Regge's space-time skeletons and the quantization of 2d gravity

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Regge's method for regularizing euclidean quantum gravity is applied to two dimensional gravity. Using topologies with genus zero and two and a scale invariant measure, we show that the Regge method fails to reproduce the values of the string susceptibilities of the continuum model.

1. INTRODUCTION

One possibility to quantize gravity is to focus on finding a regularization of the euclidean path integral. Most studies so far have used either the dynamical triangulation (DT) or the Regge approach (RA). The continuum path integral is replaced in both cases by a summation over simplicial manifolds (skeletons). In the RA the edge lengths are the dynamical degrees of freedom and the connectivity of the skeleton is kept fixed, whereas vice versa in the DT the edge length is kept fixed and the summation in the path integral is over skeletons with different connectivity. An ideal test ground for the two regularization schemes is two dimensional gravity where many exact results have been derived in the continuum. It has been shown analytically and numerically that the DT reproduces indeed the results of the continuum theory. Based on a numerical simulation it has been claimed in ref.[1] that this is also the case for the RA. It will be shown in the following that this statement is incorrect. Further evidence for the failure of the Regge approach has recently been given in ref.[2]. Let's consider now the continuum path integral for 2d pure gravity

$$Z(A) = \int \frac{Dg}{\text{Vol(Diff)}} \ e^{-S(g)} \delta(\int d^2 x \sqrt{g} - A),$$

(1)

$$S(g) = \int d^2 x \sqrt{g} (\lambda + \kappa R + \beta \frac{1}{2} R^2).$$

(2)

The $\delta$ function imposes the constraint that the total area of the surface is equal to $A$. We shall omit the Newton's term in the following since it is related to a topological invariant. It is an irrelevant constant as long as the topology is kept fixed. On a formal level, i.e. when ignoring the measure in (1) one would expect that $Z(A) \propto \exp[-\lambda A] / A$. Quantum fluctuations are seen to lead to deviations from this naive scaling relation. Using techniques of conformal field theory Kawai and Nakayama showed that the regularized path integral obeys for $\beta / A \ll 1$ the scaling relation [3]

$$Z(A) \propto A^{7 \cdot \pi^{-2}} \exp[-\lambda R A] / A,$$

(3)

where $\lambda R$ is the renormalized cosmological constant. The string susceptibility $\gamma_{str} = 2 - \frac{\beta}{2}(1 - h)$ depends on the genus $h$ of the surface. For $\beta / A \gg 1$ they derived the relation

$$Z(A) \propto A^{7 \cdot r^{-2}} \exp[-S_{R^2}^{cl}(A)] \exp[-\lambda R A] / A,$$

(4)

where $\gamma_{str} = 2 - 2(1 - h)$ is a string susceptibility that is different from $\gamma_{str}$ in eq. (3) and $S_{R^2}^{cl}(A) = 16 \pi^2 (1 - h)^2 \beta / A$ is the classical action of the $R^2$ part in (2). The renormalization of the cosmological constant and the emergence of the factors $A^{7 \cdot \pi^{-2}}$ and $A^{7 \cdot r^{-2}}$ are due to the quantum fluctuations. We will investigate in the following sections the question if the RA can reproduce the continuum results for $\gamma_{str}$ and $\gamma_{str}'$. Since relations (3) and (4) coincide for a torus with the naive scaling relation (except for $\lambda \rightarrow \lambda R$) we shall consider in the following only topologies with $h = 0$ (sphere) and $h = 2$ (bi-torus).

2. THE REGGE METHOD

Using the RA we can regularize the path integral in (1) as follows

$$Z(A, N) = \int_0^{\infty} dp(l) F(l) e^{-S(l)} \delta(\sum_i A_i - A),$$

(5)
integration is over edge lengths in the skeleton. That the triangle inequalities are fulfilled. The argument \( N_1 \) denotes the total number of links in the skeleton. The factor \( F(l) \) in (5) ensures that the triangle inequalities are fulfilled. The integration is over edge lengths in the skeleton. A serious problem is that it is not clear which measure corresponds to the gauge invariant measure in eq. (1). A measure that has been chosen so far in literature is of the very simple form

\[
d\mu(l) = \prod_k (d\xi_k^0/d\xi_k^1) \xi_k^2,
\]

which is scale invariant if the parameter \( \zeta \) is equal to zero. To study the scaling behavior of the path integral we follow [1] and consider the derivative

\[
\frac{d\log Z(A, N_1)}{dA} = -\lambda + \frac{1}{A} \left( \left[ S_{R^2} + \frac{N_1 \zeta}{2} \right] - 1 \right), (7)
\]

where \( S_{R^2} = \beta \sum_i \delta_i^2 / A_i \) and \( \lambda \beta \) and \( N_1 \) are kept fixed. A derivation of relation (7) has been given in ref. [4]. The scaling formulas (3) and (4) can only be reproduced if the sum of two terms in square brackets turns in the infinite volume limit, \( N_1 \to \infty \), into expressions of the form

\[
-\left( \lambda R - \lambda \right) A + \gamma_{str} - 2 + -\left( \lambda R - \lambda \right) A + \gamma_{str}' - 2 + S_{R^2} (\beta / A).
\]

The average action \( S_{R^2} \) is an extensive quantity and therefore both terms in square brackets in (7) are \( \propto N_1 \). This implies that \( \zeta \) should be chosen such that it cancels the term in \( S_{R^2} \) which is \( \propto N_1 \). Already at this stage strong doubts arise that the RA can reproduce the continuum results for \( \gamma_{str} \) and \( \gamma_{str}' \). Instead of the measure \( \mu(l) \) we could namely have used also a different measure where the exponent \( \zeta \) is replaced by \( \zeta_0 + \zeta_1 / N_1 \). An expansion to leading order in \( \zeta_1 / N_1 \) shows that \( \gamma_{str} \) and \( \gamma_{str}' \) get shifted by \( \zeta_1 (1 + C) \) with \( C \) the connected part of \( \left( \sum_k \log |k| \right) \left( \sum_i A_i / (\beta / A) \right) \), evaluated for \( \zeta_1 = 0 \). It is very unlikely that \( C \) is exactly equal to \( -1/2 \), such that the string susceptibilities remain unchanged. Moreover for \( \zeta_0 = \beta = 0 \) one finds \( \gamma_{str} = \gamma_{str}' = 2 + \zeta_1 / 2 \) which is independent of the genus \( h \) and completely arbitrary.

### 3. NUMERICAL RESULTS

These considerations cast serious doubts on the claim made in ref. [1] that the Regge model with scale invariant measure reproduces the continuum value of the string susceptibility \( \gamma_{str} \). To settle this, we have computed \( S_{R^2} \) numerically for the sphere \( (h = 0) \) and the bi-torus \( (h = 2) \) using the scale invariant measure, i.e. for \( \zeta = 0 \). We have constructed the sphere from the surface of a three dimensional cube. The bi-torus has been obtained by gluing together two tori along the boundary of a cut out window. Details are given in ref. [4]. Two different algorithms have been used to simulate the fixed area path integral.
form $S_{R^2} = c_0(b/a)N_2 + c_1(\beta/a) + c_2(\beta/a)/N_2$. An example for such a fit is shown in fig. 2 for $\beta/a = 100$. The string susceptibility $\gamma^s_{\text{str}}$ and the quantity $S_{R^2}/(\beta/A) = 16\pi^2(1 - h)^2$ are now given by the limits $\lim_{\beta/a \to -\infty} c_1(\beta/a)$ and $\lim_{\beta/a \to -\infty} c_2(\beta/a)/(\beta/a)$. It turns out that $c_2$ can be determined with higher accuracy than $c_1$. This is understandable because the first exponential in (4) dominates the scaling behavior. Some results for $c_1(\beta/a)$ and $c_2(\beta/a)/(\beta/a)$ are given in the following table:

<table>
<thead>
<tr>
<th>$\beta/a$</th>
<th>$c_1$</th>
<th>$c_2/(\beta/a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>-1.9(5)</td>
<td>158.9(4)</td>
</tr>
<tr>
<td>70</td>
<td>-1.9(5)</td>
<td>158.6(4)</td>
</tr>
<tr>
<td>100</td>
<td>-2.2(8)</td>
<td>158.6(4)</td>
</tr>
</tbody>
</table>

The data for $c_1$ and $c_2/(\beta/a)$ seem not to depend very much on $\beta/a$ and are presumably close to their $\beta/a = \infty$ extrapolations. The results for $c_2/(\beta/a)$ are for both topologies close to $16\pi^2 = 157.91\ldots$ Also the coefficient $c_1$ agrees for the sphere within the large error bars with the continuum result $\gamma^s_{\text{str}} - 2 = -2$. In the case of the bi-torus the deviation from the continuum value $\gamma^s_{\text{str}} - 2 = 2$ is however again substantial.

The results reported in the last two sections strongly indicate that the RA fails to reproduce the quantum effects in 2d $R^6$ gravity. Only the classical term in (4) and perhaps $\gamma^s_{\text{str}}$ for the sphere are reproduced correctly. A likely explanation for the failure of the RA is the contribution of gauge degrees of freedom which appear not to decouple from the path integral.

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REFERENCES