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Casimir theory for the piecewise uniform string:
Division into $2N$ pieces

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ABSTRACT

The Casimir energy for the transverse oscillations of a piecewise uniform string in Minkowski space is calculated. The string consists in general of $2N$ pieces of equal length, of alternating type I and type II material, endowed with different tensions T_I , T_{II} and mass densities ρ_I , ρ_{II} , but adjusted in such a way that the velocity of sound always equals the velocity of light. This string model (with $N = 1$) was introduced by the present authors in 1990. In the present paper, the governing equations are formulated in the case of arbitrary integers N , although detailed calculations are restricted to the case $N = 2$, i.e., a four-piece string. The Casimir energy is regularized using (i) the contour integration technique introduced by van Kampen, Nijboer, and Schram, and (ii) the Hurwitz ζ function technique introduced by Li, Shi, and Zhang. The energy is found as a function of the tension ratio $x = T_I/T_{II}$. The finite temperature version of the theory is also given.

I. INTRODUCTION

Consider the transverse oscillations of a piecewise uniform string in Minkowski space. The constituent pieces, each of the same length, are alternatively of type I or II material as we move around the string. The corresponding tensions T_I and T_{II} and mass densities ρ_I and ρ_{II} are required to satisfy the condition that the velocity of sound equals the velocity of light:

$$v_s = (T_I/\rho_I)^{1/2} = (T_{II}/\rho_{II})^{1/2} = c \quad . \quad (1)$$

This condition thus makes the string model relativistically invariant. We shall examine the Casimir energy for this kind of string, in particular in the case when the string of total length L is divided into four pieces of length $L/4$, i.e., two pieces of type I, and two pieces of type II, material.

The composite string model was introduced by the present authors some time ago [1], assuming the string to be divided into only two pieces. Different issues of the model, related in particular to the regularization procedures, have more recently been studied by Li, Shi, and Zhang [2], and by Brevik and Elizalde [3]. One reason why this kind of string is of interest, is that it furnishes a transparent and easily solvable model for the vacuum state in two-dimensional quantum field theories in general. Another virtue follows if we make a comparison with Casimir theories for ordinary (i.e., nonrelativistic) dielectric media having curved boundaries: the present model avoids completely in a natural way the annoying divergences that so often plague these dielectric calculations. Finally, the present string model is open to the application of different regularization methods, such as the cutoff method [1], the ζ function method [2] (the Hurwitz ζ function, not the Riemann ζ function), and the contour integral method [3] introduced by van Kampen, Nijboer, and Schram [4] in Casimir theory when describing the simple parallel two-plate geometry.

In the next section we formulate the general equations for the string when it is divided into $2N$ pieces of alternating type I and type II material, N being an integer. The eigenfrequencies are determined in principle by Eq. (9), where the matrix $\underline{M}$ follows from Eqs. (11) and (12). The general case becomes however quite complicated mathematically, and so our treatment from Sect. III onwards is restricted to a four-piece string (i.e., $N = 2$). We calculate the dispersion function and eigenvalue spectrum, find the Casimir energy at $T = 0$ using two different regularization methods, and show the result graphically in Fig. 3. Generalization of the theory to the case of finite T is also discussed.

The general theory is formulated using natural units, so that $\hbar = c = 1$. In final expressions, ordinary cgs units are reinstated.

II. GENERAL FORMALISM

Let in general the string be divided into $L/(2N)$ pieces, each of the same length. Figure 1 illustrates the situation in the case when $N = 2$. There are thus $2N$ junctions, numbered by $j = 1, 2, \dots, 2N$. Successive pieces are alternatively being of type I or type II material. The thickness of the string is negligible (for illustrative purposes it is drawn finite in the figure). Let $\psi(\sigma, \tau)$ be the transverse displacement of a point with spatial coordinate σ from its equilibrium position at time τ . For the sake of clarity we write down the basic expressions for the fundamental modes in the situation shown in Fig. 1:

$$\begin{aligned}
 \psi_I \left(0 < \sigma < \frac{1}{4}L \right) &= \xi_I^{(1)} e^{i\omega(\sigma-\tau)} + \eta_I^{(1)} e^{-i\omega(\sigma+\tau)} , \\
 \psi_{II} \left(\frac{1}{4}L < \sigma < \frac{1}{2}L \right) &= \xi_{II}^{(1)} e^{i\omega(\sigma-\tau)} + \eta_{II}^{(1)} e^{-i\omega(\sigma+\tau)} , \\
 \psi_I \left(\frac{1}{2}L < \sigma < \frac{3}{4}L \right) &= \xi_I^{(2)} e^{i\omega(\sigma-\tau)} + \eta_I^{(2)} e^{-i\omega(\sigma+\tau)} , \\
 \psi_{II} \left(\frac{3}{4}L < \sigma < L \right) &= \xi_{II}^{(2)} e^{i\omega(\sigma-\tau)} + \eta_{II}^{(2)} e^{-i\omega(\sigma+\tau)} .
 \end{aligned} \tag{2}$$

We have here included right- and left-moving waves. There occur thus eight constants, $\xi_{I,II}^{(1,2)}$ and $\eta_{I,II}^{(1,2)}$, which must be defined from the 4×2 boundary conditions at the four junctions. These conditions are that the transverse displacement ψ itself, as well as the transverse elastic force, $T\partial\psi/\partial\sigma$, must be continuous at each junction, i.e., for each value of j . Explicit calculation for the case $N = 2$ yields the following eight equations:

$$\begin{aligned}
\xi_I^{(1)} e^{i\omega L/4} + \eta_I^{(1)} e^{-i\omega L/4} &= \xi_{II}^{(1)} e^{i\omega L/4} + \eta_{II}^{(1)} e^{-i\omega L/4} , \\
T_I (\xi_I^{(1)} e^{i\omega L/4} - \eta_I^{(1)} e^{-i\omega L/4}) &= T_{II} (\xi_{II}^{(1)} e^{i\omega L/4} - \eta_{II}^{(1)} e^{-i\omega L/4}) , \\
\xi_{II}^{(1)} e^{i\omega L/2} + \eta_{II}^{(1)} e^{-i\omega L/2} &= \xi_I^{(2)} e^{i\omega L/2} + \eta_I^{(2)} e^{-i\omega L/2} , \\
T_{II} (\xi_{II}^{(1)} e^{i\omega L/2} - \eta_{II}^{(1)} e^{-i\omega L/2}) &= T_I (\xi_I^{(2)} e^{i\omega L/2} - \eta_I^{(2)} e^{-i\omega L/2}) , \\
\xi_I^{(2)} e^{3i\omega L/4} + \eta_I^{(2)} e^{-3i\omega L/4} &= \xi_{II}^{(2)} e^{3i\omega L/4} + \eta_{II}^{(2)} e^{-3i\omega L/4} , \\
T_I (\xi_I^{(2)} e^{3i\omega L/4} - \eta_I^{(2)} e^{-3i\omega L/4}) &= T_{II} (\xi_{II}^{(2)} e^{3i\omega L/4} - \eta_{II}^{(2)} e^{-3i\omega L/4}) , \\
\xi_I^{(1)} + \eta_I^{(1)} &= \xi_{II}^{(2)} e^{i\omega L} + \eta_{II}^{(2)} e^{-i\omega L} , \\
T_I (\xi_I^{(1)} - \eta_I^{(1)}) &= T_{II} (\xi_{II}^{(2)} e^{i\omega L} - \eta_{II}^{(2)} e^{-i\omega L}) .
\end{aligned} \tag{3}$$

Here the condition of solution for the coefficients $\xi_{I,II}^{(1,2)}$ and $\eta_{I,II}^{(1,2)}$ is that the 8×8 system determinant vanishes. The evaluation of this determinant is however complicated, and we therefore find it convenient to formulate the problem in a different way, via introduction of 2×2 transfer matrices: Let $\underline{\xi}$ be the column matrix whose elements are ξ and η ; thus for example

$$\underline{\xi}_I^{(1)} = \begin{pmatrix} \xi_I^{(1)} \\ \eta_I^{(1)} \end{pmatrix} , \text{ etc.} \tag{4}$$

As in Refs. [1, 3] we let x denote the ratio between the two tensions,

$$x = \frac{T_I}{T_{II}} \tag{5}$$

and for convenience we also introduce the symbol p ,

$$p = \frac{1}{2}\omega L . \tag{6}$$

Then, taking into account the Pauli spin matrices $\sigma_x, \sigma_y, \sigma_z$, we see that it is possible to write Eqs. (3) in the following compact form:

$$\begin{aligned}\xi_I^{(1)} &= M^{(1)} \xi_{II}^{(1)}, & \xi_{II}^{(1)} &= M^{(2)} \xi_I^{(2)}, \\ \xi_I^{(2)} &= M^{(3)} \xi_{II}^{(2)}, & \xi_{II}^{(2)} &= M^{(4)} \xi_I^{(1)},\end{aligned}\tag{7}$$

where

$$\begin{aligned}M^{(1)} &= 1 \frac{1+x^{-1}}{2} + \sigma_x \frac{1-x^{-1}}{2} \cos p + \sigma_y \frac{1-x^{-1}}{2} \sin p, \\ M^{(2)} &= 1 \frac{1+x}{2} + \sigma_x \frac{1-x}{2} \cos 2p + \sigma_y \frac{1-x}{2} \sin 2p, \\ M^{(3)} &= 1 \frac{1+x^{-1}}{2} + \sigma_x \frac{1-x^{-1}}{2} \cos 3p + \sigma_y \frac{1-x^{-1}}{2} \sin 3p, \\ M^{(4)} &= 1 \frac{1+x}{2} \cos 2p + \sigma_x \frac{1-x}{2} \cos 2p + \sigma_y \frac{1-x}{2} \sin 2p - i\sigma_z \frac{1+x}{2} \sin 2p.\end{aligned}\tag{8}$$

The matrices $M^{(i)}$ may be regarded as transfer matrices, since they relate the coefficients ξ and η pertaining to two neighbouring pieces of the string.

Under stationary conditions the eigenfrequencies ω of the string are all real quantities.

They are determined from the equation

$$\text{Det } (M - 1) = 0,\tag{9}$$

with $M = M^{(1)} M^{(2)} M^{(3)} M^{(4)}$.

Now all the above formulas correspond to $N = 2$. It is straightforward, at least in principle, to generalize the formalism to the case of an arbitrary value of the integer N . Defining in general p as

$$p = \omega L / N,\tag{10}$$

we find that the eigenfrequencies are determined from the same equation (9) as above, but where the composite matrix M now is

$$\underline{M} = \prod_{j=1}^{2n} \underline{M}^{(j)} , \quad (11)$$

with

$$\begin{aligned} \underline{M}^{(j)} &= 1 \frac{1+x^{(-1)^j}}{2} + \sigma_x \frac{1-x^{(-1)^j}}{2} \cos jp + \sigma_y \frac{1-x^{(-1)^j}}{2} \sin jp , \\ j &= 1, 2, \dots, (2N - 1) , \\ \underline{M}^{(2N)} &= 1 \frac{1+x}{2} \cos Np + \sigma_x \frac{1-x}{2} \cos Np + \sigma_y \frac{1-x}{2} \sin Np \\ &\quad - i\sigma_z \frac{1+x}{2} \sin Np . \end{aligned} \quad (12)$$

The formalism holds for $N = 1, 2, 3, \dots$. If $N = 1$, agreement is found with the theory given in [1,3]. Because the formalism rapidly gets involved for higher N , we shall restrict ourselves to $N = 2$ in the following sections. Let us first, however, notice the following two general properties of the system:

- (i) If $x = 1$, the system degenerates into that of a uniform string. Since the velocity of sound is required to be equal to c everywhere, it is irrelevant whether the string is composed of type I or type II material; the eigenvalue spectrum is determined from the equation

$$\cos \omega L = 1 \quad (13)$$

in either case. Thus $\omega_n L = 2\pi n$, where the integer n can be taken to be positive. The Casimir energy, defined to describe the effect from the non-uniformity of the string, is necessarily zero.

- (ii) The eigenvalue spectrum of the system is in general invariant under the transformation $x \rightarrow 1/x$. This invariance can be understood both physically and mathematically:

physically, the effect from a $I \rightarrow II$ transition at an arbitrary junction is always compensated by the effect from the $II \rightarrow I$ transition at the next junction. Mathematically, all terms in the transfer matrix $\underline{M}$ that change sign under the replacement $x \rightarrow 1/x$ become squared under the evaluation of the determinant in Eq. (9). For this reason we may henceforth restrict ourselves to the tension ratio interval $0 < x \leq 1$ in the following.

III. DISPERSION EQUATION AND CASIMIR ENERGY WHEN $N = 2$

A. Dispersion equation

This case is analytically tractable. We write the transfer matrix $\underline{M}$ on component form:

$$\underline{M} = \underline{M}^{(1)} \underline{M}^{(2)} \underline{M}^{(3)} \underline{M}^{(4)} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad (14)$$

and calculate the components M_{ik} using Eqs. (8). After some algebra we obtain

$$\begin{aligned} M_{11} &= \frac{(1-x)^2}{16x^2} [2(1+x^2) + (1+x)^2 (1-2\cos p)] - \\ &\quad - \frac{(1-x^2)^2}{8x^2} e^{-ip} + \frac{(1+x)^4}{16x^2} e^{-2ip}, \\ M_{12} &= \frac{1-x^2}{16x^2} [2(1+x^2) + (1-x)^2 - 2(1+x)^2 \cos p] - \\ &\quad - \frac{(1-x^2)(1-x)^2}{8x^2} e^{-ip} + \frac{(1+x)^2(1-x^2)}{16x^2} e^{-2ip}, \\ M_{22} &= M_{11}^*, \quad M_{21} = M_{12}^*. \end{aligned} \quad (15)$$

The dispersion equation can according to (9) and (15) be written as

$$g(\omega) = |M_{11}|^2 - 2\text{Re } M_{11} + 1 - |M_{12}|^2 = 0. \quad (16)$$

Some calculation yields the following relatively simple expression for the dispersion function:

$$\begin{aligned} g(\omega) &= - \frac{(1+x)^2}{8x^2} (3 - 10x + 3x^2) + \frac{(1-x^2)^2}{2x^2} \cos \frac{\omega L}{2} - \\ &\quad - \frac{(1+x)^4}{8x^2} \cos \omega L, \end{aligned} \quad (17)$$

where we have reinstated $\omega L/2$ for p . Written in this way, the dispersion function is

manifestly invariant under the substitution $x \rightarrow 1/x$ (it may be noted that $M_{11}(1/x) = M_{11}(x)$, $M_{12}(1/x) = -M_{12}(x)$). We assume that the oscillations of the string are stationary, and so the eigenfrequencies ω of the dispersion equation $g(\omega) = 0$ have to be real.

It is of interest to consider the reduced form of (17) in two special cases. First, if $x = 1$ we get

$$g(\omega)|_{x=1} = 2 - 2 \cos \omega L, \quad (18)$$

leading to the eigenfrequency spectrum in accordance with (13) for the uniform string. Secondly, we consider the limit of extreme string tensions. Since we have let x be restricted to the interval $0 < x \leq 1$, this extreme case corresponds to $x \rightarrow 0$. We have then

$$g(\omega)|_{x \rightarrow 0} = \frac{-1}{4x^2} \left(1 - \cos \frac{\omega L}{2} \right)^2, \quad (19)$$

which leads to the eigenvalue spectrum

$$\omega_n = 4\pi n/L, \quad n = 1, 2, 3 \dots \quad (20)$$

There are thus only half as many eigenfrequencies in this case as in the case of the uniform string. It follows that the Casimir energy is necessarily negative when $x \rightarrow 0$.

For general values of x the dispersion equation is according to (17) quadratic in $\cos \omega L/2$, with solution

$$\cos \frac{\omega L}{2} = \begin{cases} 1, \\ \frac{1-6x+x^2}{(1+x)^2} \end{cases}. \quad (21)$$

It is remarkable that the $x \rightarrow 0$ eigenfrequency spectrum, Eq. (20), appears as part of the general solution (21).

B. Casimir energy

As in earlier works [1-3] we define the Casimir energy as the zero-point energy $E_{I+\Pi}$ for the composite string minus the zero-point energy for the uniform string:

$$E = E_{I+\Pi} - E_{\text{uniform}} . \quad (22)$$

In the following we may choose to use the cutoff regularization method, as in [1], or the contour integration method, as in [3]. We shall first follow the latter approach, an important advantage of this method being that the degeneracies of the states are taken into account automatically. We first write down the expressions for E_{uniform} ; from the so-called argument principle applied to the $x = 1$ dispersion function (18) we may write

$$E_{\text{uniform}} = \frac{1}{4\pi i} \oint \omega \frac{d}{d\omega} \ln \sin^2 \frac{\omega L}{2} d\omega , \quad (23)$$

where the contour is as shown in Fig. 2. Expressing $E_{I+\Pi}$ in the same way, taking into account (17) and (21), we get for the Casimir energy

$$E = \frac{1}{4\pi i} \oint \omega \frac{d}{d\omega} \ln \left[1 - \left(\frac{1-x}{1+x} \right)^2 \frac{1}{\cos^2 \frac{\omega L}{4}} \right] d\omega . \quad (24)$$

As a corollary, we see that $E = 0$ if $x = 1$.

The contribution from the semicircle in Fig. 2 vanishes in the limit $R \rightarrow \infty$. We perform an integration by parts of the remaining integral along the imaginary axis ($\omega = i\xi$), taking into account the symmetry about the origin. In this way we obtain

$$E = \frac{1}{2\pi} \int_0^{\tilde{\tau}} \ln \left[1 - \left(\frac{1-x}{1+x} \right)^2 \frac{1}{\cosh^2 \frac{\xi L}{4}} \right] d\xi, \quad (25)$$

which may be rewritten in dimensional form as

$$E = \frac{2\hbar c}{\pi L} \int_0^{\tilde{\tau}} \ln \left[1 - \left(\frac{1-x}{1+x} \right)^2 \frac{1}{\cosh^2 y} \right] dy. \quad (26)$$

This form shows manifestly a characteristic feature of the theory: the particular combination of terms that is determined solely by the value of x is (EL).

If $x \rightarrow 0$, we calculate explicitly

$$E = \frac{4\hbar c}{\pi L} \int_0^{\tilde{\tau}} \ln \tanh y \, dy = \frac{4\hbar c}{\pi L} \int_0^1 \frac{\ln u}{1-u^2} du = -\frac{\pi\hbar c}{2L}. \quad (27)$$

For general x the integral (26) is easily calculated numerically and the result is shown in Fig.

3. The Casimir energy is negative for all $x < 1$. The transformation of an initially uniform string into a state of non-uniformity of the kind considered here thus always diminishes its zero point energy. It is to be noted that E is different from zero although all pieces of the string are of the same length. In this sense the present situation contrasts that of the two-piece string considered earlier [1-3]: in that case, equality of the piece lengths would always imply $E = 0$, irrespective of the value of x .

C. Zeta function regularization

There exists a very simple and elegant alternative method to obtain the result above.

The method consists in using the Hurwitz ζ function $\zeta(s,a)$ to regularize the initially divergent

expression for the zero point energy. The original definition of the Hurwitz function is

$$\zeta(s,a) = \sum_{n=0}^{\infty} (n + a)^{-s} \quad (0 < a \leq 1, \text{Re } s > 1) . \quad (28)$$

It reduces to the Riemann ζ function if $a = 1$. $\zeta(s,a)$ can be continued analytically to the whole s plane, and becomes regular everywhere except at $s = 1$ where it has a simple pole with residue 1. The Hurwitz ζ function was introduced in the context of Casimir string physics by Li, Shi, and Zhang [2], and the reader is referred to that paper for more details. Sufficient in our case is to note the following property of the analytically continued function:

$$\zeta(-1,a) = -\frac{1}{2} \left(a^2 - a + \frac{1}{6} \right) . \quad (29)$$

Consider now the zero-point energies: first, the uniform string corresponds according to (13) to

$$E_{\text{uniform}} = 2 \times \frac{1}{2} \sum_{n=1}^{\infty} \omega_n = \frac{2\pi}{L} \zeta(-1,1) = -\frac{\pi}{6L} , \quad (30)$$

where the prefactor 2 takes account of the degeneracy of these modes. Next, for the composite string we obtain from the first member of Eqs. (21) the spectrum

$$\omega = \frac{4\pi}{L} n , \quad n = 1, 2, 3 \dots , \quad (31)$$

whereas the second member of (21) yields

$$\omega = \frac{2}{L} \arccos \frac{1 - 6x + x^2}{(1 + x)^2} .$$

It is convenient to write this in the form

$$\omega = \frac{2\pi}{L} \left\{ \begin{array}{l} \beta + 2n, \\ 2 - \beta + 2n \end{array} \right., \quad (32)$$

where $n = 0, 1, 2 \dots$ and where β is a number required to lie in the interval $0 < \beta \leq 1$. The modes (31) are degenerate; the modes (32) are not. The zero-point energy of the composite string thus becomes

$$\begin{aligned} E_{I+II} &= \frac{4\pi}{L} \sum_{n=1}^{\infty} n + \frac{\pi}{L} \sum_{n=0}^{\infty} (2n+\beta) + \frac{\pi}{L} \sum_{n=0}^{\infty} (2n+2-\beta) \\ &= \frac{2\pi}{L} \left[2\zeta(-1, 1) + \zeta\left(-1, \frac{\beta}{2}\right) + \zeta\left(-1, 1-\frac{\beta}{2}\right) \right]. \end{aligned} \quad (33)$$

The ζ functions are evaluated from (29), and so the Casimir energy (22) can be written in dimensional form as

$$E = - \frac{\pi \hbar c}{2L} (1 - \beta)^2. \quad (34)$$

If $x = 1$ (i.e., $\beta = 1$) we recover $E = 0$, in agreement with the uniform string result above. And if $x \rightarrow 0$ (i.e., $\beta \rightarrow 0$) we get agreement with Eq. (27). For general x , it can be checked that (34) is in agreement with the integral (26). The expression (34) is thus the most compact way of expressing the result, at zero temperature.

D. Finite temperatures

The integral expression (25) for the Casimir energy at temperature $T = 0$ makes it convenient to generalize the theory to finite temperatures. The point is to replace the integral over imaginary frequencies ξ by a sum:

$$\hbar \int_0^{\infty} d\xi \rightarrow 2\pi k_B T \sum_{n=0}^{\infty} \quad , \quad (35)$$

the prime meaning that the $n = 0$ contribution is taken with half weight. The Matsubara frequencies are $\xi_n = 2\pi n k_B T / \hbar$, with $n = 0, 1, 2, \dots$ [5]. We obtain, in dimensional form, the Casimir energy

$$E(T) = k_B T \sum_{n=0}^{\infty} \ln \left[1 - \left(\frac{1-x}{1+x} \right)^2 \frac{1}{\cosh^2 \frac{\xi_n L}{4c}} \right] \quad , \quad (36)$$

valid at all temperatures. The case $x = 1$ still yields a zero result, as it should according to the definition (22) holding also for finite temperatures.

As in Ref. [3], we may note that there are two different characteristic frequencies in the system:

- (1) The thermal frequency ω_T , which can be defined such that $\hbar \omega_T = k_B T$.
- (2) The geometric frequency ω_{geom} , associated with the geometry of the string. In the present problem it seems natural to define ω_{geom} in terms of L as fundamental length:

$$\omega_{\text{geom}} = 2\pi c/L.$$

The question whether a given situation is a "high" temperature problem or a "low" temperature problem can be answered if one knows the ratio between the two temperatures above. This is seen from a rewriting of the argument in the hyperbolic function in (36):

$$\frac{\xi_n L}{4c} = \pi^2 n \frac{\omega_T}{\omega_{\text{geom}}} . \quad (37)$$

The simplest case is to assume a high temperature problem: let us assume that

$$\frac{\omega_T}{\omega_{\text{geom}}} \gg 1 . \quad (38)$$

The contribution to (36) from integers $n \geq 1$ is then according to (37) seen to be vanishingly small, and we remain with the $n = 0$ term:

$$E(T) = \frac{k_B T}{2} \ln \frac{4x}{(1+x)^2} . \quad (39)$$

This is a classical result, since it is independent of $\hbar$. We see from (39) that $E(T) \leq 0$ always. It is somewhat surprising to note that $E(T)$ tends to minus infinity in the case of extreme string tension ratios, $x \rightarrow 0$. Also, we note that the expression (39) does not contain the length L explicitly.

The opposite extreme to (38), i.e.,

$$\frac{\omega_T}{\omega_{\text{geom}}} \ll 1 , \quad (40)$$

corresponds to a low temperature problem and involves a large number of Matsubara frequencies. Numerical evaluation usually becomes necessary. We may note that the expression (36) is too complicated to permit an efficient use of the Poisson summation formula, which otherwise in some cases has turned out to be a useful means in Casimir theory (cf., for instance, Refs. [6,7]). We abstain from considering more closely the low temperature energy here.

Finally, it is tempting to dwell briefly on the possibility that strings of the present kind may have played a physical role in the early universe. Specifically, let us calculate which

string length L would correspond to equality of the characteristic frequencies, $\omega_T = \omega_{\text{geom}}$, at the beginning of the inflationary era in which it is believed that the temperature was $T \approx 10^{27}\text{K}$. We find $L = 2\pi\hbar c/(k_B T) \approx 1.4 \times 10^{-27} \text{ cm}$, which is about 1000 times the diameter of a cosmic string (10^{-30} cm). Thus for a four-piece string of this length, in thermal equilibrium with the cosmic fluid, the Casimir energy is according to (39) given by the simple formula

$$E(T) = 6.9 \times 10^{10} \ln \frac{4x}{(1+x)^2} \text{ ergs} . \quad (41)$$

IV. CONCLUSIONS

We have studied the Casimir energy, as defined in (22), for the transverse oscillations of a piecewise uniform string, divided in general into $L/(2N)$ pieces. The formal theory for this case is given in Sect. II, although we have not for general N succeeded to find the expression for the dispersion function.

The theory has been worked out in detail in the case $N = 2$, i.e., for a string divided into four pieces. The dispersion function is given in Sect. III, and the Casimir energy is found both by means of complex integration methods, and by use of Hurwitz' ζ function. Generalization of the theory to the case of finite temperatures is also made. The Casimir energy is generally found to be non-positive.

We emphasize that the piecewise uniform string model is relativistic. This is so because the velocity of sound, Eq. (1), is everywhere in the string required to be equal to c . If this requirement were removed in the theory, the regularization procedure would be more difficult. The situation seems to be quite analogous to that of the Casimir theory in material media. The case $\epsilon\mu = 1$, where ϵ is the permittivity and μ the permeability, is analogous to the case of Eq. (1), and is easily regularized to give a finite Casimir energy, as verified explicitly in Ref. [8] for the case of a spherical ball. The general case with arbitrary ϵ (or μ) is however more difficult to deal with; cf., for instance, Ref. [9].

If the string is divided into two pieces of equal length, the Casimir energy is always zero [1,3]. The division into four pieces, each of the same length, is, as we have seen in the present paper, associated with a nonvanishing Casimir energy. The main point in the present string theory thus does not depend critically on whether the type I and type II pieces are of the same length. The most fundamental property of the theory is rather the negativity

of the Casimir energy; an initially uniform string can lower the zero-point energy by transforming itself into a non-uniform state. It might even be so that this kind of "phase transitions" played a physical role in the early universe.

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Figure 1 String divided into four equal pieces.

Figure 2 Integration contour in the complex ω plane.

Figure 3 Nondimensional Casimir energy versus $x = T_l/T_r$.