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FOR THE
ZAKHAROV-SHABAT
EQUATIONS

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Variational Principle for the Zakharov-Shabat Equations

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ABSTRACT

We put forth a variational principle for the Zakharov-Shabat (ZS) equations, which is the basis of the inverse scattering transform of a number of important nonlinear PDE's. Using the variational representation of the ZS equations, we develop an approximate analytical technique for finding discrete eigenvalues of the complex spectral parameter in the ZS equations for a given pulse-shaped potential, which is equivalent to the physically important problem of finding soliton content of the given initial pulse. We apply the technique to several particular shapes of the pulse and demonstrate that the simplest version of the variational approximation, based on trial functions with one or two free parameters, proves to be fully analytically tractable, and it yields threshold conditions for appearance of the first soliton, or of the first soliton pair, which are in a fairly good agreement with available numerical results. However, more free parameters are necessary to allow prediction of additional solitons produced by the given pulse.

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1 INTRODUCTION

The inverse scattering transform (IST) is a powerful tool to solve a number of very important nonlinear PDE's [1]. As is well known, an equation amenable to solution by means of the IST must admit a representation in the form of a compatibility condition for two systems of auxiliary linear equations for the so-called Jost functions. One system has the form of a linear scattering problem. It maps a given configuration of the wave potential obeying the corresponding nonlinear PDE into the scattering data. The other system governs temporal evolution of the Jost functions, i.e., as a matter of fact, it produces a temporal evolution of the scattering data which is equivalent to the evolution of the wave potential governed by the nonlinear PDE. In the general case, the scattering data contain two components, continuous and discrete, which correspond, respectively, to the "radiation" (dispersive quasi-linear waves) and solitons. A typical problem, very important in many applications, is to find the soliton content of a given initial pulse, i.e., the discrete component of the scattering data corresponding to the given pulse. This problem is especially interesting from the viewpoint of the theory of solitons in nonlinear optical fibers, where the NLS equation, in spite of its simplicity, proves to be a very accurate model [2]. Although the problem of finding the discrete scattering data is essentially linear, its general analytical solution is not available. That is why development of both numerical and approximate analytical tools for scattering problem is of a substantial interest.

In this work, we will consider this problem for the Zakharov-Shabat (ZS) equations, which were formulated for the first time in the classical paper [3] and were employed there as an auxiliary linear system for application of the IST technique to the nonlinear Schrödinger (NLS) equation. Later [4], it has been found that the same system of linear

equations is a basis of a general formalism of the IST. That is why the ZS system is employed to construct the IST solutions of several other important nonlinear PDE's, e.g., of the self-induced transparency equations [5].

There are very few particular pulse-shaped initial configurations for which the ZS equations can be solved exactly. These include the rectangular box [6] and the sech hump [7]. The phase of the complex wave field in the above-mentioned pulses is allowed to be only a linear function of the coordinate. There is also a generalization of the sech pulse [8] when the power of sech is $1 + i\mu$, μ being an arbitrary real parameter. In all other cases, one should rely upon approximate methods, primarily numerical (see, e.g., Ref. [9] and references therein). Practically, the only attempt to develop approximate analytical methods to solve the scattering problem for the ZS equations was done in Ref. [10] by means of the WKB-like approximation.

In this work, we aim to show that the ZS equations have a natural variational representation, and to use this fact as a basis for development of an analytical approximation to finding the soliton content of a given pulse. It is very well known that the variational representation of the linear Schrödinger equation plays an important role in quantum mechanics as a basis for a powerful technique allowing to find energy spectra of many systems with a good accuracy [11]. Coming back to the NLS equation, it is relevant to mention that its variational (Lagrangian) representation was successfully used for obtaining physically meaningful approximate solutions [12], [13]. Later, a similar approach has proved to be useful for nonintegrable generalizations of the NLS equation, such as the equation with damping [14] or variable coefficients [15], and a system of two coupled equations describing a bimodal nonlinear optical fiber [16], [17], or a two-core nonlinear fiber (coupler) [18]. However, nobody, as far as we know, has ever tried to employ or

even formulate the variational principle for the ZS equations, which "underlie" the NLS equation from the viewpoint of the IST.

The paper is organized as follows. In section 2, we formulate the variational principle for the ZS equations. As it is known, each soliton contained in a given initial pulse is associated with a discrete eigenvalue of the so-called spectral parameter in the corresponding ZS equations. The eigenvalue is generally complex, its imaginary and real part giving, respectively, the amplitude and velocity of the corresponding soliton. We demonstrate that an expression for the spectral parameter in the form of a ratio of two quadratic integral functionals of the Jost functions is, simultaneously, the Lagrangian whose variation produces exactly the ZS equations. To use this variational representation for approximate analysis, one should, first of all, choose the so-called *ansatz*, i.e., a certain trial form of the Jost functions. The *ansatz* is to be constructed so that it contains some free parameters. After substitution of the chosen *ansatz* into the Lagrangian and performing the integration, one obtains an effective Lagrangian as a function of the free parameters. One should next find a set of values of those parameters at which the effective Lagrangian has an extremum. Finally, substituting the corresponding values back into the Lagrangian (which coincides with the eigenvalue), one obtains a variational value for the soliton's eigenvalue. If variation of the effective Lagrangian produces several extrema, one can predict the existence of several solitons.

In section 3, we demonstrate the simplest example of application of this technique, taking the box-shaped initial pulse and the *ansatz* based on linear Jost functions in the region of localization of the box. Actually, the *ansatz* has a single nontrivial variational parameter. This crude approximation produces not more than a single eigenvalue. The threshold for appearance of the soliton, i.e., the corresponding *minimum area* of the box,

predicted by this approximation, is $\frac{3}{2}$, while the exact solution [6] gives $\frac{\pi}{2}$. Thus, this simplest approximation yields quite an accurate result for the threshold, although it fails to predict appearance of additional solitons at larger values of the area. In section 3 we also apply the same ansatz to more sophisticated initial pulse in the form of the box with a phase jump in the middle of it. Although this generalization seems fairly simple, it is not fully exactly solvable (obviously, the exact solution would amount to solving a nasty system of transcendental equations when matching different parts of the Jost functions), and, as far as we know, it has never been considered before. In this case, application of the simplest linear ansatz for the Jost function produces an expression for the threshold area which is a certain function of the magnitude of the phase jump. Naturally, it takes a minimum value for the zero jump. Coming back to the pure box pulse (without the phase jump), one can also use the trial Jost functions in the form of cosines and sines inside the box. In this case, we obtain the exact result for the soliton content of the pulse. This is not surprising since the exact solution for the box [6] has exactly the same Jost functions.

In section 4, we consider in more detail the limiting case of the above-mentioned box with the phase jump equal π , i.e., a combination of two boxes of opposite signs, the total area of the initial pulse being thus zero. A solution of the corresponding scattering problem can be naturally reduced in this case to the transcendental matching equations. This problem was solved numerically in Ref. [19]. In section 4, we develop a variational approximation to find the eigenvalues of this pulse, the Jost function being approximated by a piece-wise linear ansatz (we admit a jump of derivatives at the point where the potential has the jump). This ansatz has two variational parameters. An advantage of this approximation is that it admits a fully analytical solution. When the amplitude of the initial pulse attains a threshold value, we predict appearance of a symmetric pair of

solitons. The solitons appear at the threshold with equal infinitesimal amplitudes and with nonzero opposite velocities. In the same section, we also demonstrate a possibility to use an alternative two-parametric ansatz, based on approximation of the Jost functions by quadratic polynomials. The results obtained by means of both approximations prove to be close to each other, and they are also found to be in a reasonable agreement with the numerical results of Ref. [19].

In section 5, we consider a Gaussian pulse (the argument of the Gaussian is complex, i.e., it contains a quadratic x -dependent term in the phase, which corresponds to the so-called chirp). The ZS scattering problem for the Gaussian was solved numerically in Ref. [20]. To apply the variational approximation, we also adopt the Gaussian ansatz for the Jost functions (the Gaussian ansatz was previously effectively used in direct applications of the variational approximation to the the NLS equation [12], [14], and to a system of coupled NLS equations [17]). This approximation again admits a fully analytical consideration, which produces a single soliton. Comparison with the numerical results obtained earlier for the Gaussian pulse [20] demonstrates that the critical area for appearance of the soliton is predicted by the variational approximation with a very good accuracy.

2 FORMULATION OF THE VARIATIONAL PRINCIPLE

The system of the ZS equations for the two-component complex Jost function $(\psi^{(1)}(x), \psi^{(2)}(x))$ has the form [3]

$$i\psi_x^{(1)} + \lambda\psi^{(1)} + u^*(x)\psi^{(2)} = 0, \quad (1)$$

$$i\psi_x^{(2)} - \lambda\psi^{(2)} + u(x)\psi^{(1)} = 0, \quad (2)$$

where $u(x)$ is the wave potential whose soliton content must be found, the asterisk stands for the complex conjugation, and λ is the spectral parameter taking values in the upper complex half-plane. Multiplying Eq. (1) by $\psi^{(2)}$ and integrating the result over dx , one can immediately obtain the following representation for the spectral parameter:

$$\lambda = -\frac{L}{N}, \quad (3)$$

where

$$L \equiv \int_{-\infty}^{+\infty} \left\{ \frac{i}{2} (\psi_x^{(1)}\psi^{(2)} - \psi_x^{(2)}\psi^{(1)}) + \frac{1}{2} [u^* (\psi^{(2)})^2 - u (\psi^{(1)})^2] \right\} dx, \quad (4)$$

$$N \equiv \int_{-\infty}^{+\infty} \psi^{(1)}\psi^{(2)} dx. \quad (5)$$

Now, one can immediately check that varying the expression (3) with respect to the functions $\psi^{(1)}$ and $\psi^{(2)}$ produces exactly the equations (1) and (2). Thus, Eqs. (3) through (5) give an effective Lagrangian for the ZS equations. It is noteworthy that the Lagrangian exactly coincides with the eigenvalue that must be sought for. A similar fact is well known in quantum mechanics [11], where the linear Schrödinger equation can be obtained by varying a functional which is nothing but the eigenvalue of energy. However, an essential difference from quantum mechanics is that the functional (3) is not real. This, of course, complies with the complex character of the eigenvalues in the ZS problem [3].

Despite the very simple derivation of the variational representation of the ZS equations, this representation, as far as we know, has never been considered before. Our aim in this work is to employ the variational representation for finding approximate values of the eigenvalues λ corresponding to the given potential $u(x)$; as it was mentioned above, each discrete eigenvalue gives rise to a soliton, its amplitude and velocity being proportional, respectively, to the imaginary and real part of λ .

The general approach to using the variational representation is based on postulating a certain *ansatz* (functional form) of the unknown Jost functions, which contains several free parameters. Next, one should insert the *ansatz* into the integrals (4) and (5), and eventually find the Lagrangian (3) as a function of the free parameters. Finally, one is looking for *critical points*, i.e., values of the parameters at which the Lagrangian has an extremum. If the underlying *ansatz* was chosen reasonably (which is the main point in any application of the variational approximation), the critical point may give rise to a soliton, whose amplitude and velocity are approximately given by the imaginary and real parts of the expression (3) taken at the critical point. In the following sections, this general idea will be applied to several different forms of the potential $u(x)$.

3 THE RECTANGULAR BOX

As the first example, we will consider the potential of the following form:

$$u(x) = \begin{cases} 0 & \text{at } |x| > 1, \\ A \exp(i\epsilon \operatorname{sgn} x) & \text{at } |x| < 1. \end{cases} \quad (6)$$

Here A is the real amplitude, which simultaneously defines the area $2A$ of the pulse (obviously, the width of the pulse can always be scaled to be 2, as it is implied in Eq. (6)), and 2ϵ is the phase jump at the central point of the box.

Now, we adopt the following *ansatz* for the Jost functions, which is, as a matter of fact, the simplest possible choice:

$$\psi^{(1)}(x) = \begin{cases} 2 \exp(-\mu(x-1)) & \text{at } x > 1, \\ x+1 & \text{at } |x| < 1, \\ 0 & \text{at } x < -1; \end{cases} \quad (7)$$

$$\psi^{(2)} = \begin{cases} 0 & \text{at } x > 1, \\ B(1-x) & \text{at } |x| < 1, \\ 2B \exp(\mu(x+1)) & \text{at } x < -1. \end{cases} \quad (8)$$

Note that, for a potential with a compact support, the components $\psi^{(1)}$ and $\psi^{(2)}$ of the Jost function must be exactly equal to zero, respectively, to the left and to the right of the support, to comply with the standard boundary conditions for the Jost functions at infinity [6] (in any case, the Jost functions corresponding to the discrete eigenvalues must vanish at infinity [3]). This property is taken into account when choosing the ansatz, as well as the obvious condition that the Jost functions must be continuous (similarly to the wave function in quantum mechanics [11]). In Eqs. (7) and (8) we have two free parameters, μ and B , although, as it immediately follows from Eqs. (4) and (5), the Lagrangian does not depend upon μ .

A straightforward calculation yields the values of the integrals (4) and (5) calculated with the trial functions (7) and (8):

$$L = 2iB + \frac{1}{6}A \left[B^2 (e^{-i\epsilon} + 7e^{i\epsilon}) - (7e^{i\epsilon} + e^{-i\epsilon}) \right], \quad N = \frac{4}{3}B. \quad (9)$$

Next, substitution of these expressions into Eq. (3) and varying in the only nontrivial parameter B yields the result that the critical points exist at

$$B = \pm i. \quad (10)$$

Inserting the critical values (10) back into Eq. (3), we obtain the following eigenvalues:

$$\text{Im}\lambda = -\frac{3}{2} \pm 2A \cos \epsilon, \quad (11)$$

$$\text{Re}\lambda = \mp \frac{3}{2}A \sin \epsilon, \quad (12)$$

the sign plus or minus being correlated with that in Eq. (10). Since only the eigenvalue with the positive imaginary part is meaningful [3], one should keep only the upper sign. Then, Eq. (11) tells us that, with the increase of the area $S \equiv 2A$ of the pulse (6), the soliton appears at the threshold value

$$S_{\text{thr}} = 3/(2 \cos \epsilon). \quad (13)$$

First of all, let us dwell on the case $\epsilon = 0$, when the ZS equations with the potential (6) have the known exact solution [6]. In this case, Eq. (13) yields the threshold area $\frac{3}{2}$, while the exact result is $\frac{\pi}{2}$ [6]. Thus, our crude approximation using the single variational parameter gives an error less than 5%. However, this approximation completely fails to predict existence of additional solitons which appear with the increase of the area, so that the total number of solitons produced by the rectangular box (without the phase jump) is [6] $[(2S - \pi)/2\pi] + 1$, [...] standing for the integer part. Actually, the ability of the ansatz based on Eqs. (7) and (8) to predict the single eigenvalue only is closely related to the presence of the single nontrivial variational parameter (B) in the ansatz.

Equation (13) demonstrates that S_{thr} monotonically increases with the increase of the phase jump 2ϵ . This seems to be quite a reasonable trend. However, Eq. (13) also predicts that S_{thr} diverges in the limit $2\epsilon = \pi$, when the potential (6) turns into the combination of two box potentials with opposite signs (Fig. 1). Evidently (see also Ref. [19]), the threshold for appearance of the solitons must remain finite in this case. In the next section, we will demonstrate that a finite threshold for this potential can be obtained analytically by means of an improved ansatz.

To conclude this section, let us mention that, to analyze the soliton content of the rectangular pulse, one can alternatively use an ansatz based on sines and cosines instead

of the linear functions in Eqs. (7) and (8). While in the general case of $\epsilon \neq 0$ this choice gives rise to rather cumbersome equations, in the particular case $\epsilon = 0$ it can be done explicitly, and yields, eventually, the exact solitonic spectrum corresponding to the box potential. This is not surprising since the exact solution of the ZS equations for this potential [6] is based on the same functions.

4 THE TWO-BOX POTENTIAL

As a next example of application of the variational approximation, in this section we aim to try an improved but still fully analytically tractable ansatz for the two-box potential shown in Fig. 1. Since this potential is discontinuous, it is natural to try a generalization of the linear ansatz of Eqs. (7) and (8), which allows discontinuities of the first derivatives of the Jost functions:

$$\psi^{(1)}(x) = \begin{cases} x + 1 & \text{at } -1 < x < 0, \\ 1 + \alpha x & \text{at } 0 < x < 1, \end{cases} \quad (14)$$

$$\psi^{(2)}(x) = \begin{cases} B(1 - x) & \text{at } 0 < x < 1, \\ B(1 - \alpha x) & \text{at } -1 < x < 0 \end{cases} \quad (15)$$

(at $|x| > 1$ the Jost functions are assumed to have the same form as in Eqs. (7) and (8), but the Lagrangian given by Eqs. (3) through (5) is insensitive to the structure of the Jost functions outside the box). This ansatz contains two variational parameters, B and α .

Substituting Eqs. (14) and (15) into Eqs. (4) and (5), one can find the values of the integrals L and N which determine the Lagrangian according to Eq. (3):

$$L = iB(1 + \alpha) - \frac{1}{6}A(1 + B^2)(2 + \alpha^2 + 3\alpha), \quad N = B(1 + \frac{1}{3}\alpha). \quad (16)$$

Insertion of these expressions into Eq. (3) and varying in B immediately reveal that, at the critical point, $B = \pm 1$ (the difference of these values from those in Eq. (10) is due to the extra factor i in front of the potential (6) at $\epsilon = \frac{\pi}{2}$), after which the eigenvalue playing the role of the Lagrangian takes the form

$$\lambda = (1 + \frac{1}{3}\alpha)^{-1} \left[-i(1 + \alpha) + \frac{1}{3}\tilde{A}(2 + \alpha^2 + 2\alpha) \right], \quad (17)$$

with $\tilde{A} \equiv BA$. Variation of the expression (17) in α leads to a quadratic equation which has the roots

$$\alpha = -3 \pm \sqrt{2 + (6/\tilde{A})}i \quad (18)$$

(the sign $\pm$ in Eq. (18) has nothing in common with $B = \pm 1$). At last, insertion of this value of α into the expression (17) yields the eventual results

$$\text{Im}\lambda = -3 \pm 6\sqrt{A} \left(A + \sqrt{A^2 + 9} \right)^{-1/2}, \quad (19)$$

$$\text{Re}\lambda = B \left[-3 + 2 \left(A^2 + A\sqrt{A^2 + 9} \right)^{1/2} \right]. \quad (20)$$

Recall that B in Eq. (20) is ± 1 , and the sign $\pm$ in Eq. (19), independent of the sign of B , is the same as in Eq. (18). As one sees from Eq. (19), only the upper sign may produce a necessary eigenvalue with the positive imaginary part. With increase of the amplitude A , the eigenvalue appears at the threshold value

$$A_{\text{thr}} = 3/2\sqrt{2}. \quad (21)$$

Actually, two different eigenvalues corresponding to $B = \pm 1$, and, thus, to nonzero opposite values of $\text{Re}\lambda$, simultaneously appear at the threshold. Obviously, they correspond to a symmetric pair of solitons with equal amplitudes and opposite velocities. The imaginary and real parts (19) and (20) of the eigenvalue are displayed, as functions of A , in Fig. 2.

One can also analyze the asymptotic behavior of the eigenvalue at $A \rightarrow \infty$. According to Eqs. (19) and (20), in this limit the imaginary part takes the constant value $(\sqrt{2}-1)3$, while the real part grows linearly. The asymptotically constant value of $\text{Im}\lambda$, as well as the failure of the present ansatz to produce more than one pair of the solitons, are the main drawbacks of the ansatz.

Qualitatively similar results for the two-box potential can be obtained on the basis of the different ansatz, which approximates the Jost functions by the quadratic polynomials instead of the piecewise linear functions in Eqs. (14) and (15) at $|x| < 1$:

$$\psi^{(1)}(x) = 1 + x + \alpha(1+x)^2, \quad (22)$$

$$\psi^{(2)}(x) = B [1 - x + \alpha(1-x)^2]. \quad (23)$$

Here B and α are the variational parameters. Calculations similar to those performed above again reveal that the parameter B takes the values ± 1 at the critical point, and then the integrals L and N can be obtained in the form (cf. Eqs. (15))

$$L = 2iB(1 + \frac{8}{3}\alpha + \frac{4}{3}\alpha^2) - \tilde{A}(2 + 7\alpha + 6\alpha^2), \quad N = \frac{4}{3}B(1 + 2\alpha + \frac{4}{5}\alpha^2), \quad (24)$$

where, as well as above, $\tilde{A} \equiv AB$. Subsequently varying the Lagrangian (3) in α , with L and N taken as per Eq. (24), leads to a quadratic equation for α . Substitution of the roots of this equation into the expression for the eigenvalue λ produces a rather cumbersome analytical expression, which we do not display here. Finally, the results for the imaginary and real parts of λ following from that expression prove to be quite similar to those displayed above the the piece-wise linear ansatz: at the threshold, there appears a symmetric pair of solitons, and no more pairs appear with the subsequent increase of the amplitude A . The plots of the corresponding imaginary and real parts of λ vs. A are shown in Fig. 3.

Comparing Figs. 2 and 3, one sees that the two different ansatze indeed produce very similar results. It is relevant to note that both ansatze are not capable of predicting the appearance of additional soliton pairs.

5 THE GAUSSIAN PULSE

Let us now consider a Gaussian shaped pulse with a chirp. We take

$$U = Ae^{-\sigma^2 x^2} e^{-i\beta x^2}, \quad (25)$$

where A is real, (β is the chirp parameter), and for trial functions

$$\psi^{(1)} = A_1 e^{-(W_1 x^2 + 2Z_1 x)}, \quad (26)$$

$$\psi^{(2)} = A_2 e^{-(W_2 x^2 + 2Z_2 x)}, \quad (27)$$

where A_1, A_2, W_1, W_2, Z_1 and Z_2 may be complex, provided that the real parts of W_1 and W_2 are positive. Then (3) and (4) yields

$$\lambda = -i(Z_1 - Z_2) - i(Z_1 + Z_2) \frac{(2i\beta + s_2 - s_1)}{2s_{12}} - \frac{i}{2} A \left[\left(\frac{s_{12}}{s_2} \right)^{1/2} \frac{A_2}{A_1} e^{x_1} + \left(\frac{s_{12}}{s_1} \right)^{1/2} \frac{A_1}{A_2} e^{x_2} \right], \quad (28)$$

where

$$s_1 = 2W_1 + \sigma^2 + i\beta, \quad (29)$$

$$s_2 = 2W_2 + \sigma^2 - i\beta, \quad (30)$$

$$s_{12} = W_1 + W_2, \quad (31)$$

$$\chi_1 = -(Z_1 + Z_2)^2 / s_{12} + 4Z_2^2 / s_2, \quad (32)$$

$$\chi_2 = -(Z_1 + Z_2)^2 / s_{12} + 4Z_1^2 / s_1. \quad (33)$$

$$s_+^2 - 2i\beta \left(\frac{\sigma^2}{s_-} \right) s_+ + (\sigma^4 - \beta^2 - s_-^2) = 0, \quad (43)$$

Varying (28) with respect to A_1 gives

$$(A_1/A_2)^2 = (s_1/s_2)^{\frac{1}{2}} e^{\chi_1 - \chi_2}. \quad (34)$$

Varying with respect to Z_1 and Z_2 allows one to solve for these variables, which gives

$$Z_1 = \frac{s_1}{4AR} (i\beta/\sigma^2 - 1), \quad (35)$$

$$Z_2 = \frac{s_2}{4AR} (i\beta/\sigma^2 + 1), \quad (36)$$

where

$$R^2 = e^{\chi} s_{12} / (s_1 s_2)^{\frac{1}{2}}, \quad (37)$$

$$\chi = \chi_1 + \chi_2. \quad (38)$$

At the extremum values of (35) and (36), χ and λ become

$$\chi = [2s_1 s_2 + 2i\beta(s_1 - s_2) + (\beta^2/\sigma^2 - \sigma^2)(s_1 + s_2)] / (4s_{12} A^2 R^2). \quad (39)$$

$$\lambda = -iAR + \frac{1}{2ARs_{12}} [s_1 s_2 + i\beta(s_1 - s_2) + \frac{1}{2}(s_1 + s_2)(\beta^2/\sigma^2 - \sigma^2)]. \quad (40)$$

What is left is to extremumize with respect to W_1 and W_2 . One finds a parametric solution of the following form. First, let σ and β be given (which are potential parameters in Eq. (25)). Then for each value of χ , solve the cubic equation,

$$r^3 + \beta r^2 + \sigma^2(2\chi^2 + 2\chi + 1)r - 2\beta\sigma^2(\chi^2 + \chi) = 0, \quad (41)$$

for the three roots of r . Then the value of $s_- = (s_1 - s_2)/2$ is

$$s_- = ir. \quad (42)$$

Next solve the quadratic equation

$$s_+^2 - 2i\beta\left(\frac{\sigma^2}{s_-}\right)s_+ + (\sigma^4 - \beta^2 - s_-^2) = 0, \quad (43)$$

for $s_+ = (s_1 + s_2)/2$. Then one has the possible values for s_1 and s_2 . Of these possible values, the only physically allowable roots are those which satisfy $\text{Re}(s_1) > 0$, $\text{Re}(s_2) > 0$ and (34). Then the value of R and λ follows from (32) and (35).

One now can easily plot the solution. We find that the physical range of χ is from 0 to 1, with $\chi = 1$ giving the threshold. The physical solution for λ is pure imaginary above the threshold. The results for $\sigma = 1$ and $\beta = 0, 10, 20, 30, 40, 50$ are shown in Fig. 4 where we have plotted the soliton amplitude vs. the absolute area in units of π . These curves are essentially the same as those obtained by numerical means in Ref.[20]. In Fig. 5, we show the threshold amplitude as a function of the chirp. Except for low chirps ($\beta < 2$), the curve is essentially linear.

6 CONCLUSIONS

In this work, we have demonstrated that the famous Zakharov-Shabat linear equations, which constitute the basis of the inverse scattering transform, admit a variational representation, which may be used to construct an effective approximate method for finding discrete eigenvalues of the scattering problem associated with the ZS equations. The method is based on presuming a reasonable functional ansatz for the Jost functions, which must contain several variational parameters. The eigenvalues generated by a given initial pulse directly give its soliton content, therefore this problem is very important for applications. We have given several explicit examples of how this approximation can be used to analyze the soliton content of physically interesting pulses. We have demonstrated that, in all the cases considered, the approximation allows to find the threshold for appearance of the first soliton or the first symmetric pair of solitons in a fully analytical

form, and comparison with the available numerical data (or with the exact solution in the case of the rectangular box) demonstrates that our approximate thresholds are very close to the exact ones. However, for being able to predict additional solitons generated by the given pulse, one should consider a more complicated ansatz with additional variational parameters.

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FIGURE CAPTIONS

Fig. 1. The two-box potential corresponding to Eq. (6) with $\epsilon = \frac{\pi}{2}$.

Fig. 2. The imaginary (solid line) and real (dashed line) parts of the eigenvalue λ (i.e., the amplitude and $\pm$ velocity of the pair of solitons produced by the two-box potential shown in Fig. 1 as predicted by the ansatz (14) and (15)) vs. the amplitude A (for $B = +1$).

Fig. 3. The same as in Fig. 2 as obtained from the ansatz (22) and (23).

Fig. 4. The imaginary part of the eigenvalue vs. the absolute area of the potential divided by π for the values of $\beta = 0, 10, 20, 30, 40, 50$.

Fig. 5. The threshold value of the absolute area divided by π vs. the chirp parameter β .

FIG. 1

FIG. 2

FIG. 3

FIG. 4

FIG. 5