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Two-dimensional magnetotransport according to the classical
Lorentz model

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Abstract

The classical Lorentz model for charged noninteracting point particles in a perpendicular magnetic field is reconsidered in 2D. We show that the standard Boltzmann equation is not valid for this model, even in the Grad limit. We construct a generalized Boltzmann equation which *is*, and solve the corresponding initial value problem exactly. The diffusion tensor follows. Away from the Grad limit a percolation problem arises. The percolation threshold is determined numerically, as are the geometrical critical exponents. The universality class is that of lattice percolation in 2D.

Since the discovery of the quantum Hall effect 15 years ago, magnetotransport in the two dimensional (2D) electron gas has been the subject of intense investigation, resulting in a number of fascinating findings, both experimental and theoretical [2]. With all this activity focused on the quantum aspects of the transport problem, it is easily taken for granted that classical magnetotransport is a closed chapter, with no surprises in store. In this Letter we shall demonstrate that this is not the case. Even the simplest classical kinetic model, the Lorentz model [3,4] with the Boltzmann equation (BE) as a basic tool, is much richer than expected. Proofs exist that, under seemingly mild restrictions, the BE is *exact* in the so-called Grad limit [5] (to be specified below). In contrast, we shall demonstrate that in 2D, and with a non-zero magnetic field, the BE is no longer valid, even in the Grad limit. Furthermore, we shall formulate a non-Markovian generalized Boltzmann equation (GBE) which *is* exact in this limit, and we shall solve the corresponding initial value problem for the spatially homogeneous case. The corresponding diffusion tensor follows immediately from this solution. Moving away from the Grad limit, we shall then show that at low but finite dimensionless densities a percolation problem arises. We determine numerically the percolation threshold for this problem and the corresponding critical exponents. The numerical results are consistent with our model belonging to the same equilibrium universality class as 2D lattice percolation. In this Letter we restrict ourselves to presenting and commenting upon the basic arguments and principle results. Further details will be published elsewhere.

FIG. 1. The Lorentz model in a magnetic field. Between collisions the “electrons” travel on arcs with the cyclotron radius.

The classical Lorentz model is defined as follows (see Fig.1): Non-interacting point particles, “electrons” with charge $-e$, move under the influence of a perpendicular magnetic field $B\hat{z}$, in a random array of stationary scatterers with short range forces. For concreteness we can think of the scatterers as being hard disks of radius a , uniformly distributed in the xy -plane, with number density n . Since the electron-electron interaction is neglected, the problem reduces to the study of one electron moving with constant speed $v = |\mathbf{v}|$ in the random array of scatterers. For small dimensionless densities, $\rho \equiv na^2 \ll 1$, the mean free path between collisions is given as (for hard disks) $\Lambda = (2an)^{-1}$. The Grad limit is defined as follows: Let $n \rightarrow \infty, a \rightarrow 0$ in such a way that Λ is fixed at a finite value. Note that although the number density, n , tends to infinity, the *dimensionless* density, ρ , tends to zero in this limit.

For simplicity, we immediately specialize to the spatially homogeneous case. The Boltzmann equation for the one-particle distribution $f(\phi, t)$, where $\phi = \angle(\mathbf{v}, \hat{x})$ is the direction of the velocity, reads:

$$\left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \phi} \right) f(\phi, t) = \nu \int_{-\pi}^{\pi} d\psi g(\psi) [f(\phi - \psi, t) - f(\phi, t)] \equiv Bf(\phi, t). \quad (1)$$

Here $\omega = eB/m$ is the cyclotron frequency (with m the effective mass), $\nu = nv\Sigma$ is the collision frequency (with $\tau = \nu^{-1}$ the mean free time between collisions), $\Sigma = \int_{-\pi}^{\pi} d\psi \sigma(\psi)$ is the total scattering cross section, $g(\psi) = \sigma(\psi)/\Sigma$ is the dimensionless differential cross section, and ψ is the scattering angle. The basic assumption underlying the Boltzmann equation is the “Stosszahlansatz”: In every collision the moving particle hits a scatterer never previously encountered, i.e., no correlations exist prior to the collision. And: There is a constant frequency, ν , of such collisions. Generally, this assumption can be proved to hold in the Grad limit [5].

However, in 2D and in a magnetic field, it is easy to see that the Stosszahlansatz fails. With a finite cyclotron radius, $R = v/\omega$, there is a finite probability, $P_0 = \exp(-2\pi R/\Lambda)$, in the Grad limit, that the electron will complete a cyclotron orbit without having suffered a collision. In that case the electron will, with certainty, remain collisionless in a cyclotron orbit forever. As an immediate consequence, the BE cannot be correct. Clearly, this is a feature peculiar to 2D. In 3D the electron has, with probability unity, a finite velocity component parallel to the magnetic field. Thus, in 3D, the collision frequency remains constant even for times greater than the cyclotron period $T = 2\pi/\omega$. The exceptional electrons in 2D that never collide, we shall call “circling”. Clearly the BE cannot be correct for those. However, even for the colliding (“wandering”) electrons, the BE fails in 2D in the Grad limit. The mechanism is similar: After a collision with a given scatterer there is, in the Grad limit, a finite probability, P_0 , that the electron suffers no further collisions until it recollides with *same* scatterer. Such recollisions introduce memory into the system and are clearly beyond the scope of the Boltzmann equation proper [6].

Before formulating the Generalized Boltzmann equation (GBE) governing the distribution function in the Grad limit, we state without proof three properties that hold *in this limit*:

- An electron either does not collide with any scatterer, or it collides (in the course of time) with infinitely many different ones. (Exceptions to this statement are of measure zero and are, consequently, ignored.)
- An electron can only recollide with a given scatterer if no other scatterer has been hit in the meantime.
- The total scattering angle after n collisions with the same scatterer is $n\psi$, where ψ is the scattering angle of the initial collision with that scatterer.

Simple geometric considerations are sufficient to prove these statements.

We are now in a position to write down the GBE:

$$\left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \phi} \right) f^G(\phi, t) = \sum_{s=0}^{\lfloor t/T \rfloor} P_0^s B_{-sT} f^G(\phi, t - sT). \quad (2)$$

Here $[t/T]$ is the number of cyclotron periods completed until t , and $B_0 \equiv B$ is the Boltzmann operator of (1). More generally, we define,

$$B_{-sT} f^G(\phi, t - sT) \equiv \nu \int_{-\pi}^{\pi} d\psi g(\psi) [f^G(\phi - (s+1)\psi, t - sT) - f^G(\phi - s\psi, t - sT)]. \quad (3)$$

This operator describes the s 'th recollision of the electron with the same scatterer. Only the initial collision is probabilistic in nature. Given the scattering angle of the first collision, all subsequent recollisions are completely determined. Whereas the initial collision, at $t - sT$, could be described by the Boltzmann operator, each subsequent recollision simply adds another ψ to the initial scattering angle. Thus, at t , the first term on the RHS of (3) is a gain term with respect to $f^G(\phi, t)$ and the second a loss term. In (2) the recollision operator (3) is weighted by the probability, P_0^s , that the electron collides with no other scatterer during the time interval $(t - sT, t)$. The superscript G on $f^G(\phi, t)$ refers to the following subtlety: For $t < T$, a cyclotron orbit has not yet been completed and it is not yet clear whether an electron is "circling" or "wandering". During this initial cyclotron period, the RHS of (2) reduces to the standard Boltzmann collision term, and $f^G = f = f^C + f^W$ is the total distribution function, including both circling (f^C) and wandering (f^W) electrons. For $t \geq T$, however, the generalized collision operator of (2) *only* applies to wandering electrons, the circling ones having been identified as those that do not suffer any collisions. I.e., for $t \geq T$, $f^G = f^W$. These facts must be kept in mind when one solves the initial value problem posed by the GBE (2).

The initial value problem is solved by appeal to the Fourier-Laplace transform,

$$F_m(p) = \int_0^\infty dt e^{-pt} f_m(t) = \int_0^\infty dt e^{-pt} \int_{-\pi}^{\pi} d\phi e^{im\phi} f(\phi, t). \quad (4)$$

Taking into account the subtlety mentioned above, one finds,

$$F_m^G(p) = \frac{(1 - e^{-(p+\nu)T}) f_m(0)}{p - im\omega + \nu \int_{-\pi}^{\pi} d\psi g(\psi) \frac{1 - e^{im\psi}}{1 - e^{-(p+\nu)T + im\psi}}}. \quad (5)$$

For a calculation of the integral over ψ , the differential cross section must be specified. For hard disks it reads, $g(\psi) = (1/4) \sin |\psi/2|$, with Fourier transform, $g_m = -(4m^2 - 1)^{-1}$.

From (5) the diffusion tensor follows. It is convenient to introduce complex notation, $\mathcal{D} = D_1 + iD_2$, where $D_1 = D_{xx} = D_{yy}$ and $D_2 = -D_{xy} = D_{yx}$. Note that only wandering electrons contribute to the diffusion process. With fixed $v = |\mathbf{v}|$, the Einstein-Kubo formula for $\mathcal{D}$ can then be written in the form

$$\mathcal{D} = \frac{1}{2} v^2 \int_0^\infty dt \langle e^{i\phi(t)} \rangle_{\phi(0)=0} = \frac{1}{2} v^2 F_1^W(0) = \frac{1}{2} v^2 F_1^G(0). \quad (6)$$

The last equality in (6) follows from the fact that the contribution from the white electrons to $\mathcal{D}$ during the first period vanishes. With hard disk scatterers the integration inherent in $F_1^G(0)$ can be performed explicitly and one finds:

$$\mathcal{D} = \frac{\frac{1}{2} v^2 \tau_D(x) (1 - x^2)}{1 - i\omega \tau_D(x)}, \quad (7)$$

with $x \equiv \exp(-\frac{1}{2}\nu T) = \exp(-\pi\nu m/eB)$ and

$$\tau_D^{-1}(x) = \nu I(x) = \nu \left[1 - \frac{1 - x^2}{2x^2} \left(\frac{1 - x^2}{2x} \ln \frac{1 + x}{1 - x} - 1 \right) \right]. \quad (8)$$

Note that the effective diffusion mean free time, $\tau_D(x)$, depends only weakly on the magnetic field. As $B \rightarrow 0$, ($x = 0$), $\tau_D(0) = (3/4)\nu^{-1} = (3/4)\tau$. As $B \rightarrow \infty$, ($x = 1$), $\tau_D(1) = \nu^{-1} = \tau$. However,

the factor $(1 - x^2) = (1 - P_0)$ in (8), the probability that an electron is wandering, varies from 1 to 0 as $\mathcal{B}$ is increased from 0 to ∞ .

FIG. 2. Paths connecting the upper edge to the lower one via a set of scatterers.

We now move away from the Grad limit, but keep both the dimensionless density and the hard disk diameter relative to the cyclotron radius small. I.e., $\rho = na^2 \equiv (a/l)^2 \ll 1$, $a/R \ll 1$. Combination of these two small parameters gives a new parameter, $R/l = R(2\Lambda a)^{-1/2} \equiv r$ of arbitrary magnitude. In the Grad limit, $r \rightarrow \infty$, and the diffusion tensor is that given by the GBE, i.e., by (7). On the other hand, for r sufficiently small, no diffusion is possible, since the electrons will, with probability unity, be trapped around a finite cluster of scatterers. In other words, there must be a percolation threshold, r_p , below which the diffusion tensor [defined by taking the thermodynamic limit *before* the limit $t \rightarrow \infty$ in (6)] vanishes.

We have determined this percolation threshold numerically by the following procedure (see Fig.2). Distribute N points (scatterers) at random on the unit square. Draw all possible paths $\mathcal{P}$ from the lower to the upper edge of the square via these points. With d_{ij} the distance between scatterers i and j along a path, the condition for the path to be percolating at magnetic field $\mathcal{B}$, i.e., at cyclotron radius R , is

$$\max_{(ij) \in \mathcal{P}} d_{ij} < 2R = 2rl = 2rN^{-1/2}. \quad (9)$$

For a given configuration of N scatterers we search through all paths, using a “relaxation” algorithm [8], and subsequently average over all configurations to find the critical value of r as [9]

$$r_c(N) = \frac{1}{2} \sqrt{N} < \min_{\mathcal{P}} \max_{(ij) \in \mathcal{P}} d_{ij} >. \quad (10)$$

Here $< \dots >$ denotes configurational average. The percolation threshold follows by extrapolation to the thermodynamic limit, $r_p = \lim_{N \rightarrow \infty} r_c(N)$. Fig.3 shows $r_c(N)$ for $100 \leq N \leq 19\,000$, averaged over a number of configurations that range from 20 000 to 1 000 (the product of N and the number of configurations averaged over is kept approximately constant). From Fig.3 we deduce that $r_p = 0.5995 \pm 0.0005$. Using more data from the same simulation, we have also determined the correlation length exponent, ν , and the order parameter exponent, β , as $\nu = 1.34 \pm 0.01$ and

$\beta = 0.14 \pm 0.01$. The exact results for lattice percolation in 2D are [7] $\nu = 4/3$, $\beta = 5/36$. Our conclusion is that the percolation problem discussed here is in the same equilibrium universality class as 2D lattice percolation.

FIG. 3. The critical radius $r_c(N)$ as a function of $N^{-3/8}$. For details, see text.

We conclude by a number of comments:

1. To our knowledge the GBE (2) is the first explicit example of an exact non-Markovian kinetic equation, valid for a physical model with well-defined dynamics. Equally important: The initial value problem has been solved exactly in the sense of (5).
2. We have also solved this problem by considering all “paths” from $\phi(0) = 0$ to $\phi(t)$ in velocity space, and summing over them with the appropriate weights. In this way we have constructed a result (for circling and wandering electrons combined), completely equivalent to (5), *without* proceeding via the GBE.
3. Note that the special role of the magnetic field and 2D here is to produce closed orbits. Any other mechanism that would produce exactly closed orbits would lead to analogous results.
4. Beyond the Grad limit, and in zero magnetic field, dynamical correlations including recollisions with the same scatterer produce algebraic tails $\sim t^{-2}$ (in 2D) in the velocity autocorrelation function [10,4]. With this in mind it is interesting to check the long time asymptotics for our case, as given by $F_1^G(p)$ of (5) for small p . However, the asymptotics found from (5) is simply exponentially damped oscillations, with parameter values somewhat shifted with respect to those given by the BE. Physically, it is not surprising that no algebraic long time tails arise here. The recollisions in GBE are of a very different nature from those connected with the ring events beyond the Grad limit.
5. The Einstein relation (also its zero temperature version of relevance in this case) proves, quite generally, that the diffusion and conductivity tensors are proportional to one another.

However, introduction of a small but finite electric field, $\mathcal{E}$, brings a qualitatively new feature into the problem. In crossed magnetic and electric fields, the center of the cyclotron orbit acquires a drift velocity perpendicular to both, and with magnitude $v_d = \mathcal{E}/B$. As a consequence, the orbits are no longer closed and the mechanism basic to our GBE no longer applies, at least in a strict sense. A rough estimate gives that an electron can recollide with the same scatterer provided that $v_d T \lesssim a$. This yields a characteristic value of the electric field, $\mathcal{E}_c = Ba/T = eB^2 a/(2\pi m)$, beyond which recollisions are no longer relevant. Clearly, $\mathcal{E}_c \rightarrow 0$ in the Grad limit. I.e., in the Grad limit, the diffusion tensor as a function of the electric field is singular at $\mathcal{E} = 0$ where, as the electric field is switched on, it jumps from the result (8) given by GBE to (presumably) the diffusion tensor which follows from the standard BE.

6. The non-linear response to small electric fields close to, but not in, the Grad limit, represents an intriguing unsolved problem. Or rather: It represents *two* unsolved problems, one above and one below the percolation threshold.
7. The above remarks indicate that the present model provides the perfect example to illustrate the dangers, forcefully pointed out by van Kampen [11], inherent in the arguments used in standard linear response theory.
8. Another intriguing unsolved problem raised by the present model is what are the effects of “small” quantum corrections? Or, put differently: Do physically relevant circumstances exist when quantum corrections are small in this context?
9. For sufficiently *large* dimensionless densities, and with overlapping scatterers, the existence of a percolation transitions was pointed out many years ago [12,4]. The mechanism (independent of a magnetic field!) is simply that the moving particle can be trapped in a cage formed by overlapping scatterers. It is, therefore, a curious fact that with overlapping scatterers and in a magnetic field, the diffusive regime as a function of the dimensionless density is bounded by a percolation threshold both from below and from above. Furthermore, with overlapping hard disks the position of the high density percolation threshold is determined by precisely the same calculation as that reported above. The only difference is that R must now be reinterpreted as the radius, a , of the overlapping hard disks.
10. In this Letter the classical 2D Lorentz model in a perpendicular magnetic field has been taken seriously on its own terms, and shown to contain a number of surprising properties. The obvious next question is then: Are some of these properties relevant in an experimental context? This questions raises non-trivial problems that we shall not try to address here. Let us confine ourselves to the identification of some of the relevant issues. The electron gas in 2D must, generally, be treated quantum mechanically. Electrons do interact, even if the Coulomb interaction is shielded. And: Weak disorder potentials, equivalent to a local electric field, are ubiquitous. These simple facts show that it will not be trivial to find an experimental realization of the present model. On the other hand, it is far from excluded that special circumstances do exist, in which the idealized classical Lorentz model is sufficiently close to the realities, for some of the results discussed here to be of relevance.

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