2^z \mathcal{M}_3' \mathcal{M}_3 \quad (105)$$

where the ordering of the map is chosen for convenience; note that $\mathcal{M}_2^{x'} \mathcal{M}_2^x$ commutes with $\mathcal{M}_2^{z'} \mathcal{M}_2^z$. Using the methods of section 3.2, we initially set $\mathcal{M}_1 = \mathcal{M}_2' = \mathcal{M}_3' = \mathcal{I}$,

which means that M_2 and M_3 are the solutions found in section 4, namely (87) and (92). Applying the arguments leading up to (43), we obtain

$$M_2^{(0)} = M_2' M_2 = M_2^{x'} R^x M_2^{z'} R^z \quad (106)$$

where $M_2^x \Rightarrow R^x$ and is given by (92) and similarly for R^z . Hence we obtain, (4),

$$\dot{M}_2' = M_2' : -\mathcal{H}_2^{T'} : \quad (107)$$

for the x and z-planes, where

$$:-\mathcal{H}_2^{T'} : = R : -\mathcal{H}_2' : R^{-1} = : -\mathcal{H}_2' (R) :. \quad (108)$$

Because $\mathcal{H}_2'$ is at least of order $\varepsilon = b_n$ and thus has rank 3, we might hope that the integral series could be truncated at the first iteration of (49). Unfortunately, the flutter is not small and we must include at least the second term. Furthermore when $\nu \approx 1$, the first term in (49) vanishes. We can find an approximate solution for f_2' by equating the formal expansion for M_2' with (49) as follows:

$$\begin{aligned} M_2' &= \left\{ \mathcal{J} + \varepsilon : f_2' : + \varepsilon^2 \frac{f_2'^2}{2} + \dots \right\} \\ &= \mathcal{J} + \int_0^\theta : -\mathcal{H}_2^{T'}(\zeta, \theta') : d\theta' + \int_0^\theta d\theta' \left\{ \int_0^{\theta'} d\theta'' : -\mathcal{H}_2^{T'}(\theta'') : \right\} : -\mathcal{H}(\theta') : + \dots \end{aligned} \quad (109)$$

Subtraction of the ε^2 term leads to

$$\varepsilon : f_2' : = \int_0^\theta : -\mathcal{H}_2^{T'}(\zeta, \theta') : d\theta' + \int_0^\theta d\theta' \left\{ \int_0^{\theta'} d\theta'' : -\mathcal{H}_2^{T'}(\theta'') : \right\} : -\mathcal{H}(\theta') : - \varepsilon^2 \frac{f_2'^2}{2} \quad (110)$$

where we write

$$\varepsilon^2 \frac{f_2'^2}{2} = \frac{1}{2} \int_0^\theta d\theta' \left\{ \int_0^{\theta'} d\theta'' : -\mathcal{H}(\theta'') : \right\} : -\mathcal{H}(\theta') :. \quad (111)$$

Use of the Jacobi identity (27) plus some manipulation leads to the result [14]

$$\varepsilon : f_2' : = \int_0^\theta : -\mathcal{H}_2^{T'}(\zeta, \theta') : d\theta' + \int_0^\theta d\theta' \int_0^{\theta'} d\theta'' : [-\mathcal{H}_2^{T'}(\theta''), -\mathcal{H}_2^{T'}(\theta')] :. \quad (112)$$

Because the arguments of the Lie-operators must be equal if the Lie-operators are equal, we can remove the colons in (112) to obtain an integral series for f_2' . This equation can be used to obtain the Lie-polynomials $f_2^{x'}$, and $f_2^{z'}$.

If we assume that $\nu_x = 1$ and $\nu_z = 0$, that is we start with a uniform field cyclotron, then $\epsilon f_2^{x'}$ has the particularly simple form

$$\epsilon f_2^{x'} = \sum_n \frac{\pi b_n^2}{4(n^2-4)} (x^2 + p_x^2), \quad (113)$$

while $\epsilon f_2^{z'}$ is the identity. Hence there is no focusing in the vertical direction at this stage. The general forms of $\epsilon f_2^{x'}$ and $\epsilon f_2^{z'}$ are much more complicated, involving terms containing $\sin(2\pi\nu)$ and $\cos(2\pi\nu)$.

The maps $M_2^{x'}$, M_2^x , $M_2^{z'}$ and M_2^z can be combined either by the use of the CBH formula, or by reducing the Lie transformations to matrix form with (29) and multiplying the matrices. In the latter case, care must be taken to ensure that the determinants of the primed matrices are unity otherwise the symplectic condition will be violated. Thus we obtain

$$M_2'' = M_2^{x'} M_2^x M_2^{z'} M_2^z = M_2^{x''} M_2^{z''} \quad (114)$$

where the x'' and z'' maps commute. Again, if we assume that initially $\nu_x = 1$ and $\nu_z = 0$, we obtain the following simple result:

$$\nu_x''^2 = 1 + \sum_n \frac{b_n^2}{4(n^2-4)}. \quad (115)$$

The map M_3' is calculated in a similar manner to (106). From (48), we find

$$\dot{M}_3^T = M_3^T : -\mathcal{H}_3^T ;, \quad \mathcal{H}_3^T = M_2'' : -\mathcal{H}_3' : M_2''^{-1} \text{ etc.} \quad (116)$$

The solution to this equation is similar to (107) except that we need only the first term as the Poisson bracket; $[\mathcal{H}_3'(\theta''), \mathcal{H}_3'(\theta')]$ is of order 4. Hence

$$\epsilon f_3^{T'} = \int_0^\theta -\mathcal{H}_3^{T'}(\zeta, \theta') d\theta' \quad (117)$$

and $M_3' = R''^{-1} M_3^{T'} R''$. The maps $M_3' M_3$ can be combined easily to form M_3'' as M_3' and M_3 commute to third order. Hence, now the map has the form

$$M^{(0)} = M_1^{(0)} M_2^{(0)} M_3^{(0)} \quad (118)$$

where $M_1^{(0)}$ is as yet undetermined.

Again, if we assume that we start with $\nu_x'' = 1$ and $\nu_z = 0$, we see that only the last term in (103) contributes, leading to the Lie-polynomial

$$f_3' = -2\pi \sum_n \left\{ b_n \frac{n^2}{n^2-1} (p_x z p_z + \pi p_x p_z^2) + 2b_n \frac{n^2}{(n^2-1)^2} x p_z^2 \right\} \quad (119)$$

for one turn.

We now substitute $\mathcal{M}_2''^{(0)} \mathcal{M}_3''^{(0)}$ into (53) to obtain the new "fed-down" Hamiltonian. We note that (100) depends only on x , so we need evaluate only the x component of the map. The result of evaluating $\mathcal{M}_2''^{(0)} \mathcal{M}_3''^{(0)} \zeta$ in (53) is

$$\mathcal{H}_1^{(1)}(\mathcal{M}_2''^{(0)} \mathcal{M}_3''^{(0)} \zeta^1) = \left[R'' \zeta^1 + R'' : f_3'' : \zeta^1 \right] b_n \cos(n\theta). \quad (120)$$

The first term in brackets in (120) is of order 1 and forms the new $\mathcal{K}_1^{(1)}$, while the second term is of order 2 and becomes the new $\mathcal{K}_2^{(1)}$. We now solve the equation

$$\dot{\mathcal{M}}_2^{(1)} = \mathcal{M}_2^{(1)} : -\varepsilon^2 \mathcal{K}_2^{(1)} : \quad (121)$$

for the "fed-down" map $\mathcal{M}_2^{(1)}$.

A new $\mathcal{H}_1^{(2)}$ is obtained in an analogous fashion to (120), which terminates the iterative process in this case. Hence, we obtain from (58) that

$$\mathcal{M}_1^{(2)} = \int_0^\theta : -\mathcal{H}_1^{(2)}(\mathcal{M}_2^{(1)} \zeta^1, \theta') : d\theta'. \quad (122)$$

where

$$\mathcal{H}_1^{(2)} = b_n x(\mathcal{M}_2^{(1)} \zeta^1) \cos(\theta)$$

and $x(\mathcal{M}_2^{(1)} \zeta^1)$ is the x -component of $\mathcal{K}_1^{(1)}(\mathcal{M}_2^{(1)} \zeta^1) = R'' R^{(1)} \zeta^1$, where $R^{(1)}$ is the matrix representation of $\mathcal{M}_2^{(1)}$. Hence, the final map has the form

$$\mathcal{M}^{(2)} = \mathcal{M}_1^{(2)} \mathcal{M}_2^{(1)} \mathcal{M}_2''^{(0)} \mathcal{M}_3''^{(0)}, \quad (123)$$

which, if we combine the $n=2$ maps, becomes

$$\mathcal{M}^{(2)} = \mathcal{M}_1^{(2)} \mathcal{M}_2^{(2)} \mathcal{M}_3''^{(0)} = e : g_1 : e : g_2 : e : g_3 : \quad (124)$$

where $\mathcal{M}_2^{(2)} = \mathcal{M}_2^{(1)} \mathcal{M}_2''^{(0)} = R'' R^{(1)}$.

In the uniform field approximation, we obtain the following approximate result for the tune shift in z resulting from the feed-down of $\mathcal{H}_3'$

$$\nu_z^2 = \sum_n b_n^2 \frac{n^2(n^2 + \nu_x''^2)}{2(n^2 - \nu_x''^2)^2}. \quad (125)$$

Thus we see that the vertical focusing is completely due to the feed-down of the third order term in the Hamiltonian that was generated by the flutter; see (119) and (103). This result should be compared with the tune shift in the radial direction, (115), which arose directly from the perturbation to the second order Hamiltonian. The vertical focusing (tune-shift) is about $\sqrt{2}n$ times stronger than the radial tune-shift.

If we compute $M_1^{(2)}$ on the midplane with $p_T = 0$, we obtain for one turn the simple result

$$g_1 = \sum_n \frac{b_n}{n^2 - \nu_x''^2} \left(\nu_x'' \sin(2\pi \nu_x'') x + (1 - \cos(2\pi \nu_x'')) p_x \right), \quad (126)$$

where ν_x'' is now the composite tune determined from g_2 . We obtain a first order estimate of the central trajectory by evaluating $M_1^{(2)} M_2^{(2)}$. The result is

$$x^c = \cos(\nu_x'' \theta) x + \sum_n \frac{b_n}{n^2 - \nu_x''^2} \left\{ \cos(n\theta) - \cos(\nu_x'' \theta) \right\} + \frac{1}{\nu_x''} \sin(\nu_x'' \theta) p_x. \quad (127)$$

The orbit becomes closed or a fixed point if we let $x = \sum b_n / (n^2 - \nu_x''^2)$ and $p_x = 0$ giving

$$x^c = \sum_n \frac{b_n}{n^2 - \nu_x''^2} \cos(n\theta). \quad (128)$$

Eq. (128) is similar to the results of Hagedoorn and Gordon [3,4], which disagree with each other in detail; (128) is identical with Hagedoorn if we set $\nu_x'' = 1$. Also, inspection of (127) shows that if the orbit is not quite closed it precesses at a rate $(\nu_x'' - 1)$ with errors in the initial values of x and p_x being $\pi/2$ out of phase. Finally, even though the flutter, b_n , can be quite large, say $1/4$, the "scallop" of the orbit is very small, being of the order of $b_n r^c / 8$ for a 3-sector cyclotron and $b_n r^c / 15$ for a 4-sector machine, where n denotes the fundamental harmonic.

In order to obtain the final form for the map, we must transform to the coordinates of the perturbed central trajectory, that is we must remove the M_1 map. From (60), we write (124) as

$$\mathcal{A}_1 M^{(2)} \mathcal{A}_1^{-1} = e^{i a_1} e^{i g_1} e^{i g_2} e^{i g_3} e^{-i a_1} \quad (129)$$

where $\mathcal{A}_1$ has the general form

$$\mathcal{A}_1 = e^{i a_1}; \quad a_1 = ax + bp_x \quad (130)$$

and we write g_1 as

$$g_1 = cx + dp_x \quad (131)$$

with

$$g_2 = -\frac{\mu_x}{2}(\gamma_x x^2 + \beta_x p_x^2) - \frac{\mu_z}{2}(\gamma_z x^2 + \beta_z p_x^2). \quad (132)$$

The constants a and b are to be determined; c and d are the coefficients of x and p_x respectively in (126). Eq. (129) can be rewritten as

$$A_1 M A_1^{-1} = e^{A_1 g_1} e^{A_1 g_2} e^{A_1 g_3} = e^{A_1 g_1} e^{h} = e^{H}, \quad (133)$$

where the CBH formula is used at each stage. First, we evaluate $A_1 g_n$ which leads to the general formula

$$A_1 g_n = \{g_n + [a_1, g_n] + \dots\} \quad (134)$$

Evaluation of $A_1 g_1$ leads to

$$A_1 g_1 = g_1 + \text{const.} \quad (135)$$

while evaluation of $A_1 g_2$ leads to

$$A_1 g_2 = g_2 + 2\pi(b\nu_x''^2 x - a p_x). \quad (136)$$

Since $\nu_x \approx 1$, the components of g_3 which depend only on (x, p_x) are very small and will be neglected; if we start with a uniform field cyclotron, they are zero. Hence, $g_3 \approx f_3'$, (119), and $A_1 g_3$ leads to

$$A_1 g_3 = g_3 - 2\pi a \sum_n b_n \frac{n^2}{n^2-1} (z p_z + \pi p_z^2). \quad (137)$$

Note that the quadratic part depends only on (z, p_z) . The final combined exponent has the form

$$H = g_2 + C_x(a, b, z, p_z)x + C_{p_x}(a, b, z, p_z)p_x + C_z(x, p_x)z^2 + C_{p_z}(x, p_x)p_z^2 \quad (138)$$

where the C 's are complicated functions of their arguments. The fixed point is found when $C_x = 0$ and $C_{p_x} = 0$. From (138), we see that, in general, the fixed point is a function of (z, p_z) . This parametric behavior also extends to the tunes, where we find that $\nu_x = \nu_x(z, p_z, p_\tau)$, and $\nu_z = \nu_z(x, p_x, p_\tau)$. However, since we are interested in a new central trajectory that preserves midplane symmetry, we set $z = p_z = 0$. This leads to the following simple solutions for the parameters of the fixed point:

$$a_n = \frac{1}{2\pi} \left\{ d_n \left(1 - \frac{1}{6} \nu_x''^2 \pi^2 \right) - c_n \pi \right\}, \quad b_n = \frac{1}{2\pi \nu_x''^2} \left\{ c_n \left(1 - \frac{1}{6} \nu_x''^2 \pi^2 \right) - d_n \nu_x''^2 \pi \right\} \quad (139)$$

where from (126)

$$c_n = \frac{b_n}{n^2 - \nu_x''^2} \nu_x'' \sin(2\pi \nu_x''), \quad d_n = \frac{b_n}{n^2 - \nu_x''^2} (1 - \cos(2\pi \nu_x'')). \quad (140)$$

Finally, we give the tune-shift resulting from the transformation to the fixed point. Setting $x = p_x = 0$, we obtain

$$\nu_z'^2 = \nu_z^2 + 2 \sum_n a_n b_n \frac{\nu_z^2 n^2}{(n^2 - 1)} - 4 \sum_n b_n b_n \frac{\nu_z^2 n^2}{(n^2 - 1)^2}. \quad (141)$$

The second and third terms in (141) are functions of c_n and d_n and scale as $b_n^2 \nu_z^2 / n^2$ and $b_n^2 \nu_z^2 / n^3$, respectively, so are very small.

6. Solution for a b_1 'decentering' harmonic

We can use the results of the previous section to obtain the amplitude of the $\nu_x = 1$ resonance. This resonance occurs in all compact cyclotrons near the extraction radius because the magnetic field is dropping and Q_x (10) becomes negative, reducing the radial focusing and increasing the vertical focusing. Inspection of (126) shows that g_1 is singular for $n = 1$ and $\nu_x = 1$. Taking the limit as $\nu_x \rightarrow 1$ in (127) leads to

$$x = \cos(\theta)x - \frac{1}{2} b_1 \theta \sin(\theta) + \sin(\theta) p_x. \quad (142)$$

The second term in (142) is the well-known formula for the amplitude of the decentering resonance. As it is proportional to θ , the amplitude increases linearly as the number of turns increases. Similarly, over one turn we find that $g_1 = -b_1 \pi x$. The tolerable strength of b_1 is very small, $\sim 10^{-4}$, and is rarely greater than 10^{-3} even when deliberately introduced locally for the purpose of 'resonant extraction'.

7. General non-linear behaviour

In this section we investigate the general non-linear behaviour of the cyclotron up to sextupole order. We will use the powerful normal form methods developed in section 3.3. Let us rewrite (66) as

$$G_3 = \sum_{n+m+k=3} \frac{A_{nm}^k}{\{1 - \exp[i(n-m) \cdot \mu]\}}, \quad (143)$$

with

$$\mu_x = \frac{2\pi \nu_x}{N}, \quad \mu_z = \frac{2\pi \nu_z}{N}. \quad (144)$$

Here N is the symmetry of the magnetic field, and k denotes the order of p_τ ; the (τ, p_τ) coordinates cannot be put in the form of (66) because they are represented by a "drift" matrix rather than a "rotation" matrix. The possible values of n and m are given in Table 1 for $n+m+k=3$.

Table 1							
$k = 0$		$k = 1$		$k = 2$		$k = 3$	
n	m	n	m	n	m	n	m
3	0	2	0	1	0	0	0
2	1	1	1	0	1		
1	2	0	2				
0	3						

We see from the table that for $N = 3$ with $\nu_x \approx 1$ the term $(n_1 - m_1) = \pm 3$ is resonant, i.e. $(n_1 - m_1)\mu_x = \pm 2\pi$, and this term cannot be removed from the normal map $\mathcal{N}$. This is also true for the chromaticity terms ($k = 1, n - m = 0$) as well as the term with $k=3$. However for $N = 4$, all terms with $k=0$ can be removed. Finally, we have the possibility that $\nu_x - 2\nu_z = 0$. This situation generates the $\nu_x = 2\nu_z$ coupling resonance, often called the Walkinshaw resonance. Otherwise, for non-rational values of the tunes, all terms except the chromaticity and p_τ^3 terms can be removed from the normal map. The properties of the Walkinshaw resonance have been described in some detail in [1,2].

Let us investigate some of the consequences of the above discussion. Assume that the tunes are not rational numbers and we have normalized the map over a unit cell. The normal map can be written in the following factored form (see (67) and (62))

$$\mathcal{N} = e^{(\mu_x + \mu'_x p_\tau) J_x} e^{\mu_z J_z} e^{ap_\tau^3} = \mathcal{R}_x \mathcal{R}_z e^{ap_\tau^3} \quad (145)$$

where μ'_x is the chromaticity; $\mu'_x = 0$ if we start with a uniform field. Now, we generate the one-turn map using (68) and obtain

$$\mathcal{M}^N = \mathcal{A}^{-1} \mathcal{N}^N \mathcal{A} = \mathcal{A}_3^{-1} e^{N(\mu_x + \mu'_x p_\tau) J_x} e^{N\mu_z J_z} e^{Nap_\tau^3} \mathcal{A}_3 \quad (146)$$

because all the factors in $\mathcal{N}$ commute. If $N(\mu_x + \mu'_x p_\tau) = 2\pi$, then $\mathcal{R}_x^N = \mathcal{I}_x$ and we obtain

$$\mathcal{M}^N = \mathcal{A}_3^{-1} \mathcal{I}_x \mathcal{R}_z^N e^{Nap_\tau^3} \mathcal{A}_3 = \hat{\mathcal{A}}_3^{-1} \mathcal{R}_z^N e^{Nap_\tau^3} \hat{\mathcal{A}}_3 \quad (147)$$

where $\hat{\mathcal{A}}_3$ contains only terms involving (z, p_z) ; i.e. all the non-linearities involving only (x, p_x) cancel. This is valid for $N = 4$, but not for $N=3$, because, as we saw above, if $\nu'_x = 1$ ($N(\mu_x + \mu'_x p_\tau) = 2\pi$) for $N=3$, we excite the "1/3-integer" resonance and the normal map cannot be written in the form of (145). Thus the effects of 3rd-order non-linearities are greatly reduced in 4-sector cyclotrons; this result is further enhanced because ν_x is nearer unity in a 4-sector cyclotron than in a 3-sector machine (see (115)).

Let us suppose, for the moment, that we could simultaneously make $\nu_x = \nu_z = 1$. Then both $\mathcal{R}_x^N = \mathcal{I}_x$ and $\mathcal{R}_z^N = \mathcal{I}_z$, and we find that $\mathcal{M}^N = \exp(:nap_\tau^3:)$, i.e. all of the non-linearities cancel to 3rd order with the exception of the p_τ^3 factor. This result was originally discovered by Karl Brown [19].

Thus, from symmetry alone, we have been able to easily derive some important results without having to evaluate the coefficients.

8. One-third Integer Resonance

In this section, we will investigate in greater detail the one-third integer resonance mentioned in the previous section. Since $\nu_x \approx 1$, somewhere near extraction, the radial phase advance per cell is $\mu_x \approx 2\pi/3$ and we will have a resonant condition. We make the transformation to normal form as per section 3.3 and restrict the analysis to the midplane so that $z = p_z = 0$. Hence,

$$N(2\pi/3) = e^{ig_2} e^{ig_3} \quad (148)$$

where

$$g_2 = (\mu_x + \mu'_x p_\tau) J_x \text{ and } g_3 = -\frac{\sqrt{2}}{6} \mathcal{Y}_x J_x^{3/2} \cos(3\varphi_x).$$

Here we have put g_3 in standard form: $\mathcal{Y}_x = \pi b_3(Q_x/3 + S_x/6)$ and $\hat{\mu}_x = \mu_x + \mu'_x p_\tau$. Only this term generated by the flutter is resonant; all other terms are incorporated into $\mathcal{A}_3$ via G_3 (66).

In order to understand the topology of this resonance it is convenient to transform to the "co-moving" frame [20]. This transformation is defined by the expression

$$\phi_1 = \mathbf{k} \cdot \boldsymbol{\varphi}; \quad (\mathbf{n} - \mathbf{m}) = \lambda \mathbf{k}, \quad \lambda = \{\text{integer}\} \quad (149)$$

and

$$K_1 = \frac{\mathbf{k} \cdot \mathbf{J}}{\|\mathbf{k}\|^2}, \quad K_2 = \frac{\mathbf{a} \cdot \mathbf{J}}{\|\mathbf{a}\|^2} \quad (150)$$

where $\mathbf{a}$ is orthogonal to $\mathbf{k}$ ($\mathbf{k} \cdot \mathbf{a} = 0$). In our case $\lambda = 1$. Hence,

$$\phi_1 = 3\varphi_x \text{ and } K_1 = J_x/3 \quad (151)$$

where $\mathbf{k} = (3, 0)$ and $\mathbf{a} = (0, 3)$; however, we have set $J_z = \varphi_z = 0$ in this case. The transformation to the co-moving frame is effected by the transformation [20]

$$N_c = e^{i2\pi\ell K_1} N = e^{i2\pi\ell K_1} e^{-i\hat{\mu} \cdot \mathbf{J}} e^{i\tilde{g}_3} e^{i\alpha p_t^3} \quad (152)$$

where $2\pi\ell = \mathbf{k} \cdot \hat{\boldsymbol{\mu}}$. We want to expand about the resonance, that is we assume that we are not quite on resonance, and write $\delta = 2\pi\ell - \mathbf{k} \cdot \hat{\boldsymbol{\mu}}$ with $\mathbf{k} = (3, 0)$, and $\ell = 1$. Hence,

$$N_c = e^{i\delta K_1} e^{i\tilde{g}_3} e^{i\alpha p_t^3} \quad (153)$$

where $\tilde{g}_3$ is in the co-moving coordinates;

$$\tilde{g}_3 = \delta \mathcal{Y}_x \sqrt{6} K_1^{3/2} \sin(\phi_1). \quad (154)$$

If we combine exponents in (153) so that

$$\mathcal{N}_c = e^{:-\delta K_1:} e^{:\tilde{g}_3:} e^{:\alpha p_t^3:} = e^{:h:} e^{:\alpha p_t^3:}, \quad (155)$$

we obtain the "pseudo-Hamiltonian" h , where

$$h \cong -\delta J_x + \frac{\sqrt{6}}{2} \varphi_x \delta \frac{\sin(\phi_1) - \sin(\phi_1 + \delta)}{1 - \cos(\delta)} + \mathcal{O}(S_x^2), \quad (156)$$

The fixed points of the resonance can be found from (59) which implies $[h, \zeta] = 0$ or

$$\frac{\partial h}{\partial K_1} = 0 \quad \text{and} \quad \frac{\partial h}{\partial \phi_1} = 0. \quad (157)$$

This leads to

$$\frac{\partial h}{\partial \varphi_x} = \cos(\phi_1) - \cos(\phi_1 + \delta) = 0, \quad (158)$$

and

$$\frac{\partial h}{\partial J_x} \cong 1 - \varphi_x \frac{3\sqrt{6K_1}}{4} \frac{\sin(\phi_1) - \sin(\phi_1 + \delta)}{1 - \cos(\delta)} = 0 \quad (159)$$

which has the solutions

$$\tan(\phi_1) = -\frac{1 - \cos(\delta)}{\sin(\delta)} \cong \frac{\delta}{2} \ll 1; \quad K_1 \cong \frac{2\delta^2}{27S_x^2}. \quad (160)$$

In terms of the original action-angle variables we obtain

$$\varphi_x \cong \{\pi/3, 2\pi/3, 5\pi/3\}, \quad J_x = \frac{2}{9S_x^2} \delta^2. \quad (161)$$

A phase space plot for the one-third integer resonance is shown in Fig. 1, for tunes just above and below the one-third integer resonance.

Inclusion of the "iterated sextupole" terms (i.e. the generation of higher order non-linearities from the repeated passage through lower order non-linear elements) are calculated to octupole order by evaluating the terms $\mathcal{O}(S_x^2)$, which were neglected in the CBH series used to obtain h (156). If we include these terms, we obtain the following pseudo-Hamiltonian

$$h' = h + 27S_x^2 K_1^2 \frac{2\sin(\hat{\mu}_x) - \sin(2\hat{\mu}_x)}{8(1 - \cos(\hat{\mu}_x))^2} - \frac{3\delta}{16} S_x^2 K_1^2. \quad (162)$$

The additional terms are "tune-shift with amplitude" terms (see Eq. 67) which affect the radius, K_1 , of the fixed points but not their angular positions. Solution of (162) leads to the result

$$J_x = \frac{2}{9S_x^2} \delta^2 - \frac{4\sqrt{3}}{27S_x^2} \delta^3 + O(\delta^4). \quad (163)$$

Thus the iterated sextupole term reduces the region of stability for positive δ and increases it for negative δ ; the ratio of the second to first terms in (163) is $2\delta/\sqrt{3}$. A true octupole component in the magnetic field is not resonant for $N = 3$ but does add a tune-shift term, which is easily added to (162) because the iterated sextupole and true octupole contributions commute.

Although these results were derived on the midplane, similar results would be obtained off the midplane except that the amplitude of the fixed point would depend on (z, p_z) as well as p_T . Thus in reality, the fixed points and separatrices have slightly different positions for different values of the "parameters" (z, p_z, p_T) .

8 Conclusions

In this paper, we have used a simplified model of the magnetic field of a cyclotron in order to illustrate the power of Lie-algebraic methods in analysing their orbit dynamics. The basic Lie-algebraic methods have been outlined including the powerful normal-form method in the context of a perturbation expansion.

Analytic results have been obtained for the motion of the central trajectory as well as the radial and vertical tunes. The rather subtle differences in the origin of radial and vertical focusing have also been discussed; the radial focusing is mainly a first-order effect, while the vertical focusing arises entirely from the feed-down of the sextupole (and higher order) component of the Hamiltonian. This feed-down is caused by the perturbation, generated by the flutter, of the original circular central trajectory to a sinusoidal trajectory. This leads to an M_1 factor in the map. Removal of the M_1 factor is effected by transforming to the fixed point at the origin. This transformation generates further tunes shifts, especially in the vertical plane.

In addition, the powerful normal form methods have been used to investigate both non-resonant and resonant behaviour. It has been demonstrated that for $\nu_x = 1$, all (x, p_x) non-linearities vanish, leaving only the chromaticities and the (z, p_z) aberrations. Because the radial tune-shift due to the flutter in a 4-sector cyclotron is about 1/2 that of a 3-sector machine, the effects of 3rd-order non-linearities are greatly reduced in 4-sector cyclotrons. In fact, it is not possible to approach $\nu_x = 1$ too closely in 3-sector cyclotrons because this excites the "one-third integer resonance", which can cause loss of beam-intensity. The driving term for this resonance is large, proportional to the b_3 component of the flutter.

Finally, analytic results were obtained for the decentering resonance, which although generally pathological, is universally used in compact cyclotrons as an aid to extracting the beam. This resonance, driven by a b_1 Fourier term in the magnetic field, is the result of unwanted errors in the magnetic field (except when introduced

locally for resonant extraction) and is a larger problem in 4-sector cyclotrons for precisely the same reason that the 3rd-order non-linearities are reduced; ν_x is very close to unity.

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Fig. 1 Phase-space plots for the "one-third integer" resonance above, a), and below, b), the resonance; $\delta = \pm 0.088$. The fixed points in b) are rotated by 180° with respect to a). Note that the angle of the fixed points are not quite $\pi/3$, $2\pi/3$, $5\pi/3$ but are offset by $\delta/2$ as shown in (160).

APPENDIX A

SOME USEFUL RELATIONS IN LIE ALGEBRA

The following collection of useful relations was assembled from references [12,13,21].

$$e^{:f:}(gh) = \left(e^{:f:}g \right) \left(e^{:f:}h \right) \quad (A.1)$$

$$e^{:f:}g(\zeta) = g \left(e^{:f:}\zeta \right) \quad (A.2)$$

$$e^{:f:}[g,h] = [e^{:f:}g, e^{:f:}h] \quad (A.3)$$

Exchange rule or transformation rule

$$e^{:f:}e^{:g:} = e^{:f:}e^{:g:}e^{-:f:}e^{:f:} = e^{:e^{:f:}g:}e^{:f:} \quad (A.4)$$

CAMPBELL-BAKER-HAUSDORFF FORMULA (CBH)

$$\begin{aligned} e^{:f:}e^{:g:} &= \exp(:f: + :g: + \frac{1}{2}\{ :f:, :g: \} + \frac{1}{12}\{ :f:, \{ :f:, :g: \} \} \\ &\quad + \frac{1}{12}\{ :g:, \{ :g:, :f: \} \} + \frac{1}{24}\{ :f:, \{ :g:, \{ :g:, :f: \} \} \} + \dots \end{aligned} \quad (A.5)$$

Making use of (24), the RHS of (A.5) can be written as

$$h = f + g + \frac{1}{2}[f,g] + \frac{1}{12}[f,[f,g]] + \frac{1}{12}[g,[g,f]] + \frac{1}{24}[f,[g,[g,f]]] + \dots \quad (A.6)$$

to fourth order.

If we introduce expansion parameters into the CBH formula we can write

$$e^{:h:} = e^{\alpha :f:} \gamma :g: \quad (A.7)$$

where if we substitute (A.7) into (A.6), it is possible to sum the series with respect to α . We obtain the result, which is not obvious, that [21]

$$h = \alpha f + \alpha :f: \left[1 - e^{-\alpha :f:} \right]^{-1} (\gamma g) + O(\gamma^2) \quad (A.8)$$

where the expression involving $:f:$ must be interpreted as

$$\alpha:f: \left[1 - e^{-\alpha:f:} \right]^{-1} = \alpha:f: \left[1 - \sum_{m=0}^{\infty} \frac{(-\alpha:f:)^m}{m!} \right]^{-1} \quad (\text{A.9})$$

$$= \alpha:f: \left[- \sum_{m=1}^{\infty} \frac{(-\alpha:f:)^m}{m!} \right]^{-1} \quad (\text{A.10})$$

$$= 1 + \frac{\alpha}{2}:f: + \frac{\alpha^2}{12}:f:2 - \frac{\alpha^4}{720}:f:4 + \frac{\alpha^6}{30240}:f:6 \quad (\text{A.11})$$

Hence, we obtain on substitution of (A.8) into (A.7) that

$$e^{\alpha:f:} e^{\gamma:g:} = \exp \left\{ \alpha:f: + : \alpha:f: \left[1 - e^{-\alpha:f:} \right]^{-1} (\gamma g): + O(\gamma^2) \right\} \quad (\text{A.12})$$

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We can define h using (A.8) and (A.9) as

$$h = \alpha:f: \left[1 - e^{-\alpha:f:} \right]^{-1} g \quad (\text{A.13})$$

This equation may be inverted to give

$$g = \left\{ \left[1 - e^{-\alpha:f:} \right] / [\alpha:f:] \right\}_h = \left\{ - \sum_{m=1}^{\infty} \frac{(-\alpha)^{m-1} :f:^{m-1}}{m!} \right\}_h \quad (\text{A.14})$$

If we substitute (A.13) into (A.12), we obtain

$$e^{[\alpha:f: + \gamma:h:]} = e^{\alpha:f:} e^{\gamma:g:} e^{O(\gamma^2):} \quad (\text{A.15})$$

Two important integral relations arise out of the above, namely

$$\left[1 - e^{-\alpha:f:} \right] / [\alpha:f:] = \int_0^1 e^{-\tau\alpha:f:} d\tau \quad (\text{A.16})$$

Hence, from (A.4)

$$g = \int_0^1 e^{-\tau\alpha:f:} h d\tau \quad (\text{A.17})$$

Finally, we can use (A.2) to obtain the formula

$$g(\zeta) = \int_0^1 d\tau \, h\left(e^{-\tau\alpha:f:\zeta}\right) \quad (\text{A.18})$$

APPENDIX B

SOME IMPORTANT FIRST-ORDER RELATIONS.

Here we evaluate three important first-order maps of the form $\mathcal{M} = \exp(:f:)$.

1. DRIFT SPACE

Let us consider

$$f = -\frac{L}{2} p^2 \quad (\text{B.1})$$

Then from (28)

$$:f:q = [f, q] = Lp \quad (\text{B.2})$$

and

$$:f:p = 0 \quad (\text{B.3})$$

We see that

$$:f:^2q = :f::f:q = :f:Lp = 0 \quad (\text{B.4})$$

Let us now find the *map*, $\mathcal{M}$, by evaluating the Lie transformation, $\exp(:f:)$. We find that

$$q^f = e^{:f:} q^i = q^i + Lp^i \quad (\text{B.5})$$

where we have substituted (B.5) into (29). Similarly we find that

$$p^f = e^{:f:} p^i = p^i \quad (\text{B.6})$$

where we have made use of (29): p^f and p^i are the final and initial values of the canonical coordinates, respectively. This is a drift and the map can be written in the usual matrix form

$$R = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \quad (\text{B.7})$$

2. THIN LENS

Now let us consider

$$f = -\frac{a}{2} q^2. \quad (\text{B.8})$$

We find that

$$:f:q = [f,q] = 0 \quad (\text{B.9})$$

and

$$:f:p = [f,p] = -aq. \quad (\text{B.10})$$

Also,

$$:f:^2q = :f:^2p = 0. \quad (\text{B.11})$$

Hence the *map*, $\mathcal{M} = \exp(:f:)$, leads to

$$q^f = e^{:f:} q^1 = q^1 \quad (\text{B.12})$$

$$p^f = e^{:f:} p^1 = -aq^1 + p^1 \quad (\text{B.13})$$

which can be written as a matrix transformation

$$R = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}. \quad (\text{B.14})$$

This is the matrix of a thin lens with focal length $f = 1/a$.

3. "ROTATION" IN PHASE SPACE

Let us consider

$$f = -\frac{\mu}{2k}(q^2 + k^2 p^2). \quad (\text{B.15})$$

The evaluation of the *map* from (B.15) can be carried out using the techniques of sections 1 and 2 above. However we will use a more elegant method that simplifies the evaluation of more complex operators. Let us evaluate $:f:$ by writing

$$:f: \begin{bmatrix} q \\ p \end{bmatrix} = \mu G \begin{bmatrix} q \\ p \end{bmatrix} \quad (\text{B.16})$$

where we mean by this notation that we form the matrix G by operating on q and p in turn with $:f:$ as was done above. We find immediately that

$$G = \begin{pmatrix} 0 & k \\ -1/k & 0 \end{pmatrix}. \quad (\text{B.17})$$

We can now evaluate the *map* by noting that

$$\mathcal{M} = e^{:f:} = e^{\mu G} = \cosh(\mu G) + \sinh(\mu G). \quad (\text{B.18})$$

If we make a formal expansion of the trigonometric functions, we obtain

$$e^{\mu G} = \left\{ I + \frac{(\mu G)^2}{2!} + \frac{(\mu G)^4}{4!} + \dots \right\} + \left\{ \mu G + \frac{(\mu G)^3}{3!} + \frac{(\mu G)^5}{5!} + \dots \right\}. \quad (\text{B.19})$$

Now, we note that

$$G^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } G^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

hence $G^4 = -G^2$. This may be obtained also from the eigenvalues of G which is perhaps easier for more complicated transforms. Making use of the above property of G , we can write (B.19) as

$$e^{\mu G} = \left\{ I + G^2 \left[\frac{\mu^2}{2!} - \frac{\mu^4}{4!} + \dots \right] \right\} + \left\{ \mu G + G^3 \left[\frac{\mu^3}{3!} - \frac{\mu^5}{5!} + \dots \right] \right\} \quad (\text{B.20})$$

The series in the first $\{ \}$ is just $[1 - \cos(\mu)]$, and the second series is $\mu - \sin(\mu)$. Hence we obtain

$$e^{\mu G} = \left\{ I + G^2 [1 - \cos(\mu)] + \mu G + G^3 [\mu - \sin(\mu)] \right\} \quad (\text{B.21})$$

This leads immediately to the result

$$e^{\mu G} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} (1 - \cos(\mu)) + \begin{pmatrix} 0 & k \\ -1/k & 0 \end{pmatrix} (\mu - \sin(\mu))$$

which reduces to

$$R = \begin{pmatrix} \cos(\mu) & k \sin(\mu) \\ -(1/k)\sin(\mu) & \cos(\mu) \end{pmatrix} \quad (\text{B.22})$$

If $k = 1$ then (B.22) represents a rotation in phase space.