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Knizhnik-Zamolodchikov Equations and the Algebraic Bethe Ansatz

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Abstract

Wave vectors of the algebraic Bethe ansatz provide modulo integrating factors solutions of difference equations and in the quasiclassical limit solutions of linear differential equations, the so called quantum and classical Knizhnik-Zamolodchikov equations. We describe the construction and show conversely how this relationship may be used to elucidate the structure of Bethe wave vectors of integrable inhomogeneous vertex models over simple Lie algebras.

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Contribution presented by the second author at the "Workshop on Symmetries and Integrability of Difference Equations, May 22 - 29, 1994, Esterel, Québec, Canada".

† Partially supported by the grant 211-5291 YPI of the German Bundesministerium für Forschung und Technologie

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BONN - TH-94-25
Bonn University
October 1994

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1 Introduction

Correlation functions of two-dimensional rational conformal field theories are governed by linear differential equations which derive from degeneracy relations of the underlying symmetry algebra. Among those theories are in particular conformally invariant sigma models on group manifolds, the Wess-Zumino-Novikov-Witten models [1]. The above mentioned differential equations were found long ago by Dashen and Frishman [2] in an important special case and were rediscovered more recently by Knizhnik and Zamolodchikov [3] in the general framework of Wess-Zumino-Novikov-Witten models. They are termed commonly Knizhnik-Zamolodchikov equations (KZE). We will in the following ignore the field theoretical context of the KZE but consider KZEs and their solutions per se.

Let g denote a simple Lie algebra with generators $t^a, a = 1 \cdots n$ and let $V^i, i = 1 \cdots N$ be vector spaces carrying representations of g with the generators being represented by operators t_i^a on V^i . The simplest version of the KZE is given in terms of a system of linear differential equations in N complex variables $z_1, \cdots z_N$ for functions ψ with values in the tensor space $V = V^1 \otimes \cdots \otimes V^N$ by

$$\kappa \frac{\partial}{\partial z_i} \psi(z_1 \cdots z_N) = \sum_{\substack{j \\ j \neq i}} \frac{(t_j \cdot t_i)}{z_i - z_j} \psi \quad (1)$$

where $(t_i t_j)$ denotes the g -invariant bilinear pairing of the generators t_i^a and t_j^a resp. κ is a coupling constant. The compatibility of the equations (1) is easily checked through direct calculation or it may be conceived as consequence of the classical Yang-Baxter equations satisfied by the matrices appearing on the r. h. s. of Eq. (1). With the notation $r_{ij}(\lambda) = \frac{(t_i t_j)}{\lambda}$ we have

$$[r_{ij}(\lambda), r_{kj}(\mu)] + [r_{ik}(\lambda - \mu), r_{ij}(\lambda)] + [r_{ik}(\lambda - \mu), r_{kj}(\mu)] = 0 \quad (2)$$

A partial classification of solutions of the preceding classical Yang Baxter equations has been achieved by Belavin and Drinfeld [4]. They present besides the rational solutions of Eq. (2) classes of trigonometric and elliptic solutions which give rise to generalized KZEs.

We concentrate in this report on the rational case, but we would like to mention in passing that the methods to be discussed below have been applied

successfully to the construction of solutions of trigonometric KZEs as well [5], [6].

It is known that Eq.(2) may be considered as the quasiclassical approximation to the (quantum) Yang-Baxter relations

$$R_{ij}(\lambda)R_{ik}(\lambda + \mu)R_{jk}(\mu) = R_{jk}(\mu)R_{ik}(\lambda + \mu)R_{ij}(\lambda) \quad (3)$$

$$R_{ij}(\cdot) : V_i \otimes V_j \longrightarrow V_i \otimes V_j$$

In fact, one recovers Eq.(2) from Eq.(3) in the case that one can expand $R_{ij}(\lambda)$ around the unit operator on $V_i \otimes V_j$ in terms of a quasiclassical expansion parameter η ,

$$R_{ij}(\lambda) = 1_i \otimes 1_j + \eta r_{ij}(\lambda) + 0(\eta^2) \quad (4)$$

by projecting in Eq.(3) onto the second order in η . As the classical Yang-Baxter relations guarantee the compatibility of the (classical) KZE, Eq.(1), so do the the quantum Yang-Baxter relations guarantee the compatibility of a system of difference equations, termed quantum Knizhnik -Zamolodchikov equations(QKZE). Let T_j denote a shift operator translating vector functions ψ in the j -th coordinate by an amount κ ,

$$(T_j\psi)(x_1, \dots, x_N) = \psi(x_1, \dots, x_j + \kappa, \dots, x_N) \quad (5)$$

The system of difference equations

$$(T_j\psi)(x_1 \dots x_N) = A_j(x_1, \dots, x_N)\psi(x_1, \dots, x_N), \quad (6)$$

$$A_j : V \longrightarrow V$$

is compatible if the operators A_j satisfy [7]

$$(T_i(A_j))A_i = (T_j(A_i))A_j \quad (7)$$

Choosing A_j in the form

$$A_j = R_{j,j-1}(x_j - x_{j-1} + \kappa) \cdots R_{j,1}(x_j - x_1 + \kappa) \cdot R_{N,j}^{-1}(x_N - x_j) \cdots R_{j+1,j}(x_{j+1} - x_j) \quad (8)$$

the conditions (7) are fulfilled in virtue of the Yang-Baxter relations (3), [7]. One is led back from the QKZE to the KZE by tuning simultaneously down

to zero the shift parameter κ and the quasiclassical expansion parameter η . Special instances of the difference equations (6) show up in the theory of form factors of integrable massive field theories, [8].

The purpose of this report is to present the connection between the KZE and QKZE resp. on one side and of the algebraic Bethe ansatz on the other side. The inhomogeneous $SU(2)$ vertex model will serve in the subsequent section as paradigm for this relation. The third section will be devoted to a discussion of the quasiclassical limit of vertex models on arbitrary simple Lie algebras. The material of section 2 is mainly taken from references [9] - [11] and the material of section 3 from [12].

2 Bethe wave vectors and (Q) KZE

We choose to discuss for the sake of its simplicity rational vertex models operating on tensor products of fundamental $SU(2)$ representations. The restriction to the fundamental representation and the choice of rational R -matrices is a mere convenience. The incorporation of arbitrary spin and the generalisation to trigonometric models could rather easily be achieved. The very choice of $SU(2)$ as underlying global symmetry algebra however is essential. Having selected instead a group of higher rank we would be forced into a discussion of the intricacies of the hierarchical Bethe ansatz. This technical point has not been mastered in a satisfactory manner in the context of the QKZE.

All spaces $V^0, V^{0'}, V^i, i = 1 \dots N$ appearing in the following either separately or as factors of tensorproducts are assumed to be proportional to C^2 . We introduce for convenience matrices $R_{00'}(\Lambda), R_{0i}(\Lambda), R_{ij}(\Lambda)$, all of which operate on spaces $C_2 \otimes C_2$ (more precisely on $V^0 \otimes V^{0'}, V^0 \otimes V^{0'}$ and $V^i \otimes V^j$ resp.) differing from each other by normalization factors,

$$R_{00'}(\Lambda) = \mathbf{1}_0 \otimes \mathbf{1}_{0'} + \frac{\eta}{\eta - 2\Lambda} \sum \sigma_0^a \otimes \sigma_{0'}^a \quad (9)$$

$$R_{0i}(\Lambda) = \frac{\eta - 2\Lambda}{2\Lambda} (\mathbf{1}_0 \otimes \mathbf{1}_i + \frac{\eta}{\eta - 2\Lambda} \sum \sigma_0^a \otimes \sigma_i^a) \quad (10)$$

$$R_{ij}(\Lambda) = \left(\frac{4\Lambda^2 - \eta^2}{4\Lambda^2 - 4\eta^2} \right)^{1/2} (\mathbf{1}_i \otimes \mathbf{1}_j + \frac{\eta}{\eta - 2\Lambda} \sum \sigma_i^a \otimes \sigma_j^a) \quad (11)$$

where $\mathbf{1}_x$ and σ_x^a denote the unit matrix and the Pauli matrices resp. acting on V^x . We will make also use of an alternative representation of R_{0i} ,

$$R_{0i}(\Lambda) = \begin{pmatrix} a_i(\Lambda) & b_i(\Lambda) \\ c_i(\Lambda) & d_i(\Lambda) \end{pmatrix} \quad (12)$$

$$a, b, c, d, : V^i \rightarrow V^i$$

$$a_i(\Lambda) = \frac{\eta - 2\lambda}{2\lambda} \left(\mathbf{1}_i + \frac{\eta}{\eta - 2\lambda} \sigma_i^3 \right)$$

etc. The normalisations in (10) and (11) are chosen s. t. holds

$$R_{ij}(\lambda) R_{ij}(-\lambda) = \mathbf{1}_i \otimes \mathbf{1}_j \quad (13)$$

and

$$d_i(\Lambda) v_i = v_i \quad (14)$$

where $v_i \in V^i$ is highest weight vector of the $SU(2)$ algebra, i. e.

$$\sigma_i^3 v_i = v_i, \quad \sigma_i^\pm v_i = 0.$$

Of crucial importance for the following will be the Yang-Baxter relations

$$R_{ij}(\lambda) R_{ik}(\lambda + \mu) R_{jk}(\mu) = R_{jk}(\mu) R_{ik}(\lambda + \mu) R_{ij}(\lambda) \quad (15)$$

$$R_{00'}(\lambda) R_{0i}(\lambda + \mu) R_{0'i}(\mu) = R_{0'i}(\lambda) R_{0i}(\lambda + \mu) R_{00'}(\lambda) \quad (16)$$

$$R_{0i}(\lambda) R_{0j}(\lambda + \mu) R_{ij}(\mu) = R_{ij}(\mu) R_{0j}(\lambda + \mu) R_{0i}(\lambda) \quad (17)$$

Eq's (15), (16) and (17) are isomorphic to each other since we deal exclusively with the fundamental representation of $SU(2)$. We nevertheless quote these equations separately, since they are used in different contexts.

We are now prepared to give a short account of the algebraic Bethe ansatz, for more details see ref. [13].

Following [13] we introduce the monodromy operator,

$$J_0(\lambda, z) = R_{0N}(\lambda - z_N) R_{0N-1}(\lambda - z_{N-1}) \cdots R_{01}(\lambda - z_1) \equiv \begin{pmatrix} A(\lambda, z) & B(\lambda, z) \\ C(\lambda, z) & D(\lambda, z) \end{pmatrix} \quad (18)$$

and the transfer matrix $T(\lambda, z)$ as the trace of J_0 with respect to the space V^0 ,

$$T(\Lambda, z) = \text{tr}_0 J_0(\lambda, z) = A(\Lambda, z) + D(\Lambda, z) \quad (19)$$

The notation used on the r. h. s. of Eq. (18) is the analogue of the notation used in Eq. (12). It may easily be verified that the local relation (16) can be lifted to the global relation

$$R_{00'}(\lambda - \mu) J_0(\lambda, z) J_{0'}(\mu, z) = J_{0'}(\mu, z) J_0(\lambda, z) R_{00'}(\lambda - \mu) \quad (20)$$

The last equation can be used to extract the commutation relations

$$\begin{aligned} [A(\lambda, z), A(\mu, z)] &= [D(\lambda, z), D(\mu, z)] = 0 \\ [B(\lambda, z), B(\mu, z)] &= [C(\lambda, z), C(\mu, z)] = 0 \end{aligned} \quad (21)$$

$$A(\mu)B(\lambda) = f(\mu - \lambda)B(\lambda)A(\mu) + g(\lambda - \mu)B(\mu)A(\lambda) \quad (22)$$

$$D(\mu)B(\lambda) = f(\lambda - \mu)B(\lambda)D(\mu) + g(\mu - \lambda)B(\mu)D(\lambda) \quad (23)$$

Let Ω be the tensor product of highest weight vectors v_i , introduced in Eq. (14),

$$\Omega = v_N \otimes \cdots \otimes v_1$$

Eq.'s (21) - (23) are sufficient to conclude that the wave vectors

$$X(\lambda_1 \cdots \lambda_m; z) = \prod_{i=1}^m B(\lambda_i, z) \Omega$$

of the algebraic Bethe ansatz satisfy a generalized eigenvalue equation of the form

$$\begin{aligned} T(\lambda, z) X(\lambda_1 \cdots \lambda_m; z) &= \\ = \Lambda(\lambda, \lambda_1 \cdots \lambda_m; z) X(\lambda_1 \cdots \lambda_m; z) - \sum_{\alpha} \frac{F_{\alpha} X_{\alpha}}{\lambda - \lambda_{\alpha}} \end{aligned} \quad (24)$$

where F_{α} denote some c -number functions and X_{α} are some vectors which are not parallel to X . The next steps in the algebraic Bethe ansatz consists in a specification of the spectral parameters $\lambda_1 \cdots \lambda_m$ s.t. the functions F_{α} vanish. Eq. (29) will then become a true eigenvalue equation. We will not follow this line but keep the parameters $\lambda_1 \cdots \lambda_m$ in general position. The 'off shell Bethe vector' $X(\lambda_1 \cdots \lambda_m, z)$ will be the basic building block for the subsequent construction of solutions of the Q KZE.

It is convenient to rewrite Eq. (6), [11], with the help of Eq. (13) as

$$R_{1,j}(X_1 - X_j - \kappa) \cdots R_{j-1,j}(X_{j-1} - X_j - \kappa) \varphi(X_1 \cdots X_j + \kappa_1 \cdots X_N) = \\ R_{j,N}(X_j - X_N) \cdots R(X_0 - X_{j+1}) \varphi(X_1 \cdots X_N) \quad (25)$$

The system (6) and therefore (25) is consistent, as mentioned above, because of the Yang-Baxter relations (15). Reshetikhin [11] proposes for (25) the ansatz

$$\varphi(X_1 \cdots X_N) = \\ = \sum_{\mu_1 \cdots \mu_m} F(\mu_1 \cdots \mu_m; X_1 \cdots X_N) \prod_{i=1}^m B(\mu_i; X) \quad (26)$$

with F a suitable chosen c-number function and the operators B taken from Eq. (18) acting on the highest weight vector Ω . The sum on the right hand side of (26) is meant to represent an additive version of a Jackson integral, cf. [7], s. t. the sums become invariant with respect to shift separately in the arguments $\mu_1 \cdots \mu_m$ by an amount κ . Inserting the ansatz (26) into Eq. (25) one can push, applying Eq. 17, the matrices $R_{x,j}$ and $R_{j,x'}$ on the l. h. s. and r. h. s. resp. through the operators $B(\mu_i)$ to the right. The ensuing expressions can be brought with the commutation relations (21) - (22) into the same operator order. The remaining difference of the two sides can be arbsobed into shifts of the suitably chosen c-number functions F . That means that the Jackson integral of the Bethe wave vector together with a well chosen integrating factor render a solution of the QKZE. This access to the KZE via the Bethe ansatz was first used in the quasiclassical context in [10].

3 The structure of quasiclassical Bethe wave vectors

We turn around now the strategy followed in the previous section. That is, we use knowledge about solutions of KZEs to disentangle the structure of Bethe wave vectors in the the quasiclassical approximation of vertex models where these models go over into Gaudin magnets [14] . The general vector valued solution of the KZE for an arbitrary simple Lie algebra was found by

Schechtman and Varchenko (SV) [15] . To describe their result we have to introduce some notations. Let $E_{\pm\alpha}$ and h_α denote the generators of a simple Lie algebra corresponding to simple root vectors α and to generators of a Cartan subalgebra respectively, with commutation relations

$$\begin{aligned} [E_\alpha, E_{-\alpha}] &= h_\alpha \\ [E_\alpha, E_\beta] &= N_{\alpha,\beta} E_{\alpha+\beta} \\ [h_\alpha, E_\beta] &= (\alpha, \beta) E_{\alpha+\beta} \end{aligned} \quad (27)$$

where $N_{\alpha,\beta}$ are constants and (α, β) refers to an appropriate inner product in the space of root vectors. The representation spaces $V^i, i = 1, \dots, N$ introduced above before Eq.(1) are supposed to be characterized by weight vectors ρ^i . $E_{\pm\alpha}^i, h_\alpha^i$ are representatives on V^i of the abstract generators of Eq.(27).

The operators t_i^a on the r. h. s. of Eq. (1) are supposed to comprise the latter ones. We define

$$E_{\pm\alpha} = \sum_{j=1}^N \frac{E_{\pm\alpha}^j}{\mu - z_j}$$

$$h_\alpha(\mu) = \sum_{j=1}^N \frac{h_\alpha^j}{\mu - z_j}$$

and introduce the scalar function

$$\chi(\mu_1 \cdots \mu_m; z_1 \cdots z_N) \equiv \chi(\mu, \lambda) =$$

$$\prod_{i < j}^m (\mu_i - \mu_j)^{\frac{(\alpha_i, \alpha_j)}{\kappa}} \prod_{a < b}^N (z_a - z_b)^{\frac{(\rho^a, \rho^b)}{\kappa}}$$

$$\prod_{j,a}^{m,N} (\mu_j - z_a)^{\frac{\alpha_j \rho^a}{\kappa}}$$

where $(\alpha_i, \alpha_j), (\rho^a, \rho^b)$ and (α_j, ρ^a) denote the same inner product as in Eq. (27). The SV solution of the KZE (1) can be represented as multiple contour integral,

$$\psi_m(z_1 \cdots z_N) = \oint \cdots \oint d\mu_1 \cdots d\mu_m \chi(\mu, z) \varphi_m(\mu, z)$$

where the vector valued function $\varphi_m(\mu, z)$ can be determined recursively as follows:

$$\varphi_m(\mu, z) = X^m(\mu, z)\omega$$

where ω denotes the highest weight state in V , i.e.

$$E_\alpha^j \omega = 0, \quad h_\alpha^j \omega = \rho_\alpha^j \omega, \quad j = 1, \cdots N.$$

$$X^1(\mu_1) = E_{-\alpha_1}(\mu_1),$$

$$X^m(\mu_1 \cdots \mu_m) = X^{m-1}(\mu_1 \cdots \mu_{m-1})X^1(\mu_m) + D(\mu_m)X^{m-1}(\mu_1 \cdots \mu_{m-1}), \quad (28)$$

$$X^{m-1} = X^{m-1}(E_{-\alpha_1}, \cdots E_{-\alpha_{m-1}}), \quad D(\mu_m)X^{m-1} = \sum_{i=1}^{m-1} \frac{1}{\mu_m - \mu_i}.$$

$$X^{m-1}(E_{-\alpha_1}, \cdots, [E_{-\alpha_m}, E_{-\alpha_i}], \cdots E_{-\alpha_{m-1}})$$

The recursion relations can be solved [12], [17]. φ_m is solution of the off-shell Bethe ansatz equation for the Gaudin magnet:

$$H_k \psi_m = h_k \psi_m + \sum_{j=1}^m f_{\mu_j} \bar{\varphi}_j^k, \quad a = 1, \cdots N \quad (29)$$

$$H_k = \sum_{j \neq a}^N \frac{(t_j t_k)}{(z_k - z_j)}$$

where h_k, f_{μ_j} are given by

$$h_k = \sum_{i \neq k} \frac{(\rho^k \rho^i)}{z_k - z_i} + \sum_{j=1}^m \frac{(\alpha_j) \rho^k}{z_k - \mu_j}$$

$$f_{\mu_j} = \sum_{i \neq j}^m \frac{(\alpha_j \alpha_i)}{\mu_j - \mu_i} - \sum_{k=1}^l \frac{(\alpha_j \rho^k)}{\mu_j - z_k}$$

and $\bar{\varphi}_k^j$ is given as sum of a certain subset of terms defining φ_k , *cf* [12]. The relations (29) were proven in [12] for the algebra of SU (3). It is rather obvious from the structure of this proof that the same holds for general simple Lie algebras. The result is confirmed by the work of Feigin, Frenkel and Reshetikhin [16] who use methods quite different from ours. The verification of the KZE is rather straightforward if once Eqs (29) is established. We refer for the details of the argument to [12].

The identification of the vector part of the integrand of the VS expression as off shell Bethe wave vector of Gaudin magnets displays the remarkable fact that the latter can be represented in a universal form equally valid for any simple Lie algebra.

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