The Electroweak Phase Transition: A non-perturbative Lattice Investigation *

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We present results obtained from a numerical investigation of the electroweak phase transition in the SU(2)-Higgs model. The simulations are performed at two values of the Higgs boson mass, $M_H \approx 20$ GeV and $M_H \approx 50$ GeV. While the phase transition is of strongly first order at the smaller value of the Higgs mass it weakens rapidly when the Higgs mass is increased. This is in qualitative agreement with perturbation theory as the comparison of various physical quantities shows.

1. Introduction

The number of baryons is not conserved in the minimal standard model [1]. Also CP is not an exact symmetry. If, in addition, the cosmological finite temperature *electroweak phase transition* would be of strong enough first order, all three of Sakharov's conditions for a generation of an asymmetry between baryons and antibaryons would be fulfilled. This offers the intriguing possibility that the observed baryon asymmetry of the universe can be explained within the minimal standard model alone [2]. The resolution of this question is therefore a major challenge for elementary particle physics.

The standard calculational method for the study of the symmetry restoring electroweak phase transition is resummed perturbation theory [3–6]. While in the Higgs phase perturbation theory is expected to work well for Higgs boson masses smaller than $M_H \sim 80$ GeV, in the symmetric phase irreparable infrared singularities oc-

cur which prevent a quantitative control of graph resummation [7]. Indeed, the results of perturbation theory show bad convergence for physical quantities depending on characteristics of both phases.

One non-perturbative approach for the study of the electroweak phase transition, which we will adopt, is based on numerical simulations. In the work described below, fermions and the U(1) gauge fields are omitted, which can be expected on general grounds to be a reasonable first approximation. One is therefore left with the SU(2)-Higgs model. We will stay in the original four-dimensional theory without dimensional reduction. This has the advantage of keeping the number of bare parameters small and not introducing any further approximations beyond the lattice regularization. The results presented in these talks have been published recently [8–10]. There a much more detailed discussion and a more extended list of references can be found. We restrict the present calculations to smaller Higgs boson masses below 50 GeV. Since this region of parameters of the minimal standard model is already excluded by experiments, our present scope is merely theoretical because we would like to check

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the validity of some other theoretical approximation schemes, e.g. resummed perturbation theory.

The lattice action of the SU(2) Higgs model is conventionally written as

$$\begin{aligned}
S[U, \varphi] = & \beta \sum_{pl} \left(1 - \frac{1}{2} \text{Tr } U_{pl} \right) \\
& + \sum_x \left\{ \frac{1}{2} \text{Tr} (\varphi_x^\dagger \varphi_x) + \lambda \left[\frac{1}{2} \text{Tr} (\varphi_x^\dagger \varphi_x) - 1 \right]^2 \right. \\
& \left. - \kappa \sum_{\mu=1}^4 \text{Tr} (\varphi_{x+\hat{\mu}}^\dagger U_{x\mu} \varphi_x) \right\}. \quad (1)
\end{aligned}$$

Here $U_{x\mu}$ denotes the SU(2) gauge link variable, U_{pl} is the product of four U 's around a plaquette and φ_x is a complex $2 \otimes 2$ matrix in isospin space describing the Higgs scalar field and satisfying $\varphi_x^\dagger = \tau_2 \varphi_x^T \tau_2$. The bare parameters in the action are $\beta \equiv 4/g^2$ for the gauge coupling, λ for the scalar quartic coupling and κ for the scalar hopping parameter related to the bare mass square μ_0^2 by $\mu_0^2 = (1 - 2\lambda)\kappa^{-1} - 8$. Throughout this paper we set the lattice spacing to one ($a = 1$), therefore all the masses and correlation lengths etc. will always be given in lattice units, unless otherwise stated.

Since we are interested in the study of the symmetry restoring phase transition as a function of temperature, we use asymmetric lattices: the small temporal extensions $L_t = 2, 3, \dots$ represent the discretized inverse temperature $L_t = 1/(aT)$. The other three (spatial) extensions of the lattice have to be much larger, for reaching the thermodynamical limit. As stated before, we will stay at small Higgs boson masses. Hence for the bare quartic coupling we have chosen values near $\lambda = 0.0001$ (corresponding to $M_H \sim 18$ GeV) and $\lambda = 0.0005$ (corresponding to $M_H \sim 50$ GeV). In the present paper the inverse temperature in lattice units will be restricted to $L_t = 2$ and $L_t = 3$.

2. Monte Carlo simulation

The simulations have been performed on the APE (Alenia Quadrics) computers at DESY-IFH. It turned out that in the SU(2) Higgs model

one encounters large autocorrelation times already on small lattices of size 8^4 if one uses a standard Metropolis algorithm. We found that the expectation value of the Higgs field length $\rho_x^2 = \frac{1}{2} \text{Tr} (\varphi_x^\dagger \varphi_x)$ shows the largest autocorrelation of all investigated quantities. By adopting an overrelaxation method for the radial mode of the Higgs field [8] in combination with standard overrelaxation and heatbath methods for the gauge fields and the SU(2)-angle of the scalar field we were able to reach a substantial reduction of the autocorrelation time as fig. 1 demonstrates.

Figure 1. The logarithm of the autocorrelation function for the length of the scalar field ρ . Shown are different algorithms and their combinations, i.e. Metropolis (Met), heatbath (HB) and overrelaxation (OR). Clearly a mixture of HB and OR gives the smallest autocorrelation time.

In [10] we performed a search for an optimized mixing ratio of the various types of algorithms. This mixing ratio was then used for the simulations described below.

3. Phase transition points

A numerical simulation of the SU(2)-Higgs model should start by first determining the phase transition points. A change of κ is reflected

mainly in a change of the lattice spacing a . Therefore if one crosses the transition at fixed β , λ by changing κ , the essential change is in the physical temperature $T = 1/(aL_t)$. (The physical volume is assumed to be large enough such that its change with a^3 is not important.) Thus we are looking for the phase transition in the hopping parameter at $\kappa = \kappa_c$, for fixed β , λ .

We found that for searching the transition point the gauge invariant effective potential depending on the square of the Higgs field length $\bar{\rho}^2$ can be very helpful. The gauge invariant effective potential to 1-loop order in the broken phase is given by

$$V_{1-loop} = V_{tree} \quad (2)$$

$$+ \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \left\{ \frac{9}{2} \ln(\hat{k}^2 + m_g^2) + \frac{1}{2} \ln(\hat{k}^2 + m_\phi^2) \right\} ,$$

where the masses are related to the parameters in the lattice action (1) and $\bar{\rho}^2$ by

$$m_g^2 = \frac{1}{2} \kappa g^2 \bar{\rho}^2 , \quad m_\phi^2 = \frac{4}{\kappa} \lambda \bar{\rho}^2 \quad (3)$$

and the momenta in the lattice integrals are $\hat{k}^2 = \sum_\mu [2 - 2 \cos(k_\mu)]$. The tree level potential reads

$$V_{tree}(\bar{\rho}^2) = (1 - 8\kappa) \bar{\rho}^2 + \lambda(\bar{\rho}^2 - 1)^2. \quad (4)$$

For the computation of the gauge invariant effective potential on a finite lattice of size $L_x \cdot L_y \cdot L_z \cdot L_t$ the integrals in (2) have been replaced by the corresponding lattice sums. Following the experience in QCD, we also used the mean field improved gauge coupling $g \rightarrow g/\sqrt{U_{pl}}$, with U_{pl} the measured plaquette value. This choice for the gauge coupling appeared to be very useful to achieve better agreement with simulation results.

An alternative method for the determination of the phase transition points is the so called two-coupling method. Here the lattice in the z -direction is divided into two halves, each of which is equipped with a different hopping parameter κ such that one of them corresponds to the symmetric and the other to the broken phase. If the z -direction of the lattice is long enough to accommodate a pair of interfaces and the configuration stays for many autocorrelation times in the mixed

state the free energies are such that the situation is stable against transitions to a unique phase. Thus the parameter sets of the two phases give a lower and an upper bound on the transition parameters. The critical hopping parameter is then determined by the average of the best lower and upper κ -value. The values of κ_c computed in this way are in very good agreement with the one obtained from the gauge invariant potential at $\lambda = 0.0001$. For $\lambda = 0.0005$ we observe a L_T -independent shift by $\Delta\kappa_c = 0.00025(1)$ (see table 2 in [10]).

By far the most accurate determination of the critical value of κ is obtained from the equal height criterion of the canonical distribution of the action density. The distributions have been obtained by a combination of the multicanonical method with κ -reweighting. We show the distribution at $\kappa_c = 0.128307$ in fig. 2. Note that the equal height criterion is roughly equal to the equal area condition in this case.

Figure 2. Distribution of the action density.

4. Masses

The physical Higgs mass M_H can be extracted from correlators of quantities as the site variable

$$R_x \equiv \frac{1}{2} \text{Tr} (\varphi_x^\dagger \varphi_x) \equiv \rho_x^2 . \quad (5)$$

There are alternative choices which have also been used. The W-boson mass M_W can be obtained similarly from the composite link fields ($r, k = 1, 2, 3$)

$$W_{xrk} \equiv \frac{1}{2} \text{Tr} (\tau_r \alpha_{x+\hat{k}}^+ U_{xk} \alpha_x) . \quad (6)$$

Due to lattice symmetry only the diagonal correlators in (r, k) are non-zero and they can be averaged before extracting masses.

Masses were extracted from the connected correlators by least square fits by a single cosh or cosh + constant. Constant contributions are possible in the Higgs channels but were most of the time negligible, because they are of the order $\exp(-M_H L_t)$ and our lattices usually have large enough time extension L_t .

Statistical errors on masses were always estimated by subdividing the data sample into subsamples. Performing the fits in subsamples gives estimates of standard deviations of fit parameters. This is particularly straightforward on the Quadrics Q16, because these lattices can be simulated on 8 nodes which is repeated 16 times with independent sequences of random numbers. The 16 parallel sets of statistically independent results can be used for the estimate of fit parameter errors.

An important effect of finite temperatures is the change in correlation lengths. This is particularly interesting at a first order phase transition, where correlation lengths stay finite and may eventually differ in the two metastable states.

In our numerical simulations we make use of the fact that on a large enough lattice the metastability at the first order phase transition becomes so strong that in a finite amount of computer time the system stays in the phase it started from. Therefore, the masses can be extracted in the symmetric and the broken phase separately.

There is a clear signal for a discontinuity of m_W and m_H between the two sides of the phase transition which shrinks with increasing (zero temperature) Higgs mass as shown in fig. 3 for the case of m_H . Thus we have a first indication that the phase transition becomes weaker for larger Higgs masses. The values of $m_{W,H}$ are rather small, therefore the lattice volumes may be too small

and the infinite volume values may still be somewhat different.

Figure 3. Finite temperature mass gap $\Delta = |m_{sym} - m_{br}|$ with m_{sym} and m_{br} the finite temperature Higgs masses in the symmetric and the broken phase, respectively. Triangles correspond to $L_t = 2$ and squares to $L_t = 3$.

5. Renormalization group trajectories

The continuum limit of quantum field theories on the lattice is taken along renormalization group trajectories, also called *lines of constant physics* (LCP's). Going to the continuum limit along such lines in bare parameter space, the same physical theory is reproduced with increasing precision. In the SU(2) Higgs model, in order to define the LCP's, one has to keep fixed the values of two independent renormalized couplings, the renormalized gauge (g_R^2) and the renormalized quartic ($\lambda_R = M_W^2/(32M_H^2g_R^2)$) couplings. The parameter characterizing the points along LCP's can be chosen as $\tau \equiv \log M_W^{-1}$. (Remember that in this paper the lattice spacing is set to $a = 1$, therefore M_W is measured here in lattice units.)

Since our numerical simulations are performed in the weak coupling region of parameter space, the change of the bare couplings $g^2 \equiv 4/\beta$ and

$\lambda_0 = \lambda/4\kappa^2$ along LCP's should be well approximated by the solutions of the one-loop perturbative renormalization group equations:

$$\begin{aligned} \frac{dg^2(\tau)}{d\tau} &= -\frac{43}{48\pi^2}g^4 + \mathcal{O}(\lambda_0^3, \lambda_0^2g^2, \lambda_0g^4, g^6) , \\ \frac{d\lambda_0(\tau)}{d\tau} &= \frac{1}{16\pi^2} \left[96\lambda_0^2 + \frac{9}{32}g^4 - 9\lambda_0g^2 \right] \\ &+ \mathcal{O}(\lambda_0^3, \lambda_0^2g^2, \lambda_0g^4, g^6) . \end{aligned} \quad (7)$$

The integration of these equations can be started at the transition points of the $L_t = 2$ lattices. At a distance $\Delta\tau = \log(L_t/2)$ ($L_t = 3, 4, 5, \dots$) in parameter τ one obtains the corresponding values of (g^2, λ_0) for the transition points of lattices with temporal extension L_t . To give some impression of how the bare parameters change, going from $L_t = 2$ to $L_t = 3$ changes $\lambda = 0.0001$ to $\lambda = 0.00011$ and $\beta = 8.0$ to $\beta = 8.147$, respectively.

In order to have a perturbative prediction also for the change of the third bare parameter κ , one can make use of the one-loop perturbative invariant effective potential. The measured values of κ_c are very well reproduced by the one-loop invariant effective potential for $\lambda = 0.0001$ and show a parallel shift between prediction and measurements: the prediction is too low by $\Delta\kappa_c = 0.00025(1)$ at $\lambda = 0.0005$. As a consequence, the derivatives of κ along the LCP's can be well determined. The estimates of the LCP's by one-loop perturbation theory work amazingly well. We determined the renormalized gauge coupling as usual from the static potential computed from Wilson loops measured at zero temperature. At $\lambda = 0.0001$ we find for $L_t = 2$ and $L_t = 3$ the values of g_R^2 to be $0.564(6)$ and $0.5597(10)$, respectively. At $\lambda = 0.0005$ we obtained $g_R^2(L_t = 2) = 0.5763(10)$ and $g_R^2(L_t = 3) = 0.5651(13)$. Thus we see that the renormalization is first of all quite small and secondly that scaling violations are tiny thus confirming the prediction of the 1-loop renormalization group equations. A similar picture is obtained from the ratio R_{HW} of the zero temperature Higgs and W-masses: $R_{HW}(L_t = 2) = 0.222(12)$, $R_{HW}(L_t = 3) = 0.201(5)$ for $\lambda = 0.0001$ and $R_{HW}(L_t = 2) = 0.593(19)$, $R_{HW}(L_t = 3) = 0.606(24)$ for $\lambda = 0.0005$.

6. Latent heat

An important characteristic feature of first order phase transitions is the latent heat, i.e. the discontinuity of the energy density ϵ . As it has been shown in [9], the latent heat $\Delta\epsilon$ in the SU(2) Higgs model is given by

$$\begin{aligned} \frac{\Delta\epsilon}{T_c^4} &= L_t^4 \left\langle \frac{\partial\kappa}{\partial\tau} \cdot 8\Delta L_{\varphi,x\mu} \right. \\ &\quad \left. - \frac{\partial\lambda}{\partial\tau} \cdot \Delta Q_x - \frac{\partial\beta}{\partial\tau} \cdot 6\Delta P_{pl} \right\rangle . \end{aligned} \quad (8)$$

Here we used the gauge invariant variables

$$L_{\varphi,x\mu} \equiv \frac{1}{2} \text{Tr}(\varphi_{x+\hat{\mu}}^\dagger U_{x\mu} \varphi_x) ,$$

$$Q_x \equiv (\rho_x^2 - 1)^2 \quad \text{and} \quad P_{pl} \equiv 1 - \frac{1}{2} \text{Tr} U_{pl} . \quad (9)$$

The jumps of the above quantities can be obtained by performing the simulations on large enough lattices, where the strong metastability of phases can be exploited. For $\Delta\epsilon$ in eq. (8) besides the discontinuities of global quantities the derivatives of bare parameters along the LCP's are needed. These have been determined in the way discussed in the previous section. We want to note that the main contribution to the error of the latent heat comes from the error of $\partial\kappa/\partial\tau$ as evaluated from the one-loop invariant effective potential and from numerical simulation data. The results for the latent heat read for $\lambda = 0.0001$

$$\frac{\Delta\epsilon}{T_c^4} = 1.81(29) \quad , \quad \frac{\Delta\epsilon}{T_c^4} = 1.57(13)$$

and for $\lambda = 0.0005$,

$$\frac{\Delta\epsilon}{T_c^4} = 0.132(17) \quad , \quad \frac{\Delta\epsilon}{T_c^4} = 0.122(9) .$$

Here the first numbers correspond to $L_t = 2$ and the second numbers to $L_t = 3$. We see here that within errors the results on $L_t = 2$ and $L_t = 3$ lattices coincide, again pointing in the direction that scaling violations are small.

7. Interface tension

At the transition point of the electroweak phase transition mixed states can appear, where different bulk phases are separated by interfaces.

The interface tension, σ , is the free energy per unit area associated with these interfaces. The dynamics of a first order phase transition is to a large extent determined by the latent heat and the interface tension. In this subsection we present our results for the interface tension obtained by the histogram method for $\lambda = 0.0001$ and $L_t = 2$.

At κ_c the probability distribution of an order parameter (e.g. action density $s \equiv S/\Omega$ defined by (1), or link variable L_φ in (9)) develops two peaks. They correspond to pure phases and the suppressed configurations between the peaks are dominated by mixed states, where the phases are separated by interfaces. Defining κ_c by the equal height signal, the suppression at infinite volume is given by the interface tension

$$\sigma_\infty = \lim_{\Omega \rightarrow \infty} \sigma_\Omega, \quad \sigma_\Omega = \frac{1}{2L_x L_y L_t} \log \frac{p_{max}}{p_{min}}, \quad (10)$$

where p_{max} corresponds to the heights of the peaks and p_{min} to the minimum in between. In the case of elongated lattices, as was used here, the finite size corrections can be given explicitly

$$\sigma_\Omega = \sigma_\infty + \frac{1}{L_{xy}^2 L_t} (c + \frac{3}{4} \log L_z - \frac{1}{2} \log L_{xy}), \quad (11)$$

where $L_x = L_y \equiv L_{xy}$, L_z is the longest extension and c is an unknown constant. The obvious advantage of this choice is that practically all mixed configurations contain two surfaces perpendicular to the z direction. As is demonstrated in fig. 2, the distribution of the action density can be well measured, using the multiccanonical simulation technique. As a result, we find the finite volume interface tensions to be $\sigma_\Omega/T_c^3 = 0.63(3)$ for $\Omega = 2 \cdot 4^2 \cdot 128$ and $\sigma_\Omega/T_c^3 = 0.80(2)$ for $\Omega = 2 \cdot 8^2 \cdot 128$, respectively. Combining these two values one gets from (11)

$$\left(\frac{\sigma_\infty}{T_c^3} \right) = 0.83(4). \quad (12)$$

This result is in very good agreement with the results of the two- κ method discussed in [10].

8. Conclusion

The non-perturbative lattice computations discussed in this paper demonstrate the first order nature of the electroweak phase transition for

Higgs boson masses $M_H < 50$ GeV. The strongest indication for this is the two-peak structure of order parameter distributions (see e.g. fig. 2) showing a pronounced two peak structure. The lattice data follow the trend that the phase transition weakens substantially when the Higgs mass is increased. This can be seen in quantities characterizing the phase transition like the latent heat or the surface tension. The rate of decrease is qualitatively the same as given by two-loop resummed perturbation theory [6,9,10].

Another interesting observation is the smallness of the scaling violations. These can be estimated by studying physical quantities as a function of the temporal extension of the lattice. We find that, within the error bars, the renormalized couplings λ_R and g_R^2 stay constant as predicted by the 1-loop renormalization group equation. In addition, dimensionless quantities like $\Delta\epsilon/T_c^4$ or T_c/M_H (see [10] for the latter quantity) remain almost constant. The apparent general tendency of small scaling violations already at such small temporal lattice extensions is consistent with the expected dominance of the lowest Matsubara modes motivating the dimensional reduction.

REFERENCES

1. G. 't Hooft, Phys. Rev. Lett. **37** (1976) 8; Phys. Rev. **D14** (1976) 3432.
2. V.A. Kuzmin, V.A. Rubakov, M.E. Shaposhnikov, Phys. Lett. **B155** (1985) 36.
3. M.E. Carrington, Phys. Rev. **D45** (1992) 2933.
4. W. Buchmüller, Z. Fodor, T. Helbig, D. Waliser, Ann. Phys. (NY) **234** (1994) 260.
5. P. Arnold, O. Espinosa, Phys. Rev. **D47** (1993) 3546.
6. Z. Fodor, A. Hebecker, preprint DESY-94-025, Nucl. Phys. B to appear.
7. A.D. Linde, Phys. Lett. **96B** (1980) 289.
8. Z. Fodor, K. Jansen, Phys. Lett. **B331** (1994) 119.
9. F. Csikor, Z. Fodor, J. Hein, K. Jansen, A. Jaster, I. Montvay, Phys. Lett. **B334** (1994) 405.
10. Z. Fodor, J. Hein, K. Jansen, A. Jaster, I. Montvay, DESY preprint, DESY 94-159.