Planckian Scattering of Non-abelian
Gauge Particles

Supriya Kar¹ and Jnanadeva Maharana²
Institute of Physics, Bhubaneswar-751 005, India.

Abstract

We present a systematic study of high energy scattering of non-abelian gauge particles in (3+1) dimensional Einstein gravity using semi-classical techniques of Verlinde and Verlinde. It is shown that the BRST gauge invariance of the Yang-Mills action in presence of quantum gravity at Planckian energy regime is maintained and the vertex operator is invariant under the BRST transformations. The presence of gravitational shock wave describing the gauge particles is discussed in the resulting (3+1) dimensional effective theory of Yang-Mills gravity.

¹e-mail: supriya@iopb.ernet.in
²e-mail: maharana@iopb.ernet.in
1 Introduction

In recent years the high energy scattering in quantum gravity at Planckian energy has attracted considerable attention. Several attempts have been undertaken to understand the phenomena and it is argued that a clear understanding of such processes will have to account for the effects of quantum gravity. It is felt that a non-perturbative treatment of the quantum gravity will be essential to envisage collisions at the Planckian energies. On the other hand, now there are convincing results which show that the semi-classical techniques\(^1\)\(^2\) can be used to compute the Planckian scattering amplitudes when the momentum transfer during the scattering is small compared to the Planck scale. In this kinematical regime a simple physical picture emerges, when one considers the collision of two neutral particles\(^1\), of masses negligible compared to the Planck mass, \(M_P\), at centre of mass energy of the order of \(M_P\). One of the in-coming particle finds a shock wave\(^1\) approaching and the effect of this shock wave is to add a phase to the wave function. The scattering amplitude thus computed corresponds to the one in eikonal approximation to leading order\(^3\)\(^4\). This scenario was first realised by 't Hooft\(^1\). Furthermore it is pointed out in Ref. 5 that the effective action describing the scattering processes has one piece which is in close resemblance with the string action with an imaginary string tension and has a vertex operator which is analogous to the one encounters in string theory\(^6\). In this context, it is worthwhile to note that the scattering processes at Planckian energy have been envisaged in the frame-work of string theory\(^7\). The string scattering amplitudes have been analysed in Ref. 8 for new symmetries that might reveal at high energy. Furthermore the superstring Planckian scattering in four dimensions, in the leading order eikonal, has been discussed in Ref. 4 which has a semi-classical shock
wave geometry. It is recognised that the string theory is capable of providing answers to some of the deep questions in quantum gravity; therefore, it is natural to expect that string theory will be able to predict phenomena at the Planck scale. The results of string theory$^{4-8}$ and those of the semi-classical approach$^{1-3,5}$ give similar results in the common domains of the kinematical variables as expected.

In the recent past, Verlinde and Verlinde$^9$ have undertaken a systematic investigation of scattering processes at Planckian energy. Their starting point is to write the full action as the sum of Einstein-Hilbert action for gravity and the matter part in 3+1 dimensions. It is argued that the Einstein-Hilbert (E-H)action can be decomposed into sum of two parts, since in an appropriate kinematic region, the small angle scattering processes are characterized by two length scales of very different orders of magnitude corresponding to longitudinal and transverse directions. Therefore, it is natural to introduce two coupling constants associated with the two parts of the gravitational action. It was demonstrated$^9$ that one of the two actions becomes a topological sigma model whereas the other one can be treated as a semi-classical gravitational action. Furthermore it was shown that the topological part of the action can be set to zero when one considers Planckian scattering and the semi-classical piece of the action can be expressed as a surface term. Similarly when the masses of the in-coming particles are negligible compared to $M_P$ the matter part of the action can also be expressed as a surface term. Indeed, the effective action that describes the high energy scattering is demonstrated to have a term which is analogous to the string action and the matter-gravity interaction part bears a strong similarity with the vertex operators in string theory. This remarkable result was first presented by 't Hooft in his investigations of Planckian scattering$^5$. Verlinde and Verlinde$^9$ have
derived this result in the semi-classical approximations for quantum gravity on a rigorous field theoretical frame-work. An alternative geometrical description of the topological field theory describing the Planckian scattering\textsuperscript{9} has also been discussed in Ref. 10. Recently, the semi-classical shock wave picture of various high energy scattering processes describing point particles, \textit{e.g.} point magnetic monopole and point charge particles\textsuperscript{11}, have been discussed in literature. Furthermore a new semi-classical approach to four dimensional Planckian energy scattering has been proposed in Ref. 12 to calculate the S-matrix to leading order.

In an another interesting paper\textsuperscript{13}, Verlinde and Verlinde have studied high energy scattering processes in quantum chromodynamics (QCD). At asymptotic energy a simple intuitive picture emerges where the longitudinal length scales are squeezed due to the Lorentz contraction whereas the transverse directions remain unaffected. The final effective action describing the high energy processes in QCD corresponds to a chiral Lagrangian. Consequently, using the techniques of chiral Lagrangian for the effective theory they were able to reproduce several results of the high energy scattering processes in QCD.

Motivated by this semi-classical shock wave solutions of vacuum Einstein’s eq.\textsuperscript{14}, we study the high energy collision of non-abelian gauge particles in 3+1 dimensions at Planckian energy. However in presence of matter fields and cosmological constant the shock wave geometry is also discussed in Ref. 15 recently. It is well known that at the Planckian energies gravity plays a dominant role in describing interactions among the particles. Here we present a systematic study of high energy processes involving the gauge particles in the frame-work of semi-classical approximation adopted in Ref. 9. Since the coupling of the Yang-Mills to Einstein gravity is conformally invariant
at the classical level, we have to specify the gauge fixing for the gravitational part as well as for the Yang-Mills action. The gauge choice, ghost action and the BRST transformations for the gravitational part have been presented in Ref. 9.

It is shown, in the present investigation, that the Yang-Mills ghost Lagrangian also decomposes into two parts when one introduces two length scales in the theory. We argue that each of the gauge fixed action and the ghost action decompose into pieces associated with the space-time metric. Moreover the BRST transformations reflect these effects explicitly. We present a systematic analysis for the effective theory and write down the vertex operators satisfying the requirements of BRST invariance ensuring the gauge invariance of the theory.

We outline our article as follows: In section 2, we discuss the Verlinde and Verlinde's approach to construct the topological field theory and in section 3, we consider the Yang-Mills coupling to the gravity in the Planckian energy limit and we show explicitly the BRST invariance of the Yang-Mills action in presence of gravity. The section 4, deals with the Einstein-Hilbert action coupled to Yang-Mills in high energy limit and we write down the canonical quantization relations. In section 5, we discuss the semi-classical gravitational shock wave picture for the Planckian energy scattering of non-abelian gauge particles and we write down the S-matrix. Finally we conclude with the discussions in section 6.
2 Effective theory of gravity in the Planckian energy limit

Let us choose two light-cone coordinates \( x^\alpha \equiv (x^+, x^-) \) and two transverse coordinates \( y^i \equiv (y, z) \) for \( i = 1, 2 \) with \( x^\pm = x \pm t \) to describe the collision of two fast moving particles with very large longitudinal momenta in \( x^\alpha \) plane and relatively at a large transversal distance \( (y^i) \). The gravitational field of each of these fast moving particles, as experienced by the other particle, takes the form of a shock wave. Thus the interactions between the particles will occur at the instant each particle passes through the shock waves of the other and as a result confined to the two dimensional transverse plane \( (y^i) \).

Under a gauge choice of the metric \( \mathcal{G}_{i\alpha} = 0 \), the metric can be written as\(^9\)

\[
\mathcal{G}_{\mu\nu} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & h_{ij} \end{pmatrix}
\]  

(1)

where \( g_{\alpha\beta} \) has the Lorentzian signature and \( h_{ij} \) is Euclidean. Notice that the Einstein-Hilbert, E-H, action

\[
S_{E-H} = \frac{1}{8\pi G_N} \int d^4 x \sqrt{-\mathcal{G}} R
\]

(2)

can be decomposed under the above gauge choice as a sum of two terms

\[
S_{E-H} = S_L(g, h) + S_T(h, g)
\]

(3)

where

\[
S_L(g, h) = \int d^2 x^\alpha d^2 y^i \sqrt{-\mathcal{G}} \sqrt{h} \left( R_h + \frac{1}{4} h^{i\gamma} \partial_{\gamma} g_{\alpha\beta} \partial_{\beta} g_{\gamma\epsilon} e^{\alpha\gamma} e^{\beta\epsilon} \right)
\]

(4)

and

\[
S_T(h, g) = \int d^2 x^\alpha d^2 y^i \sqrt{-\mathcal{G}} \sqrt{h} \left( R_g + \frac{1}{4} g^{\alpha\beta} \partial_{\alpha} h_{ij} \partial_{\beta} h_{kl} e^{ik} e^{jl} \right).
\]

(5)
If we adopt a convention, following Ref. 9, where the coordinates do not have dimensions of length and the metric carries the dimension of \((\text{length})^2\); we can write

\[
g_{\alpha\beta} = l_L^2 \hat{g}_{\alpha\beta}, \quad h_{ij} = l_T^2 \hat{h}_{ij}
\]  

(6)

Then it is easy to see that

\[
S_L(l_L^2 \hat{g}, l_T^2 \hat{h}) = l_L^2 S_L(\hat{g}, \hat{h}),
\]

\[
S_T(l_T^2 \hat{h}, l_L^2 \hat{g}) = l_T^2 S_T(\hat{h}, \hat{g}).
\]  

(7)

We recall that the E-H action is always divided by the Newton’s constant \(G_N = l_p^{-2}\). Now it is clear that in this kinematical regime the E-H action splits into a strongly coupled part \(S_L\) with the coupling constant \(g_L = \frac{\hat{g}}{l_p} \approx 1\) and a weakly coupled part \(S_T\) with coupling \(g_T = \frac{\hat{h}}{l_T} \ll 1\). It is argued\(^9\) that the full E-H action can be written as a sum of a theory to leading order in \(g_T\) and a non-perturbative theory with coupling \(g_L\). Furthermore the gauge choice \(\hat{G}_{i\alpha} = 0\) gives rise to ghost action which can be written as a sum of two terms

\[
S_{\text{ghost}} = S_L(b, c) + S_T(b, c)
\]  

(8)

where

\[
S_L(b, c) = \int d^2 x^\alpha d^2 y^i \sqrt{\hat{g}} \sqrt{\hat{h}} \hat{h}^{ij} b_{i\alpha} \partial_j c^\alpha
\]  

(9)

and

\[
S_T(b, c) = \int d^2 x^\alpha d^2 y^i \sqrt{\hat{h}} \sqrt{\hat{g}} \hat{g}_{\alpha\beta} b_{i\alpha} \partial_\beta c^i.
\]  

(10)

One can check that \(S_L\) and \(S_T\) scale with \(l_L^{-2}\) and \(l_T^{-2}\) respectively as in the case for \(S_L(g, h)\) and \(S_T(h, g)\). We conclude that \(S_L\) corresponds to a strong coupling and therefore requires a non-perturbative treatment. On the other hand \(S_T\) is in the
weak coupling regime. Consequently, the dominant configurations contributing the partition function are at the saddle point $S_T = 0$ which corresponds to

$$R_g = 0, \quad \partial_a h_{ij} = 0.$$  \hspace{1cm} (11)

The general solutions of eq. (11) are

$$g_{a\beta} = \partial_a X^a \partial_\beta X^b \eta_{ab}$$  \hspace{1cm} (12)

where $X^a = X^a(x^a, y^i)$ is the diffeomorphism of $g_{a\beta}$ to the digonal metric $\eta_{ab}$ and the euclidean metric, $h_{ij}$ is independent of longitudinal coordinates $(x_a)$. Notice that Eq. (12) reminds us of the vierbein formalism and we are tempted to make the identification $\epsilon^a_a = \partial_a X^a$. Indeed, this identification helps us in carrying out calculation later. Now the full action in Eq. (2) in this energy regime can be written in terms of the boundary values $\tilde{X}^a(\tau, y^i)$ by using Stoke’s theorem as

$$S_{bM} = \int d^a x \int d^i y^i \sqrt{h} \epsilon_{ab}(R_b \tilde{X}^a \partial_a \tilde{X}^b + \partial_b \tilde{X}^a \partial_a \tilde{X}^b).$$  \hspace{1cm} (13)

It has been shown in Ref. 9 that this boundary values $\tilde{X}^a(y^i)$ take arbitrary values and enable one to incorporate matter into the theory and hence responsible for the coupling of external matter to the gravitational field. Furthermore it has been argued that $S_T(h, g)$ together with the ghost Lagrangian $S_T(b, c)$ describe a topological sigma model. However, the strong coupling theory described by $S_L$, can be treated semi-classically. The stress energy momentum tensor,

$$T_{\alpha\beta} = \frac{2}{\sqrt{g}} \frac{\delta S_m}{\delta g^{\alpha\beta}},$$  \hspace{1cm} (14)

is traceless as the transversal components $T_{ij}$ are assumed to be negligible compared to the longitudinal components $T_{a\beta}$, when one considers the Planckian energy scale.
3 BRST invariance of the effective Yang-Mills action

Consider the coupling of $SU(2)$ gauge theory, for example W-bosons, to gravity which is conformally invariant at the classical level. The Yang-Mills action is

$$S_{YM} = -\frac{1}{4} \int d^4 x \sqrt{-g} G^{\mu \nu} G_{\rho \lambda} F_{\mu \nu}^a F_{\rho \lambda}^a$$

where the field strength $F_{\mu \nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c$, here $A_\mu^a \equiv A_\mu^a T^a$, being the gauge potential with ‘$a$’ taking value in the gauge group; ‘$\epsilon$’ is the Yang-Mills coupling constant and $\epsilon^{abc}$ are the structure constants evaluated when the generators, $T^a$, belong to the adjoint representation of the group. Now with the gauge choice (1) of the metric and the scaling arguments of $g_{\alpha \beta}$ and $h_{ij}$ as in Eq.(6), we write down the Eq.(15) as a sum of three terms

$$S_{YM} = -\frac{1}{4} \frac{l_T^2}{l_L^2} \int d^2 x^a d^2 y^i \sqrt{-\tilde{g}} \tilde{h}^{\alpha \gamma} \tilde{h}^{\beta \delta} F^{a}_{\alpha \beta} F^{a}_{\gamma \delta}$$

$$-\frac{1}{4} \frac{l_T^2}{l_T^2} \int d^2 x^a d^2 y^i \sqrt{-\tilde{g}} \tilde{h}^{i j} \tilde{h}^{j i} F^{a}_{i j} F^{a}_{k l}$$

$$-\frac{1}{2} \int d^2 x^a d^2 y^i \sqrt{-\tilde{g}} \tilde{h}^{i j} \tilde{h}^{\alpha \beta} F^{a}_{i j} F^{a}_{\alpha \beta}.$$  

Since the Yang-Mills coupling constant is dimensionless in four dimensions, we show that the three terms have coefficients which are ratios of length scales. Furthermore, the first two terms have coefficients $(\frac{l_T^2}{l_T^2})$ and $(\frac{l_L^2}{l_T^2})$, where as the last term does not have a coefficient as ratios of $l_T$ and $l_L$. The situation is analogous to the scenario considered by Verlinde and Verlinde in the context of QCD$^{13}$. If we identify $\frac{l_T^2}{l_T^2} = \lambda$ of Ref. 13, then the Lagrangian has the exact decomposition, however we have not incorporated the quark fields for the sake simplicity. One can argue that the first term
in (16), in this limit, is dominated by the trivial vacuum, i.e. $F_{\alpha\beta}^a = 0$ so that $A^\alpha_a$ is a pure gauge. In the high energy limit when $\frac{\hbar^2}{4\pi} \to 0$, the magnetic term (second term of Eq.(16) ) gives negligible contribution. As a result the truncated action is represented by the last term.

So far, we have not incorporated the gauge fixing and the ghost terms for the Yang-Mills theory. In order to present our arguments in a transparent manner, let us consider the case of flat space. The gauge field effective action is the sum of the Yang-Mills part, the gauge fixed part $L_{gf}$ and the ghost part $L_{ghost}$,

$$L = L_{YM} + L_{gf} + L_{ghost},$$

(17)

where $L_{YM}$ is given by eq.(16) with the metric replaced by the flat space metric. We choose Lorentz gauge to write down the gauge fixed Lagrangian as

$$L_{gf} = - \frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2$$

(18)

and the ghost part as

$$L_{ghost} = \partial^\mu \eta^a D^{ab}_\mu \omega^b$$

(19)

where $\eta^a$ and $\omega^b$ are the independent ghost fields, which are in the adjoint representation of the gauge group. The BRST transformations,

$$\delta A^\mu_a = -\frac{\epsilon}{e} D^\mu \omega^a,$$

$$\delta \eta^a = -\frac{\epsilon}{ae} \partial^\mu A^a_\mu,$$

$$\delta \omega^a = -\frac{\epsilon}{2} f^{abc} \omega^b \omega^c.$$  

(20)

leave the action (16) invariant, where $\epsilon$ is infinitesimal Grassmann parameter. Now, let us apply the arguments of Ref. 13 for QCD to the full action (16) with a flat
metric, where the high energy limit is implemented by the scaling argument given by

\[ x^\alpha \rightarrow \lambda x^\alpha, \quad y^i \rightarrow y^i, \]
\[ \partial_\alpha \rightarrow \lambda^{-1} \partial_\alpha, \quad \partial_i \rightarrow \partial_i, \]
\[ A_\alpha \rightarrow \lambda^{-1} A_\alpha \quad \text{and} \quad A_i \rightarrow A_i. \]

(21)

Immediately, one can see that the gauge fixed and the ghost part of the action under the above coordinate scaling can be written as

\[ \mathcal{L}_{gf} = -\frac{1}{2\alpha} \lambda^{-4} (\partial^\alpha A_\alpha^a)^2 + (\partial^i A_i^a)^2 \]

(22)

and the ghost part as

\[ \mathcal{L}_{ghost} = \lambda^{-2} \partial^\alpha \eta^a D_\alpha^{\ ab} \omega^b + \partial^i \eta^a D_i^{\ ab} \omega^b. \]

(23)

Thus, we observe that in the high energy limit \( i.e. \lambda \rightarrow 0 \), the term \( (F_{a\alpha\beta})^2 \) should be sufficiently small and only \( (F_{i\alpha})^2 \) survives. Further \( A_\alpha^a \) being a pure gauge can be set to zero. In this limit, the BRST transformations (20) are

\[ \delta A_\alpha^a = -\frac{\epsilon}{\alpha} D_\alpha^{\ ab} \omega^b = 0, \]
\[ \delta A_i^a = -\frac{\epsilon}{\alpha} D_i^{\ ab} \omega^b, \]
\[ \delta \eta^a = -\frac{\epsilon}{\alpha \epsilon} \partial^\alpha A_i^a, \]
\[ \delta \omega^a = -\frac{\epsilon}{2} F_i^{\ ab} \omega^b \omega^c \]

(24)

and the effective Lagrangian

\[ \mathcal{L}_{eff} = -\frac{1}{4} F_{i\alpha}^a F_i^\alpha_a \]

(25)
is BRST invariant. It is evident that in this approach to high energy limit the non-abelian gauge fields behave effectively like abelian in the flat space. Since $F_{ia}^a = \partial_i A_a^a - \partial_a A_i^a + \epsilon^{abc} A_b^i A_c^a$ goes over to $-\partial_a A_i^a$, when $A_a$ can be set to zero by the gauge choice. Now we consider the case of more general metric $\mathcal{G}_{\mu \nu}$, in the curved spacetime. The full action can be written as a sum of the Yang-Mills action $S_{YM}$, gauge fixed part $S_{gf}$ and the ghost part $S_{ghost}$. The Yang-Mills action is given in Eq.(16), the gauge fixed part and the ghost part can be written as

$$S_{gf} = -\frac{1}{2\alpha} \int d^4 x \sqrt{-g} \mathcal{G}^{\mu \nu} \partial_{\mu} A^a_{\nu} \mathcal{G}^{\lambda \rho} \partial_\lambda A^a_\rho$$

and

$$S_{ghost} = \int d^4 x \sqrt{-g} \mathcal{G}^{\mu \nu} \partial_{\mu} \eta^a D_{\nu} a A^a.$$ 

Now under the scaling of the metric $g_{\alpha \beta}$ and $h_{ij}$ (6) the gauge fixed and the ghost part of the action can be written as

$$S_{gf} = -\frac{1}{2\alpha} \frac{l_T^2}{l_p^2} \int d^2 x^2 d^2 y^2 \sqrt{-g} \sqrt{h} \mathcal{G}^{\alpha \beta} \partial_{\alpha} A_{\beta} \partial_{\gamma} A_{\gamma}$$

and

$$S_{ghost} = \frac{l_T^2}{2\alpha} \int d^2 x^2 d^2 y^2 \sqrt{-g} \sqrt{h} \mathcal{G}^{\alpha \beta} \partial_{\alpha} A_{\beta} D_{\gamma} a A^a.$$ 

In the Planckian energy limit $i.e. l_T \approx l_p$ and $l_T \gg l_p$, since $A_\alpha$ is in a pure gauge, as we argued earlier ( see the discussion after Eq.(16) ). Thus dropping the terms which have $l_T$ as the coefficient, the effective action is the sum of Eqs.(16), (28) and (29) can be written as

$$S^{eff}_{YM} = -\frac{1}{2} \int d^2 x^2 d^2 y^2 \sqrt{-g} \sqrt{h} \mathcal{G} a A_{\alpha} F_{ia}^a F_{ia}^a.$$
and the BRST transformations are

\[
\begin{align*}
\delta A^a &= -\frac{e}{\alpha} D^a_{\alpha} \omega^b = 0, \\
\delta A^a_i &= -\frac{e}{\alpha} D^a_{\alpha} \omega^b, \\
\delta \eta^a &= -\frac{e}{\alpha} \hat{h}^{ij} \partial_i A^a_j, \\
\delta \omega^a &= -\frac{e}{2} f^{abc} \omega^b \omega^c. 
\end{align*}
\]

(31)

It is noteworthy that the Yang-Mills effective action given in Eq.(30) respects the BRST symmetry as is evident from the transformation (31), in the Planckian energy limit.

4 Canonical quantization of the effective theory

In order to have a Planckian scattering picture using the semi-classical technique, we consider the Yang-Mills action (30) describing non-abelian gauge particles (e.g. W-bosons) of negligible mass in comparison to \( M_P \) as the external matter coupled to the E-H action (2)

\[
S_{eff} = S_{eff}^{E-H} + S_{eff}^{YM}.
\]

(32)

These W-bosons give rise to the energy-momentum stress tensor, which is covariantly conserved. In the present investigation, momenta of the particles in the longitudinal plane are of the order of Planckian energy. The transversal momenta are negligible in comparison to \( M_P \) but much larger than the particle momenta. In this energy regime, the mass of W-bosons are negligible and the scattering process is very well calculable as graviton exchange between the gauge particles dominate, with \( G_N \) playing the role of dimensionless coupling constant. For asymptotically large energies, all the particle
momenta are in the $x^a$ direction, as a result at the boundary $\partial M$ of the longitudinal plane $M$, all components except $T_{a\beta}$ vanish. The Yang-Mills energy-momentum stress tensor $T_{a\beta}$ can be calculated from Eq.(30) in the Planckian energy limit ($A_a$ is in pure gauge) is given by

$$T_{a\beta} = \hat{h}^{ij} \left[ \partial_a A_i \partial_\beta A_j - \frac{1}{2} \hat{g}_{a\beta} \hat{g}^{\rho\sigma} \partial_\rho A_i \partial_\sigma A_j \right]. \quad (33)$$

Since the in-coming and the out-going W-bosons in the Planckian energy regime are interacting with the E-H gravity at the boundary $\partial M$ of the longitudinal Lorentzian plane $M$, it is convenient to represent the matter particles of a momentum flux $P_{aa}$, defined in terms of stress-energy tensor $T_{a\beta} = P_{aa} e^{a}_{\beta}$, where $e^{a}_{\beta} = \partial_\beta X^a$ is the vierbein. Now, the momentum flux can be written as

$$P_{aa} = \hat{h}^{ij} \left[ \partial_a A_i \frac{\partial A_j^a}{\partial X^a} - \frac{1}{2} \partial_a X^b \eta_{ba} \partial_\rho A_i \partial_\rho A_j \right]. \quad (34)$$

The variation of the Yang-Mills action (30) with respect to $X^a$ is

$$\delta S_{YM} = 2 \int_{\partial M} dx^\beta \int d^2 y \sqrt{h} \epsilon^{a}_{\beta} P_{aa} \delta X^a \quad (35)$$

where $\bar{X}^a$ take the asymptotic values of $X^a$. Using eq.(34), we write down the effective Yang-Mills action in terms of the gauge fields as

$$S_{YM}^{\text{eff}} = 2 \int_{\partial M} dx^\beta \int d^2 y \sqrt{h} \epsilon^{a}_{\beta} \hat{h}^{ij} \left[ \partial_a A_i \frac{\partial A_j^a}{\partial X^a} \bar{X}^a - \frac{1}{2} \bar{X}^a \partial_a \bar{X}^a \right] \quad (36)$$

In order to determine the scattering amplitudes of W-bosons in presence of gravity, we write down the full effective action at the boundary $\partial M$ of the longitudinal plane by introducing a “time” variable $\tau$ parametrizing the coordinates $x^\beta(\tau)$ on the boundary. Since the boundary $\partial M$ is closed, $\tau$ is periodic. Now from Eqs.(13) and
(36) the full effective action on the boundary \( \partial M \) in \((2+1)\) dimensions is

\[
S_{\text{eff}}^{\partial M} = \int d\tau \int d^2 y \sqrt{\hat{h}} \epsilon_{ab} \partial_\tau \hat{X}^a (\Delta_\hat{h} - R_\hat{h}) \hat{X}^b + 2 \int d\tau \int d^2 y \sqrt{\hat{h}} \epsilon_{\beta} \hat{h}^{ij} [\partial_\tau A_i \frac{\partial A_j}{\partial \hat{X}^a} \hat{X}^a - \frac{1}{2} \hat{X}^a \partial_\tau \hat{X}_a \partial_\rho A_i \partial^\rho A_j] \tag{37}
\]

where \( \Delta_\hat{h} \) is the scalar laplacian in the transversal \( y^i \) plane. Here the action is written in terms of the new coordinate fields \( \hat{X}^a \). Note that the time derivative of the coordinate and the gauge fields appear as linear terms and the action is quadratic in the fields \( \hat{X}^a \) as is seen in Eq.(37). Thus the canonical conjugate momenta corresponding to the coordinate fields \( \hat{X}^a \) and the non-abelian gauge fields \( A_i \) are

\[
P_a = \epsilon_{ab} (\Delta_\hat{h} - R_\hat{h}) \hat{X}^b - \hat{h}^{ij} \hat{X}_a \partial_\rho A_i \partial^\rho A_j \tag{38}
\]

and

\[
\Pi_j = 2 \frac{\partial A_j}{\partial \hat{X}^a} \hat{X}^a \tag{39}
\]

by considering the parameter \( \tau \) as the quantization time variable. The canonical conjugate momenta \( P_a \) thus derived is invariant under the BRST transformations given by eq.(31), since \( T_{\alpha \beta} \) is also BRST invariant. In the absence of gauge fields \( (A_i = 0) \) the canonical quantization of \( \hat{X}^a \) in the 2+1 is discussed in Ref. 9 and the equal time commutation relations become

\[
[\hat{X}^b(y_1), P_b(y_2)] = i \delta^{(2)}(y_1, y_2). \tag{40}
\]

However in the presence of non-abelian gauge fields the equal time commutator for the coordinate fields can be written as

\[
[\hat{X}^a(y_1), \hat{X}^b(y_2)] = i \epsilon^{ab} \hat{f}(y_1, y_2), \tag{41}
\]
where $\bar{f}((y_1, y_2))$ is the Green function for the interacting W-bosons in presence of gravity. Now we write down classical equations of motion from Eq.(37) by taking the variation with respect to the coordinate field $\bar{X}^a$ as

$$\partial_\tau \bar{X}^a(\triangle^a - R^a_h) = \epsilon^{ab} P_{a,\tau}$$

where

$$P_{a,\tau} = -2\epsilon_\beta^\alpha \hbar \delta_{ij} \left[ \partial_\tau A_i \frac{\partial A_j}{\partial \bar{X}^a} - \frac{1}{2} \partial_\tau \bar{X}^a \partial_\rho A_i \partial^\rho A_j \right].$$

This expression (42) represents the Einstein equation in presence of Yang-Mills source at the Planckian energy and relates coordinate fields to their canonical conjugate momenta.

5 Semi-classical picture of scattering

Now in order to have an intuitive semi-classical picture we consider two W-bosons scattering process. The momentum flux $P_{a,\tau}$ for the W-bosons is the quantum mechanical operator in the Hilbert space of the gauge theory and is divided into left and right movers with momenta $p_\omega^i$ and $p_\omega^j$ respectively as W-boson masses are assumed to be negligible compared to the $M_P$. As a result the boundary $\partial M$ is divided into four different asymptotic regions of B space-time ($I^+$ and $I^-$). Considering the in-coming momentum flux at asymptotic past $I^-$ in the following form

$$P_{-in}(x, y) = p_{-1}^+ - \delta(x_+ - x_{1+})\delta(y - y_1),$$

$$P_{+in}(x, y) = p_{+2}^+ + \delta(x_- - x_{2-})\delta(y - y_2)$$

where $p_{-1}^+$ and $p_{+2}^+$ are the longitudinal momentum of the B in-coming left and right moving W-bosons in $x^-_1$ and $x^+_2$ directions respectively. The dynamics is assured
by the $\delta$-functions in the Eq. (44). Now the classical solutions of the reduced Einstein Eq. (42) are given by

$$X_{\pm}^{in} = x^\pm - p_1^\pm \theta(x_+ - x_{1+}) \bar{f}(y_1, y_2),$$

$$X_{\pm}^{in} = x^\pm - p_2^\pm \theta(x_+ - x_{2+}) \bar{f}(y_1, y_2).$$

(45)

The classical field configurations (45) exhibit a discontinuity known as gravitational shock wave at the $x^n$ trajectory of the particles are incompatible according to the commutation relations (41) and may give rise to the quantum mechanical effect of the gravitational shock waves. Thus in the quantum mechanical description it might be possible to consider two colliding shock waves. However in this frame-work, two W-bosons scattering by the exchange of gravitons has a semi-classical picture of gravitational shock wave colliding with a slow moving W-boson. This two particle scattering amplitude has been derived in an elegant way way by 't Hooft, considering the effect of shock wave of one of the two colliding particles on the wave packet of the other.

In order to determine $S$-matrix for the W-bosons scattering due to the exchange of gravitons, we write down the effective Yang-Mills coupling to Einstein gravity at Planckian energy, using Eqs. (34) and (37) in terms of “vertex operators” defined as

$$V(P) = \exp[2i \int d\tau \int d^2y \sqrt{\hbar} \bar{\cal X}^a(y)].$$

(46)

Here ‘P’ represents the path order product, as the $\bar{\cal X}^a(y)$ do not mutually commute. It is note-worthy that the vertex operators (46) do satisfy the BRST invariance (31). Furthermore the presence of four asymptotic regions of space-time correspond to four vertex operators describing in-coming and out-going momentum flux at $\mathcal{I}^-$ and $\mathcal{I}^+$. A brief study of the scattering amplitude for scalar particles has been discussed in
Ref.[9]. Now, we write down the four point vertex function representing the scattering amplitude for the W-bosons which may be computed by calculating the expectation value of the vertex operators

\[< V(P_+^{\text{in}}) V(P_-^{\text{in}}) V(P_+^{\text{out}}) V(P_-^{\text{out}}) > = \exp[i \int d^2 y_1 \sqrt{\hbar} \int d^2 y_2 \sqrt{\hbar} P_+^{\text{in}}(y_1) \bar{f}(y_1, y_2) P_-^{\text{in}}(y_2)]. \tag{47}\]

The mutually commuting operators \(P_+^{\text{in/out}}(y)\) can be used to characterize the incoming and out-going states of W-bosons. Furthermore by momentum conservation of out-flux and in-flux the scattering amplitude can be calculated using Eq.(44). In this approach\(^9\) the general covariance is broken in the transverse \(y^i\) direction. However in some physical situations the general covariance remains unbroken, \(e.g.\) the flat transversal metric \((h_{ij} = \delta_{ij})\) and analogy with string amplitude is very close. In this special case of flat transversal metric, the string amplitude calculation may be carried out giving rise to the scattering amplitudes. It is shown in Ref. 5 that the S-matrix element can also be written in a form similar to the string amplitude. This can be argued as, at distances much smaller than Planck length, one requires an explicit cut off because of the difficulty of renormalizability of quantum theory of Einstein gravity as a result it is believed that in this regime the conventional field theory should be replaced by string theory. Further 't Hooft's proposal\(^5\) for black hole S-matrix is analogous to the expression (47), apart from the fact that the transversal coordinates are replaced by angular coordinates. However in this semi-classical approach, it is not possible to consider angular coordinates in the transversal palne, as in the limit radial coordinate \(r \to 0\), the momentum transfer in the transverse plane is also large.
6 Discussions

To conclude, in this article we have studied small angle scattering of non-abelian gauge particles in the presence of nontrivial gravity. We have shown that in the Planckian energy limit, the gauge particles described by the Yang-Mills action is gauge invariant even in the absence of ghost and gauge fixed terms. The relevant gravitational modes mediating between the gauge particles at Planck energy are described by means of a topological field theory and the dynamics is described by the boundary values of $X^a$. The scattering amplitude can be calculated explicitly for the non-abelian gauge particles which is valid only to leading order in transverse coupling constant $g_T$ and is non-perturbative in the longitudinal coupling constant $g_L$. It would be interesting to consider the high energy scattering of the particles that appear as massless excitations of string theory and which might acquire masses due to various mechanisms in this frame-work. Since the string effective action is endowed with a rich symmetry structure, we expect that the high energy scattering amplitudes derived from such an action will have several novel features. We hope to address these issues in future.

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