Comment on the Black-Scholes pricing problem

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In a recent paper [1] Bouchaud and Sornette presented an interesting analysis of the Black-Scholes derivative pricing model. They made an attempt to rederive the model in terms more familiar to workers in statistical physics and to extend it to the cases in which the price of the stock (or any other security) underlying the given derivative is a random process distributed differently from the log-normal law generally accepted in mathematical finance [2].

Here I would like to point that the simple rederivation of the Black-Scholes model given in [1] overlooks a major achievement of that theory: its ability to price the derivatives without making assumptions about the *average return* on the stock. All arguments of [1] assume that both parameters characterizing the log-normal distribution of the stock price, the volatility (i. e. the diffusion coefficient of the logarithm of the price) *and* the average return (the drift velocity of the logarithm of the price), are known; the average return is taken to be equal to the risk-free interest rate. The authors then define the price of the derivative by calculating its average over the given distribution of the stock price at the expiration of the derivative, and then proceed to recover the Black-Scholes risk-free portfolio (combination of selling the derivative with buying a proper number of the shares) as the "optimal portfolio" which minimizes the risk to the seller of the derivative.

This presentation of the Black-Scholes theory inverts its logic. Pricing a derivative via calculating its expectation with respect to the imaginary log-normal distribution of the stock price, in which the average return is set equal to the risk-free interest rate, is in fact well known in mathematical finance under the name of "risk-neutral valuation" [2]. Its possibility however is a consequence of the existence of the risk-free portfolio constructed by Black and Scholes.

A crucial element of the real market, incorporated into modern mathematical finance, is the trade-off between risk and return. Different securities are generally characterized by very different average returns and risks; higher returns are expected from risky securities (risk bears its price for the seller). While the return and the volatility are positively correlated, no quantitative relationship between them can be established without making additional assumptions [3], except that a risk-free portfolio (combination of securities) must return the risk-free rate.

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Empirically, extrapolation of volatility from the past data into the future is much more reliable than that of the average return.

By construction, in the Black-Scholes risk-free portfolio both the noise *and* the average drift are completely compensated. Consequently the Green's function for the linear partial differential equation describing the time evolution of the price of the derivative does not contain the average return parameter. Convolution of the boundary condition, representing the known value of the derivative at maturity, with this Green's function turns out to be equivalent to the risk-neutral valuation. Note that if the average return on the stock were known and the expectation value of the price of the derivative were calculated with the real distribution of the future price of the stock, the method used by the authors would lead to an error (i. e. arbitrage would be possible): the correct price of the derivative is *not* equal to the current expectation of its value at expiration. Correspondingly, the results obtained in [1] for models with time-correlated fluctuations of the stock price, in which case constructing a riskless portfolio turns out to be impossible, apply only to the imaginary risk-neutral world, but not to the real, risk-averse market. Also, in view of this discussion, calling the Black-Scholes portfolio "optimal" seems misleading (*fully hedged* would be more appropriate): zero risk is optimal only under condition of equal returns; in the market in which taking higher risks generally leads to higher returns absolute optimization is impossible.

In summary, applicability of the results obtained in [1] is restricted to the imaginary risk-neutral market; the risk-return trade-off underlying the Black-Scholes analysis requires more elaborate treatment of the problem.

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References

- [1] Bouchaud J.-P. and Sornette D., *J. Phys. I France* 4 (1994) 863.
- [2] Hull J. C., *Options, Futures, and other derivative securities* (Prentice-Hall, Englewood Cliffs, N. J., 1993).
- [3] The *capital asset pricing model* [2] yields linear relationship between the *excess* average return of a security over that of the market as a whole and the susceptibility of the fluctuations in the price of the security to those of the market (known as *beta* of the security). It is very hard to predict the return of the market, however.