THE ADLER-SHIOTA-VAN MOERBEKE FORMULA
FOR THE BKP HIERARCHY

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Abstract
We study the BKP hierarchy and prove the existence of an Adler–Shiota–van Moerbeke formula. This formula relates the action of the $BW_{1+\infty}$-algebra on tau-functions to the action of the "additional symmetries" on wave functions.

1. Introduction and main result
1.1. Adler, Shiota and van Moerbeke [ASV1-2] obtained for the KP and Toda lattice hierarchies a formula which translates the action of the vertex operator on tau-functions to an action of a vertex operator of pseudo-differential operators on wave functions. This relates the additional symmetries of the KP and Toda lattice hierarchy to the $W_{1+\infty}$, respectively $W_{1+\infty} \times W_{1+\infty}$-algebra symmetries. In this paper we investigate the existence of such an Adler–Shiota–van Moerbeke formula for the BKP hierarchy.

1.2. The BKP hierarchy is the set of deformation equations

$$\frac{\partial L}{\partial t_k} = [(L^n)_+, L], \quad k = 3, 5, \ldots$$

for the first order pseudo-differential operator

$$L \equiv L(x, t) = \partial + u_1(x, t)\partial^{-1} + u_2(x, t)\partial^{-2} + \cdots,$$

here $\partial = \frac{\partial}{\partial x}$ and $t = (t_1, \ldots)$. It is well-known that $L$ dresses as $L = P\partial P^{-1}$ with

$$P \equiv P(x, t) = 1 + a_1(x, t)\partial^{-1} + a_2(x, t)\partial^{-2} + \cdots = \frac{\tau(x - 2\partial^{-1}, t - 2[\partial^{-1}])}{\tau(x, t)},$$

where $\tau$ is the famous $\tau$-function, introduced by the Kyoto group [DJKM1-3] and $[\tau] = (\frac{3}{\alpha}, \frac{3}{\alpha}, \ldots)$.

The wave or Baker–Akhiezer function

$$w \equiv w(x, t, z) = W(x, t, \partial)e^{\xi},$$

where

$$W \equiv W(x, t, z) = P(x, t)e^{\xi(x, t, z)} \quad \text{with} \quad \xi(x, t, z) = \sum_{k=1}^{\infty} t_{2k+1}\partial^{2k+1}$$

* The research of Johan van de Leur is financially supported by the “Stichting Fundamenteel Onderzoek der Materie (F.O.M.)”. E-mail: vdeur@math.utwente.nl
is an eigenfunction of $L$, viz.,
\[ Lw = zw \quad \text{and} \quad \frac{\partial w}{\partial t_k} = (L^k)_+ w. \]

Now introduce, following Orlov and Schulman [OS], the pseudo-differential operator $M \equiv M(x,t) = W x W^{-1}$ which action on $w$ amounts to
\[ Mw = \frac{\partial w}{\partial z}. \]
then $[L, M] = 1$ and
\[ \frac{\partial M}{\partial t_k} = [(L^n)_+, M], \quad k = 3, 5, \ldots \]
Let
\[ Y(y, w) = \sum_{\ell=0}^{\infty} \frac{(y - w)^\ell}{\ell!} \sum_{k \in \mathbb{Z}} w^{-k-\ell-1}(M^\ell L^{k+\ell} - (-)^{k+\ell} L^{k+\ell-1} M^\ell L), \quad (1.1) \]
then one has the following main result.

**Theorem 1.1.**

\[ 2(w - y)Y(y, w)w(x, t, z) = (w + y)(e^{-\eta(x,t,z)} - 1) \left( \frac{X(y, w)\tau(x,t)}{\tau(x,t)} \right) w(x, t, z), \quad (1.2) \]

where $X(y, w)$ is the following vertex operator
\[ X(y, w) = w^{-1} \exp(x(y - w)) + \sum_{j > 2, \text{odd}} t_j (y^j - w^j) \exp(-2 \frac{\partial}{\partial x}(y^{-1} - w^{-1}) - 2 \sum_{j > 2, \text{odd}} \frac{\partial}{\partial t_j} \frac{y^{-j} - w^{-j}}{j}). \quad (1.3) \]

Formula (1.2) is the Adler–Shiota–van Moerbeke formula for the BKP hierarchy, we will give a proof of this formula in section 6. This formula relates the “additional symmetries” of the BKP hierarchy, generated by $Y(y, w)$, to the $BW_{1+\infty}$ algebra, generated by $X(y, w)$. This $BW_{1+\infty}$ algebra is a subalgebra of $W_{1+\infty}$, which is defined as the $-1$-eigenspace of an anti-involution on $W_{1+\infty}$. 

2
2. The Lie algebras $o_\infty$, $B_\infty$ and $BW_{1+\infty}$

2.1. Let $\mathfrak{gl}_\infty$ be the Lie algebra of complex infinite dimensional matrices such that all nonzero entries are within a finite distance from the main diagonal, i.e.,

$$\mathfrak{gl}_\infty = \{(a_{ij})_{i,j \in \mathbb{Z}} | a_{ij} = 0 \text{ if } |i-j| >> 0\}.$$

The elements $E_{ij}$, the matrix with the $(i,j)$-th entry 1 and 0 elsewhere, for $i,j \in \mathbb{Z}$ form a basis of a subalgebra $\mathfrak{gl}_\infty \subset \mathfrak{gl}_\infty$. The Lie algebra $\mathfrak{gl}_\infty$ has a universal central extension $\Lambda_\infty = \mathfrak{gl}_\infty \oplus \mathbb{C}e_A$ with the Lie bracket defined by

$$[a + \alpha e_A, b + \beta e_A] = ab - ba + \mu(a,b)e_A,$$

for $a,b \in \mathfrak{gl}_\infty$ and $\alpha, \beta \in \mathbb{C}$; here $\mu$ is the following 2-cocycle:

$$\mu(E_{ij}, E_{kl}) = \delta_{i,j} \delta_{k,l} (\theta(i) - \theta(j)),$$

where the function $\theta : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\theta(i) = \begin{cases} 0 & \text{if } i > 0, \\ 1 & \text{if } i \leq 0. \end{cases}$$

The Lie algebra $\mathfrak{gl}_\infty$ and $\bar{\mathfrak{gl}}_\infty$ both have a natural action on the space of column vectors, viz., let $C^\infty = \bigoplus_{k \in \mathbb{Z}} e_k$, then $E_{ij} e_k = \delta_{j,k} e_i$. By identifying $e_k$ with $t^{-k}$, we can embed the algebra $D$ of differential operators on the circle, with basis $-t^{i+j}(\frac{\partial}{\partial t})^k$ ($j \in \mathbb{Z}, k \in \mathbb{Z}_+$), in $\bar{\mathfrak{gl}}_\infty$:

$$\rho : D \rightarrow \bar{\mathfrak{gl}}_\infty,$$

$$\rho(-t^{i+j}(\frac{\partial}{\partial t})^k) = \sum_{m \in \mathbb{Z}} -m(m-1) \cdots (m-k+1)E_{-m-j,-m}.$$ 

(2.4)

It is straightforward to check that the 2-cocycle $\mu$ on $\mathfrak{gl}_\infty$ induces the following 2-cocycle on $D$:

$$\mu(-t^{i+j}(\frac{\partial}{\partial t})^i, -t^{k+l}(\frac{\partial}{\partial t})^l) = \delta_{i,k} \delta_{j,l} (\theta(i) - \theta(l)).$$

(2.5)

This cocycle was discovered by Kac and Peterson in [KP] (see also [R], [KR]). In this way we have defined a central extension of $D$, which we denote by $W_{1+\infty} = D \oplus \mathbb{C}e_A$, the Lie bracket on $W_{1+\infty}$ is given by

$$[-t^{i+j}(\frac{\partial}{\partial t})^j + \alpha e_A, -t^{k+l}(\frac{\partial}{\partial t})^l + \beta e_A] = \sum_{m=0}^{\max(i,j)} (-1)^{m} \binom{i+j}{m} \binom{k+l}{m} \binom{j}{m} \binom{i}{m} \binom{j+l+1}{m} \binom{i+j}{j+l+1} c_A.$$ 

(2.6)

Let $D = t \frac{\partial}{\partial t}$, then we can rewrite the elements $-t^{i+j}(\frac{\partial}{\partial t})^j$, viz.,

$$-t^{i+j}(\frac{\partial}{\partial t})^j = -t^j D(D - 1)(D - 2) \cdots (D - j + 1).$$

(2.7)

Then

$$\rho(t^j f(D)) = \sum_{j \in \mathbb{Z}} f(-j) E_{j-k,j},$$

(2.8)

and the 2-cocycle is as follows [KR]:

$$\mu(t^k f(D), t^l g(D)) = \begin{cases} \sum_{-k \leq j \leq -1} f(j)g(j+k) & \text{if } k = -\ell \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$

(2.9)
hence the bracket is
\[ [t^k f(D), t^\ell g(D)] = t^{k+\ell} f(D + \ell) g(D) - g(D + k) f(D)) + \rho(t^k f(D), t^\ell g(D)). \] (2.10)

2.2. Define on \( \overline{gl_\infty} \) the following linear anti-involution:
\[ \iota(E_{jk}) = (-)^{j+k} E_{-k,-j}. \] (2.11)

Using this anti-involution we define the Lie algebra \( \overline{o_\infty} \) as a subalgebra of \( \overline{gl_\infty} \):
\[ \overline{o_\infty} = \{ a \in \overline{gl_\infty} | \iota(a) = -a \}. \] (2.12)

The elements \( F_{jk} = E_{-j,k} - (-)^{j+k} E_{-k,j} \) with \( j < k \) form a basis of \( \overline{o_\infty} = \overline{o_\infty} \cap \overline{gl_\infty} \). The 2-cocycle \( \mu \) on \( \overline{gl_\infty} \) induces a 2-cocycle on \( \overline{o_\infty} \), and hence we can define a central extension \( B_\infty = \overline{o_\infty} \oplus \mathbb{C} c_B \) of \( \overline{o_\infty} \), with Lie bracket
\[ [a + \alpha c_B, b + \beta c_B] = ab - ba + \frac{1}{2} \mu(a, b)c_B, \] (2.13)

for \( a, b \in \overline{o_\infty} \) and \( \alpha, \beta \in \mathbb{C} \). It is then straightforward to check that the anti-involution \( \iota \) induces
\[ \iota(t) = -t, \quad \iota(D) = -D. \] (2.14)

Hence, it induces the following anti-involution on \( D \):
\[ \iota(t^k f(D)) = f(-D)(-t)^k. \] (2.15)

Define \( D^B = D \cap \overline{o_\infty} = \{ w \in D | \iota(w) = -w \} \), it is spanned by the elements
\[ W_k(f) := t^k f(D) - f(-D)(-t)^k = t^k(f(D) - (-)^k f(-D - k)). \] (2.16)

It is straightforward to check that
\[ \rho(W_k(f)) = \sum_{j \in \mathbb{Z}} f(-j) F_{k-j,j}. \]

The restriction of the 2-cocycle \( \mu \) on \( D \), given by (2.5) or (2.9), induces a 2-cocycle on \( D^B \), which we shall not calculate explicitly here. It defines a central extension \( BW_1 = D^B \oplus \mathbb{C} c_B \) of \( D^B \), with Lie bracket
\[ [a + \alpha c_B, b + \beta c_B] = ab - ba + \frac{1}{2} \mu(a, b)c_B, \]

for \( a, b \in D^B \) and \( \alpha, \beta \in \mathbb{C} \).
3. The spin module

3.1. We now want to consider highest weight representations of $o_\infty$, $B_\infty$ and $BW_{1+\infty}$. For this purpose we introduce the Clifford algebra BCI as the associative algebra on the generators $\phi_j$, $j \in \mathbb{Z}$, called neutral free fermions, with defining relations

$$\phi_i \phi_j + \phi_j \phi_i = (-)^j \delta_{i,-j}. \quad (3.1)$$

We define the spin module $V$ over BCI as the irreducible module with highest weight vector the vacuum vector $|0\rangle$ satisfying

$$\phi_j |0\rangle = 0 \quad \text{for } j > 0. \quad (3.2)$$

The elements $\phi_{j_1} \phi_{j_2} \cdots \phi_{j_p} |0\rangle$ with $j_1 < j_2 < \cdots < j_p \leq 0$ form a basis of $V$. Then

$$\pi(F_{jk}) = \frac{(-)^j}{2} (\phi_j \phi_k - \phi_k \phi_j),$$

$$\tilde{\pi}(F_{jk}) = (-)^j : \phi_j \phi_k :,$$

$$\tilde{\pi}(c_B) = I, \quad (3.3)$$

where the normal ordered product $:\quad :$ is defined as follows

$$: \phi_j \phi_k : = \begin{cases} \phi_j \phi_k & \text{if } k > j, \\ \frac{1}{2} (\phi_j \phi_k - \phi_k \phi_j) & \text{if } j = k, \\ -\phi_k \phi_j & \text{if } k < j, \end{cases} \quad (3.4)$$

define representations of $o_\infty$, respectively $B_\infty$.

When restricted to $o_\infty$ and $B_\infty$, the spin module $V$ breaks into the direct sum of two irreducible modules. To describe this decomposition we define a $\mathbb{Z}_2$-gradation on $V$ by introducing a chirality operator $\chi$ satisfying $\chi |0\rangle = |0\rangle$, $\chi \phi_j + \phi_j \chi = 0$ for all $j \in \mathbb{Z}$, then

$$V = \bigoplus_{\alpha \in \mathbb{Z}_2} V_\alpha \quad \text{where } V_\alpha = \{ v \in V | \chi v = (-)^\alpha v \}.\)$$

Each module $V_\alpha$ is an irreducible highest weight module with highest weight vector $|0\rangle$, $|1\rangle = \sqrt{2} |0\rangle$ for $V_0$, $V_1$, respectively, in the sense that $\tilde{\pi}(c_B) = 1$ and

$$\pi(F_{-i,j})|\alpha\rangle = \pi(F_{-i,j})|\alpha\rangle = 0 \quad \text{for } i < j,$$

$$\pi(F_{-i,j})|\alpha\rangle = (-)^j \frac{1}{2} |\alpha\rangle \quad \text{for } i > 0,$$

$$\tilde{\pi}(F_{-i,j})|\alpha\rangle = 0. \quad (3.5)$$

Clearly $V_\alpha$ is also a highest weight module for $BW_{1+\infty}$, viz

$$\tilde{\pi} \cdot \rho(W_k(f)) = \sum_{j \in \mathbb{Z}} (-)^{k+j} f(-j) : \phi_{k-j} \phi_j :,$$

$$\tilde{\pi} \cdot \rho(W_k(f))|\alpha\rangle = 0 \quad \text{for } k \geq 0. \quad (3.6)$$

From now on we will omit $o_\infty$, $\tilde{\pi}$ and $\tilde{\pi} \cdot \rho$, whenever no confusion can arise.
4. Vertex operators

4.1. Using the boson fermion correspondence (see e.g. [DJKM 3], [K], [tKL] and [Y]), we can express the fermions in terms of differential operators, i.e. there exists an isomorphism \( \sigma : V \rightarrow \mathbb{C}[\theta, t_1, t_2, \cdots] \), where \( \theta^2 = 0, t_it_j = t_jt_i, \theta t_j = t_j \theta \) and \( V_{\alpha} = \theta^\alpha \mathbb{C}[t_1, t_2, \cdots] \), such that \( \sigma(|0\rangle) = 1 \). Define the following two generating series (fermionic fields):

\[
\phi^\pm(z) = \sum_{j \in \mathbb{Z}} \phi^\pm_j z^{-j} = \sum_{j \in \mathbb{Z}} (\pm)^j \phi^\pm_j z^{-j},
\]

then one has the following vertex operator for these fields:

\[
\sigma \phi^\pm(z) \sigma^{-1} = \frac{\theta + \frac{\partial}{\partial \theta}}{\sqrt{2}} \exp(\pm \sum_{j > 0, \text{odd}} t_j z^j) \exp(\mp 2 \sum_{j > 0, \text{odd}} \frac{\partial}{\partial t_j} z^{-j}).
\]

4.2. Define

\[
W(y, w) = \sum_{\ell = 0}^{\infty} \frac{(y-w)^\ell}{\ell!} W^{(\ell+1)}(w)
\]

\[
= \sum_{\ell = 0}^{\infty} \frac{(y-w)^\ell}{\ell!} \sum_{k \in \mathbb{Z}} W_k^{(\ell+1)} w^{-k-\ell-1}
\]

\[
= \frac{\phi^+(y) \phi^-(w)}{w}.
\]

then

\[
W^{(\ell+1)}(z) = \frac{\partial^\ell \phi^+(z) \phi^-(z)}{\partial z^\ell} : \frac{1}{z} :
\]

and

\[
W_k^{(\ell+1)} = W_k(-\ell \left( \begin{array}{c} \ell \\ \ell \end{array} \right) )
\]

\[
= -\ell^k D(D-1) \cdots (D-\ell+1) + (\ell-1) \cdots (\ell-\ell+1) (-1)^k
\]

\[
= -\ell^k \ell! \frac{\partial}{\partial \ell} + (-1)^k \ell! \frac{\partial}{\partial \ell} \ell^k \ell!.
\]

Using (4.2), we find that for \(|w| < |y|

\[
W(y, w) = \frac{1}{2} \frac{y+w}{y-w} (X(y, w) - w^{-1}),
\]

where \(X(y, w)\) is the vertex operator defined in (1.3). Hence,

\[
W^{(\ell)}(z) = \frac{w \partial^\ell X(y, z)}{\partial z^\ell} \bigg|_{y=x} + \frac{1}{2} \frac{\partial^{\ell-1} X(y, z) - z^{-1}}{\partial z^{\ell-1}} \bigg|_{y=x}.
\]

Define

\[
\alpha_j(z) = \begin{cases} 
\frac{1}{2} \phi^j \exp(\lambda) & \text{if } j = -1, \\
\frac{1}{2} \phi^j \exp(\lambda) & \text{if } j < 2 \text{ odd}, \\
\frac{1}{2} \phi^j \exp(\lambda) & \text{if } j = 1, \\
\phi^j \exp(\lambda) & \text{if } j > 2 \text{ odd}, 
\end{cases}
\]

and their generating series by

\[
\alpha(z) = \sum_{j \in \mathbb{Z}} \alpha_j z^{-j-1},
\]

then \([\alpha_j, \alpha_k] = \frac{1}{2} \delta_{j+k, -1}. Since \ X(z, z) = z^{-1}, one finds the following expression for \(W^{(\ell)}(z)\):

\[
W^{(\ell)}(z) = \frac{2}{\ell} : (2\alpha(z) + \frac{\partial}{\partial z})^n \alpha(z) : + \frac{1}{z} : (2\alpha(z) + \frac{\partial}{\partial z})^{n-2} \alpha(z) :.
\]
For \( \ell = 1, 2, 3 \) one finds respectively
\[
\begin{align*}
W^{(1)}(z) &= 2\alpha(z), \\
W^{(2)}(z) &= 2 : \alpha(z)^2 : + \frac{\partial \alpha(z)}{\partial z} + \frac{\alpha(z)}{z}, \\
W^{(3)}(z) &= \frac{8}{3} \alpha(z)^3 : + \frac{8}{3} \alpha(z) \frac{\partial \alpha(z)}{\partial z} : + \frac{2}{z} \alpha(z)^2 : + \frac{\partial \alpha(z)}{\partial z} + \frac{2}{3} \frac{\partial^2 \alpha(z)}{\partial z^2}.
\end{align*}
\]

5. The BKP hierarchy
5.1. The BKP hierarchy is the following equation for \( \tau = \tau(t_1, t_2, \ldots) \) (see e.g. [DJM3], [K], [L2], [Y]):
\[
\text{Res}_{z=0} \frac{dz}{z} \phi^+(z) \tau \odot \phi^-(z) \tau = \frac{1}{2} \theta \tau \odot \theta \tau. \tag{5.1}
\]

Here \( \text{Res}_{z=0} dz \sum_j f_j z^j = f_{-1} \). We assume that \( \tau \) is any solution of (5.1), so we no longer assume that \( \tau \) is a polynomial in \( t_1, t_2, \ldots \).

We proceed now to rewrite (3.1) in terms of formal pseudo-differential operators. We start by multiplying (5.1) from the left with \( \frac{\partial}{\partial \theta} \odot \frac{\partial}{\partial \theta} \) and divide both the first and the last component of the tensor product by \( \tau(t) \). Let \( x = t_1 \) and \( \theta = \frac{\partial}{\partial t} \), then (5.1) is equivalent to the following bilinear identity:
\[
\text{Res}_{z=0} \frac{dz}{z} w(x, t, z) w(x', t', -z) = 1, \tag{5.2}
\]
where
\[
w(x, t, \pm z) = W(x, t, \pm z) e^{\xi z} = W(x, t, \partial) e^{\xi z} \quad \text{with}
\]
\[
W(x, t, z) = P(x, t, 0) e^{\xi(t, z)} = \sum_{i \geq 3} t_i z^i 
\]
and
\[
P(x, t, z) = \frac{e^{-\eta(x, t, z)} \tau(x, t)}{\tau(x, t)} = \frac{\tau(x - \frac{\partial}{\partial x}, t_3 - \frac{2}{3} t_5 - \frac{2}{15} t_7 - \cdots)}{\tau(x, t)} - \overline{\tau(x, t)}.
\tag{5.4}
\]

where \( \eta(x, t, z) = 2(\frac{\partial}{\partial z} e^{-z} - \sum_{j > 2} \frac{\partial^j}{\partial z^j} e^{-z}) \), for convenience we also define \( \xi(x, t, z) = \xi(t, z) e^{\xi z} \).

5.2. As usual one denotes the differential part of a pseudo-differential operator \( P = \sum_j P_j \partial^{-j} \) by \( P_{\pm} = \sum_{j \geq 0} P_j \partial^{-j} \) and writes \( P_{\pm} = P - P_{+} \). The anti-involution \( * \) is defined as follows \( (\sum_j P_j \partial^{-j} )^* = \sum_{j} (-\partial)^{-j} P_j \). One has the following fundamental lemma.

**Lemma 5.1.** Let \( P(x, t, \partial) \) and \( Q(x, t, \partial) \) be two formal pseudo-differential operators, then
\[
(P(x, t, \partial) Q(x, t', \partial)) = \pm \sum_{i > 0} R_i(x, t, t') \partial^{-i}
\]
if and only if
\[
\text{Res}_{z=0} dz P(x, t, \partial) e^{\pm \xi z} Q(x', t', \partial) e^{\mp \xi' z} = \sum_{i > 0} R_i(x, t, t') \frac{(x - x')^{-i}}{(i - 1)!}. \tag{5.5}
\]

The proof of this lemma is analogous to the proof of Lemma 4.1 of [L1] (see also [KL]).

5.3. Now differentiate (5.2) to \( t_k \), where we assume that \( x = t_1 \), then we obtain
\[
\text{Res}_{z=0} \frac{dz}{z} \frac{\partial P(x, t, z)}{\partial t_k} + P(x, t, z) z^k e^{\xi(x, t, z)} P(x', t', -z) e^{-\xi(x', t', z)} = 0. \tag{5.5}
\]
Now using lemma 5.1 we deduce that

\[
\left( \frac{\partial P}{\partial t_k} + P \frac{\partial k}{\partial P} \right) \frac{1}{P} = 0.
\]

From the case \( k = 1 \) we then deduce that \( P^* = \partial P^{-1} \partial^{-1} \), if \( k \neq 1 \), one thus obtains

\[
\frac{\partial P}{\partial x_k} = -\left( \frac{P \partial k P^{-1} \partial^{-1}}{P} \right) \frac{1}{P}.
\]  

(5.6)

Since \( k \) is odd, \( \partial^{-1} (P \partial k P^{-1})^* \partial = -P \partial k P^{-1} \partial^{-1} \) and hence \( (P \partial k P^{-1} \partial^{-1}) \partial = (P \partial k P^{-1}) \partial \). So (5.6) turns into Sato’s equation:

\[
\frac{\partial P}{\partial t_k} = -\left( \frac{P \partial k P^{-1}}{P} \right) \frac{1}{P}.
\]  

(5.7)

5.4. Define the operators

\[
L = W \partial W^{-1} = P \partial P^{-1}, \quad \Gamma = x + \sum_{j \geq 2} j t_j \partial^{j-1},
\]

\[
M = W x W^{-1} = P \Gamma P^{-1} \quad \text{and} \quad N = ML.
\]  

(5.8)

Then \([L, M] = 1\) and \([L, N] = L\). Let \( B_k = (L^k)_+ \), using (5.7) one deduces the following Lax equations:

\[
\frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial M}{\partial t_k} = [B_k, M] \quad \text{and} \quad \frac{\partial N}{\partial t_k} = [B_k, N].
\]  

(5.9)

The first equation of (5.9) is equivalent to the following Zakharov Shabat equation:

\[
\frac{\partial B_j}{\partial t_k} - \frac{\partial B_k}{\partial t_j} = [B_k, B_j],
\]

(5.10)

which are the compatibility conditions of the following linear problem for \( w = w(x, t, z) \):

\[
L w = z w, \quad M w = \frac{\partial w}{\partial z} \quad \text{and} \quad \frac{\partial w}{\partial t_k} = B_k w.
\]  

(5.11)

5.5. The formal adjoint of the wave function \( w \) is (see [DJKM]):

\[
w^* = w^*(x, z) = P^* e^{-\xi(x, t, z)} = \partial P^{-1} e^{-\xi(x, t, z)}.
\]  

(5.12)

Now \( L^* = -\partial L \partial^{-1} = -\partial P \partial P^{-1} \partial^{-1} \) and \( M^* = \partial \Gamma \partial P^{-1} \partial P^{-1} \partial^{-1} \), so \([L^*, M^*] = -1\) and

\[
L^* w^* = z w^*, \quad M^* w^* = -\frac{\partial w^*}{\partial z} \quad \text{and} \quad \frac{\partial w^*}{\partial x_k} = -(L^k)_+ w^* = -B_k^* w^*.
\]  

(5.13)

Finally, notice that by differentiating the bilinear identity (5.2) to \( x'_1 \) we obtain

\[
\text{Res}_{x=0} dz w(x, t, z) w^*(x', t, z) = 0.
\]  

(5.14)
6. Proof of Theorem 1

6.1. In this section we prove Theorem 1.1. We start from the bilinear identity (5.14) and multiply it by \( \tau(x, t) \), which gives

\[
\text{Res}_{z=0} dz e^{-\eta(x, t, z)} \tau(x, t) e^{\xi(x, t, z)} \frac{\partial}{\partial x'} \left( \frac{e^{\eta(x', t') \tau(x', t')}}{\tau(x', t')} e^{-\xi(x', t', z)} \right) = 0. \tag{6.1}
\]

Now let \((1 - w/y)^{-1} (1 + w/y) Y(y, w)\) act on this identity, then one obtains

\[
\text{Res}_{z=0} \frac{dz}{w} \left( \frac{1 + w/y}{1 - w/y} \right) e^{-\eta(x, t, z)} \tau(x, t) e^{\xi(x, t, z)} \frac{\partial}{\partial x'} \left( \frac{e^{\eta(x', t') \tau(x', t')}}{\tau(x', t')} e^{-\xi(x', t', z)} \right) = 0. \tag{6.2}
\]

Next use the fact that \((1 - u)^{-1} (1 + u) = 2 \delta(u, 1) - (1 - u^{-1})^{-1}(1 + u^{-1})\), where \(\delta(u, v) = \sum_{j \in \mathbb{Z}} u^{-j} v^{j-1}\), then (6.2) is equivalent to

\[
- \text{Res}_{z=0} \frac{dz}{w} \left( \frac{1 + w/y}{1 - w/y} \right) e^{-\eta(x, t, z)} \tau(x, t) e^{\xi(x, t, z)} \frac{\partial}{\partial x'} \left( \frac{e^{\eta(x', t') \tau(x', t')}}{\tau(x', t')} e^{-\xi(x', t', z)} \right) = 2 \delta(x, t) \tau(x, t) e^{\xi(x, t, z)} \frac{\partial}{\partial x'} \left( \frac{e^{\eta(x', t', w) \tau(x', t')} e^{-\xi(x', t', z)}}{\tau(x', t')} \right).
\]

Divide this formula by \( \tau(x, t) \), then it turns into

\[
- \text{Res}_{z=0} \frac{dz}{x} e^{-\eta(x, t, z)} \left( \frac{1 + w/y}{1 - w/y} \right) X(y, w) \tau(x, t) e^{\xi(x, t, z)} \frac{\partial}{\partial x'} \left( \frac{e^{\eta(x', t') \tau(x', t')}}{\tau(x', t')} e^{-\xi(x', t', z)} \right) = 2 \delta(z, x) \frac{\partial}{\partial x'} \left( \frac{e^{\eta(x', t', w) \tau(x', t')} e^{-\xi(x', t', z)}}{\tau(x', t')} \right).
\]

Now define

\[
\sum_{j=0}^{\infty} \epsilon_j(x, t, y, w) z^{-j} = e^{-\eta(x, t, z)} \left( \frac{1 + w/y}{1 - w/y} \right) X(y, w) \tau(x, t), \tag{6.5}
\]

then the first line of (6.4) is equal to

\[
- \text{Res}_{z=0} dz \sum_{j=0}^{\infty} \epsilon_j(x, t, y, w) L^{-j-1}(W(x, t, z) \epsilon^{x_j}) \frac{\partial}{\partial x'} (W(x', t', z) \epsilon^{-x_j}).
\]
Now using Lemma 5.1 with \( t = t' \), one deduces that

\[
\frac{1}{2} \sum_{j=1}^{\infty} e_j(x, t, y, \omega) L^{-j} = \\
- \sum_{k=0}^{\infty} \frac{(y - \omega)^k}{k!} \sum_{k \in \mathbb{Z}} w^{-k-\ell-1}(W(x, t, z)x^\ell \partial^{k+\ell} W(x, t, z) - (-)^{k+\ell} \partial^{k+\ell-1} x^\ell \partial W(x, t, z)^{-1}).
\]

(6.6)

So finally one has

\[
\frac{1}{2} (e^{-\eta(x, t, z)} - 1) \left( \frac{X(y, y) \tau(x, t)}{\tau(x, t)} \right) w(x, t, z) = \\
- \sum_{k=0}^{\infty} \frac{(y - \omega)^k}{k!} \sum_{k \in \mathbb{Z}} w^{-k-\ell-1}(M^\ell L^k + (-)^{k+\ell} M^\ell L_{k+\ell-1} - W(x, t, z) =
\]

which is equal to the Adler–Shiota–van Moerbeke formula (1.2) for the BKP case:

\[
(w + y)(e^{-\eta(x, t, z)} - 1) \left( \frac{X(y, y) \tau(x, t)}{\tau(x, t)} \right) w(x, t, z) = 2(w - y)Y(y, w) w(x, t, z).
\]

6.2. Since the left-hand-side of (1.2) is also equal to

\[
(w + y)(e^{-\eta(x, t, z)} - 1) \left( \frac{X(y, y) \tau(x, t)}{\tau(x, t)} \right) w(x, t, z),
\]

we have the following corollary of Theorem 1.1:

**Corollary 6.1.** For \( k \in \mathbb{Z} \) and \( f \) some polynomial one has

\[
\frac{\sigma \cdot \tau \cdot \rho(W_k(f)) \tau(x, t)}{\tau(x, t)} = \frac{(f(N)L^k - (-L)^k f(-N))_\omega w(x, t, z)}{w(x, t, z)}.
\]

(6.7)

**References**


[L2] J. van de Leur, The $n$-th reduced BKP hierarchy, the string equation and $BW_{1+\infty}$-constraints. hep-th 9411067.


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